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Research note

Elastic relaxation coefficients for a spherical cavity in a prestressed medium of arbitrary orientation

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Summary. Archambeau gave elastic relaxation coefficients for a spherical cavity introduced into a pure shear prestress field. The technique is generalized to a stress field for which only the trace of $\sigma_{ij}^{(0)}$ is zero. The coefficients are given for a general deviatoric prestress field of arbitrary orientation. They are then specialized to the case of a pure shear stress expressed in terms of the orientation angles commonly used in fault plane descriptions, i.e. dip and slip angle. The extension of this technique to an arbitrary homogeneous prestress field and its limitations are discussed.

Introduction

Archambeau (1964, 1968 and 1972) formulated an initial value problem of a spherical cavity introduced into an initially prestressed homogeneous pure or plane shear field. The dynamic fields were functions of differences between the initial and final states of the dilatation and vector rotation fields. These difference fields were represented by a spherical harmonic expansion whose coefficients were called the relaxation coefficients.

The relaxation coefficients were obtained from the static displacements caused by the presence of a spherical cavity in a deviatoric stress field at infinity. In simplifying the displacements for a deviatoric stress field (first invariant zero) to a pure field (first and third invariants zero), Archambeau imposed the condition that all normal stress components were identically zero. There are coordinate orientations for which this is true in a pure shear field (but none for a general deviatoric field). However, even for pure shear fields, it is overly restrictive for reference frames based on equivalent fault plane solutions.

In this note, we extend the relaxation coefficient expressions to those for a cavity in an arbitrary stress field at infinity. The relaxation coefficients are shown to depend only on the deviatoric part of the stress field. Thus a dynamic solution based on the relaxation coefficient technique is incomplete, and other methods must be used to obtain the radiation contribution due to the isotropic part of the prestress field.

Theory

From Landau & Lifshitz (1959), the change in the static displacement field, u_i^*, between an elastic space under a homogeneous stress field $\sigma_{ij}^{(0)}$ and a space with a spherical free surface cavity of radius R_0, centred at the origin with the same stress field $\sigma_{ij}^{(0)}$ at infinity is

$$u_i^* = \frac{R_0^3}{\mu(7-5\sigma)}\left\{5(1-\sigma)\tau_{ik}^{(0)}\frac{\partial}{\partial x_k}\left(\frac{1}{R}\right) - \frac{R_0^2}{4}\tau_{kl}^{(0)}\frac{\partial^3}{\partial x_i \partial x_k \partial x_l}\left(\frac{1}{R}\right) - {}^5\!/\!_4\,\tau_{kl}^{(0)}\frac{\partial^3(R)}{\partial x_i \partial x_k \partial x_l} + \frac{(7-5\sigma)}{12}\sigma_{ll}^{(0)}\frac{\partial}{\partial x_i}\left(\frac{1}{R}\right)\right\} \tag{1}$$

where the deviatoric stress components, $\tau_{ij}^{(0)}$, of the homogeneous stress field are given by

$$\tau_{ij}^{(0)} = \sigma_{ij}^{(0)} - {}^1\!/\!_3\,\delta_{ij}\sigma_{ll}^{(0)}$$

with (2)

$$\sigma_{ll}^{(0)} = \sigma_{11}^{(0)} + \sigma_{22}^{(0)} + \sigma_{33}^{(0)}.$$

The corresponding change in dilation, θ^*, and rotation field, Ω_i^*, defined by

$$\theta^* \equiv \frac{\partial u_l^*}{\partial x_l} \quad \text{and} \quad \Omega_i^* \equiv {}^1\!/\!_2\, e_{ijk}\frac{\partial u_k^*}{\partial x_j}$$

where e_{ijk} is the permutation tensor, is

$$\theta^* = \frac{5R_0^3(1-2\sigma)}{2\mu(7-5\sigma)}\left\{\tau_{kl}^{(0)}\frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{R}\right)\right\}$$

and (3)

$$\Omega_i^* = \frac{5R_0^3(1-\sigma)}{2\mu(7-5\sigma)}\left\{e_{ijk}\tau_{kl}^{(0)}\frac{\partial^2}{\partial x_j \partial x_l}\left(\frac{1}{R}\right)\right\}$$

It is interesting to note that the changes in dilation and rotation are independent of the isotropic part of the stress field and depend only on the deviatoric stress components of the initial homogeneous stress field.

Applying Archambeau's (1964) relations between the Cartesian derivatives of $(1/R)$ and spherical harmonics, equations (3) can be expressed as

$$\chi_\alpha^*(R) = \frac{1}{R^3}\sum_{m=0}^{2}\left(a_{2m}^{(\alpha)}\cos m\phi + b_{2m}^{(\alpha)}\sin m\phi\right)P_2^m(\cos\theta);$$

$\alpha = 1, 2, 3$, and 4 (4)

where

$$\chi_j^* \equiv \Omega_j^* \quad \text{and} \quad \chi_4^* \equiv \theta^*; \quad j = 1, 2, \text{and } 3$$

with

$$a_{2m}^{(\alpha)} = \frac{5[(1-\sigma)-\delta_{\alpha 4}\sigma]R_0^3}{\mu(7-5\sigma)} \begin{bmatrix} -\tfrac{3}{2}\tau_{23}^{(0)} & -\tfrac{1}{2}\tau_{12}^{(0)} & -\tfrac{1}{4}\tau_{23}^{(0)} \\ \tfrac{3}{2}\tau_{13}^{(0)} & \tfrac{1}{2}(\tau_{11}^{(0)}-\tau_{33}^{(0)}) & -\tfrac{1}{4}\tau_{13}^{(0)} \\ 0 & \tfrac{1}{2}\tau_{23}^{(0)} & \tfrac{1}{2}\tau_{12}^{(0)} \\ \tfrac{3}{2}\tau_{33}^{(0)} & \tau_{13}^{(0)} & \tfrac{1}{4}(\tau_{11}^{(0)}-\tau_{22}^{(0)}) \end{bmatrix}$$

$$b_{2m}^{(\alpha)} = \frac{5[(1-\sigma)-\delta_{\alpha 4}\sigma]R_0^3}{\mu(7-5\sigma)} \begin{bmatrix} 0 & -\tfrac{1}{2}(\tau_{22}^{(0)}-\tau_{33}^{(0)}) & \tfrac{1}{4}\tau_{13}^{(0)} \\ 0 & \tfrac{1}{2}\tau_{12}^{(0)} & -\tfrac{1}{4}\tau_{23}^{(0)} \\ 0 & -\tfrac{1}{2}\tau_{13}^{(0)} & \tfrac{1}{4}(\tau_{22}^{(0)}-\tau_{11}^{(0)}) \\ 0 & \tau_{23}^{(0)} & \tfrac{1}{2}\tau_{12}^{(0)} \end{bmatrix} \qquad (5)$$

and where m is the column index, α is the row index

$$\tau_{ll}^{(0)} = 0.$$

Setting

$$\tau_{11}^{(0)} = \tau_{22}^{(0)} = \tau_{33}^{(0)} = 0$$

the relaxation coefficients a_{2m} and b_{2m} reduce to those given in Archambeau (1972) after correction for misprints. Even for orientations for which the diagonal pure shear stress elements are zero, the off diagonal elements in Archambeau (1972) are not independent. They are related from the second invariant by

$$({}^s\sigma_{12}^{(0)})^2 + ({}^s\sigma_{13}^{(0)})^2 + ({}^s\sigma_{23}^{(0)})^2 = ({}^s\sigma^{(0)})^2$$

where ${}^s\sigma^{(0)}$ is the magnitude of the pure shear field.

Relaxation coefficients for equivalent seismic fault parameters

An important application of Archambeau (1972) and Randall (1966, 1971, 1972), is the seismic displacement field due to the tectonic release from an explosion in an initially prestressed pure shear field of arbitrary orientation. By a pure shear field, we mean a homogeneous stress field for which two of the principal stresses are equal and opposite and the third is zero. Thus, a homogeneous stress field with stress invariants of trace and determinant equal to zero is a pure shear stress field. This prestress problem can be considered fundamental since any homogeneous stress field can be resolved into the sum of an isotropic stress field and two pure shear fields oriented about any two of the principal stress axes. A ±45° rotation of the pure shear field about its null principal stress axis from the nonzero principal stress axes results in five of its six independent stress components being zero. In terms of the rotation reference frame (x_1^0, x_2^0, x_3^0)

$${}^s\sigma_{ij}^{(0)} = {}^s\sigma^{(0)} \neq 0 \qquad \begin{cases} i \neq j \\ i \neq k \\ j \neq k \end{cases}$$

where x_k^0 is the axis of the null principal stress.

The tectonic release radiation is the same as that due to a point shear dislocation or double couple in an unstressed elastic space with the exception that unlike a double couple the P and S waves have different source histories. A natural coordinate system for this event shown in Fig. 1 is to have x_1^0 and x_2^0 lie in the orthogonal fault plane solutions obtained from its seismic radiation. The null principal stress axis x_3^0 lies along the intersection of the fault planes. This alignment causes the two non-zero principal stress axes to be in the directions of maximum compression and tension.

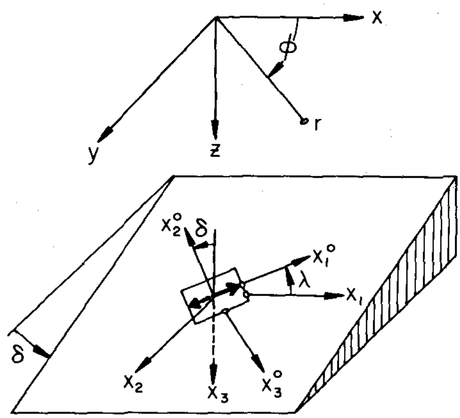

Figure 1. Spatial parameters for the equivalent fault plane solution.

The advantage of evaluating the tectonic release radiation field in this source coordinate system is that with the pure shear stress field being a deviatoric stress field all the deviatoric stress components in equation (5) are zero except

$$\tau_{12}^{(0)} = {}^s\sigma_{12}^{(0)} = {}^s\sigma^{(0)}.$$

This substantially reduces the effort needed to evaluate the tectonic release radiation field expressions given in Archambeau (1972). The seismic displacement field in the observer or hypocentral reference frame (x_1, x_2, x_3) can then be obtained by a coordinate transformation of these results (Harkrider 1976).

An alternate procedure is to resolve the pure shear stress field in the observer's coordinates. The radiation field can then be evaluated directly by substitution in equation (5)

For the latter method, the components used in the generalized relaxation coefficient formulation are given in terms of the fault plane parameters, dip δ and slip angle λ, by

$$\begin{aligned}
\tau_{11}^{(0)} &= 0 \\
\tau_{22}^{(0)} &= -\sin\lambda\sin 2\delta\ {}^s\sigma^{(0)} \\
\tau_{33}^{(0)} &= \sin\lambda\sin 2\delta\ {}^s\sigma^{(0)} \\
\tau_{12}^{(0)} &= \cos\lambda\sin\delta\ {}^s\sigma^{(0)} \\
\tau_{13}^{(0)} &= -\cos\lambda\cos\delta\ {}^s\sigma^{(0)} \\
\tau_{23}^{(0)} &= \sin\lambda\cos 2\delta\ {}^s\sigma^{(0)}.
\end{aligned} \tag{6}$$

The source and observer systems are the same for $\delta = 90^\circ$ and $\lambda = 0^\circ$. $\tau_{11}^{(0)}$ is zero for all values of δ and λ since by their definition x_1 always lies in the equivalent seismic fault plane (x_1^0, x_3^0), which is a plane of zero normal prestress.

Discussion

An interesting consequence of the fact that the generalized relaxation coefficients are independent of the isotropic part of the homogeneous stress field is that a direct application of Archambeau's theory of stress wave radiation from explosions in prestressed media would result in no seismic radiation from the formation of a cavity in an isotropic prestress field. In other words his results predict tectonic release radiation only from the relaxation of the deviatoric components of the prestress field.

For some prestress fields, the resolution of the deviatoric part of the stress field into pure shear fields is less convenient than a direct substitution of the deviatoric stress components into the generalized form of the relaxation coefficients. An example of this is an explosion in an uniaxial normal stress field.

For those who prefer the relaxation coefficient technique over the more general formulations of Randall (1964, 1966, 1971, 1972), it must be remembered that this technique does not include the contribution due to the isotropic part of the prestress field. This part of radiation must be obtained by other methods such as Randall (1964, 1972), in order to obtain a complete solution. The exact solution to the pure shear problem can be found in Burridge (1975).

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