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TWISTED RELATIVE TRACE FORMULAE WITH A VIEW TOWARDS UNITARY GROUPS

By JAYCE R. GETZ and ERIC WAMBACH

Abstract. We introduce a twisted relative trace formula which simultaneously generalizes the twisted trace formula of Langlands et. al. (in the quadratic case) and the relative trace formula of Jacquet and Lai. Certain matching statements relating this twisted relative trace formula to a relative trace formula are also proven (including the relevant fundamental lemma in the “biquadratic case”). Using recent work of Jacquet, Lapid and their collaborators and the Rankin-Selberg integral representation of the Asai L -function (obtained by Flicker using the theory of Jacquet, Piatetskii-Shapiro, and Shalika), we give the following application: Let E/F be a totally real quadratic extension with $\langle \sigma \rangle = \text{Gal}(E/F)$, let U^σ be a quasi-split unitary group with respect to a CM extension M/F , and let $U := \text{Res}_{E/F} U^\sigma$. Under suitable local hypotheses, we show that a cuspidal cohomological automorphic representation π of U whose Asai L -function has a pole at the edge of the critical strip is nearly equivalent to a cuspidal cohomological automorphic representation π' of U that is U^σ -distinguished in the sense that there is a form in the space of π' admitting a nonzero period over U^σ . This provides cohomologically nontrivial cycles of middle dimension on unitary Shimura varieties analogous to those on Hilbert modular surfaces studied by Harder, Langlands, and Rapoport.

Contents.

1. Introduction.
- 1.1. Distinction.
- 1.2. A result on unitary groups.
- 1.3. Outline of a proof.
2. Selected notation and conventions.
- 2.1. Algebraic groups.
- 2.2. Adèles.
- 2.3. Harish-Chandra subgroups.
3. Twisted relative classes and norm maps.
- 3.1. Twisted relative classes.
- 3.2. Norms of stable classes: definitions.
- 3.3. Existence of norms.
4. Matching of functions.
- 4.1. Local orbital integrals: definitions.
- 4.2. Stable local orbital integrals.
- 4.3. Matching of functions: definition.
- 4.4. Local relative κ -orbital integrals.

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- 4.5. Proof of the fundamental lemma: first case.
 - 4.6. Matching at the E -split places.
 - 5. Matching at the ramified places.
 - 5.1. Local constancy of the relative orbital integrals.
 - 5.2. Proof of Theorem 5.1.
 - 6. Spherical characters.
 - 7. Prestabilization of a single stable relative orbital integral.
 - 7.1. Global relative orbital integrals.
 - 7.2. Some cohomology groups.
 - 7.3. Prestabilization of a single relatively elliptic term.
 - 8. Grouping relatively elliptic terms.
 - 8.1. Haar measures.
 - 8.2. Elliptic kernels.
 - 8.3. Stable geometric expansions.
 - 8.4. The unitary case.
 - 9. Relative trace formulae.
 - 9.1. A simple relative trace formula.
 - 9.2. Comparison.
 - 10. Application.
 - 10.1. Statement of theorem.
 - 10.2. Proof of Corollary 10.4.
 - 10.3. Preparations for the proof of Theorem 10.3.
 - 10.4. Proof of Theorem 10.3: separating Hecke characters.
 - 10.5. Proof of Theorem 10.3: nonvanishing.
- References.

1. Introduction.

1.1. Distinction. Let F be a number field and let G be a reductive F -group with automorphism σ over F of order 2. Write $G^\sigma \leq G$ for the reductive subgroup whose points in an F -algebra R are given by

$$G^\sigma(R) := \{g \in G(R) : g = g^\sigma\}.$$

Let π be a cuspidal unitary automorphic representation of $G(\mathbb{A}_F)$. For smooth ϕ in the π -isotypic subspace of the cuspidal subspace $L^2_0(G(F) \backslash {}^1G(\mathbb{A}_F)) \leq L^2(G(F) \backslash {}^1G(\mathbb{A}_F))$, the period integral

$$(1.1.1) \quad \mathcal{P}_{G^\sigma}(\phi) := \int_{G^\sigma(F) \backslash (G^\sigma(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F))} \phi(x) dx$$

converges (see [AGR, Proposition 1, Section 2]). Here ${}^1G(\mathbb{A}_F) \leq G(\mathbb{A}_F)$ is the Harish-Chandra subgroup (see Section 2.3) and dx is induced by a choice of Haar measure on $G^\sigma(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F)$. If $\mathcal{P}_{G^\sigma}(\phi) \neq 0$ for some smooth ϕ in the π -isotypic

subspace of $L_0^2(G(F) \backslash {}^1G(\mathbb{A}_F))$, we say that π is G^σ -*distinguished*, or simply distinguished.

In [HLR], Harder, Langlands, and Rapoport introduced the notion of distinction as a tool for producing cohomologically nontrivial cycles on Hilbert modular surfaces. Their work was based on the following observation: If $G(F \otimes_{\mathbb{Q}} \mathbb{R}) \cong G^\sigma(F \otimes_{\mathbb{Q}} \mathbb{R}) \times G^\sigma(F \otimes_{\mathbb{Q}} \mathbb{R})$ and if π has nonzero cohomology with coefficients in $\mathbb{C}$ (see Definition 10.2) and is G^σ -distinguished then there is a differential form ω on a locally symmetric space attached to G that is π^∞ -isotypic under the action of certain Hecke correspondences such that ω has a nonzero period over a sub-symmetric space defined by G^σ . Harder, Langlands and Rapoport investigated the case where $G = \text{Res}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} \text{GL}_2$ for a square-free positive integer d and σ is the automorphism of G induced by the nontrivial element of $\text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$. As a consequence of their investigation they proved the Tate conjecture for the “non-CM” part of the cohomology of Hilbert modular surfaces. The key fact that allowed them to link Galois invariant elements in an étale cohomology group to the existence of cycles was a characterization of GL_2 -distinguished cuspidal automorphic representations of $\text{Res}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} \text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ in terms of a certain invariance property of π under σ .

The characterization of $\text{GL}_{2/\mathbb{Q}}$ -distinguished representations used in [HLR] can be proven using a modification of the Rankin-Selberg method [A]. More generally, let E/F be a quadratic extension of number fields. A variation of Jacquet, Piatetskii-Shapiro and Shalika’s interpretation of the Rankin-Selberg method similarly gives some characterization of those automorphic representations of $\text{Res}_{E/F} \text{GL}_n$ distinguished by $\text{GL}_{n/F}$ in terms of Asai L -functions [Fl3]. However, in most situations a characterization of the automorphic representations of G distinguished by G^σ in terms of arithmetic properties of the automorphic representation or analytic properties of L -functions attached to the automorphic representation is unknown.

To address these situations, Jacquet has introduced relative trace formulae in order to provide a (mostly conjectural) conceptual understanding of distinction in great generality. Though the theory still is far from complete, many interesting results are known, even in higher rank. For example, if E/F is a quadratic extension as above, Jacquet, Lapid, and their collaborators have provided a characterization of those representations of $\text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F)$ that are distinguished by a quasi-split unitary group in n -variables attached to E/F [J1, J2].

1.2. A result on unitary groups. Since the locally symmetric spaces attached to GL_n are never hermitian for $n > 2$, the higher rank results of Jacquet and Lapid mentioned above cannot be directly applied to the study of the Tate conjecture for Shimura varieties as in [HLR]. Despite this, in this paper we provide one means of relating distinction of automorphic representations of GL_n to distinction of automorphic representations on unitary groups (including those defining hermitian locally symmetric spaces). We now state an application. Let E/F be a

quadratic extension of totally real number fields, let $\langle \sigma \rangle = \text{Gal}(E/F)$, and let M/F be a CM extension. Let U^σ be a quasi-split unitary group in n variables attached to the extension M/F and let $U := \text{Res}_{E/F} U^\sigma$. We have the following result:

THEOREM 1.1. *Let π be a cuspidal automorphic representation of $U^\sigma(\mathbb{A}_E) = U(\mathbb{A}_F)$. Suppose that π satisfies the following assumptions:*

(1) *There is a finite-dimensional representation V of U_{F_∞} such that π has nonzero cohomology with coefficients in V .*

(2) *There is a finite place v_1 of F totally split in ME/F such that π_{v_1} is supercuspidal.*

(3) *There is a finite place $v_2 \neq v_1$ of F totally split in ME/F such that π_{v_2} is in the discrete series.*

(4) *For all places v of F such that ME/F is ramified and M/F , E/F are both nonsplit at v the weak base change Π of π to $U(\mathbb{A}_{ME}) \cong \text{GL}_n(\mathbb{A}_{ME})$ has the property that Π_v is relatively τ -regular.*

The representation π admits a weak base change Π to $\text{GL}_n(\mathbb{A}_{ME})$. If the partial Asai L -function $L^S(s, \Pi; r)$ has a pole at $s = 1$ then some cuspidal automorphic representation π' of $U(\mathbb{A}_F)$ nearly equivalent to π is U^σ -distinguished. Moreover, we can take π' to have nonzero cohomology with coefficients in V .

Here we say two automorphic representations are nearly equivalent if their local factors are isomorphic at almost all places. Moreover, S is a finite set of places of M including all infinite places, all places where ME/M is ramified, and all places below places of ME where Π is ramified, and

$$r : {}^L \text{Res}_{ME/M} \text{GL}_n \longrightarrow \text{GL}_{n^2}(\mathbb{C})$$

is the twisted tensor representation attached to the extension ME/M (see [F13] or [R, Section 6] for the definition of this representation). We are also using the fact that $L^S(s, \Pi; r)$ has a meromorphic continuation to a closed right half plane containing $s = 1$ (see [F13, FIZ]).

Theorem 1.1 is proved as Corollary 10.4 below. For the definition of a relatively τ -regular representation, see Section 6 below. For other unexplained notation and terminology, see Section 10. We indicate the role of some of the assumptions in the theorem:

Remarks. (1) The assumption that π is cohomological allows us to use the proof of [HL, Theorem 3.1.4] to conclude that the weak base change Π exists.

(2) The assumptions at the places v_1 and v_2 are made for two reasons. First, they allow us to apply known results on base change liftings from [HL]. Second, they allow us to use the simple form of the relative trace formula proved in [H]. If a relative version of the Arthur-Selberg trace formula were known, together with a theory of relative endoscopy, then it would probably be possible to remove the restrictions on v_1 and v_2 .

(3) Assumption (4) allows us to use test functions supported on the relatively τ -regular semisimple set at places where we have not proven satisfactory matching statements for arbitrary functions. The definition of a relatively τ -regular admissible representation is given in Section 6 below.

(4) The assumption that U^σ is quasi-split is relaxed somewhat in Corollary 10.4 below.

Given all of these simplifying assumptions, one might be surprised that this paper is still rather long. As an explanation, we point out that to prove our theorem we have to provide, from scratch, analogues of many standard constructions in the theory of the usual trace formula (for example, we need analogues of many of the results of [K1, K3, K4, K5]). Unfortunately, this takes some space. We have tried to streamline the presentation by not working in the greatest possible generality.

1.3. Outline of a proof. Let π be a cuspidal automorphic representation of $U(\mathbb{A}_F)$ satisfying the hypotheses of Theorem 1.1 and admitting a weak base change Π to $\text{Res}_{ME/F} \text{GL}_n(\mathbb{A}_F)$ (see Section 10.1 for our conventions regarding weak base change). Let τ be the automorphism of $G := \text{Res}_{ME/F} \text{GL}_n$ fixing U , and let $\theta = \sigma \circ \tau$. We prove Theorem 1.1 in two steps. First, we prove that if Π is distinguished by $\text{Res}_{M/F} \text{GL}_n$ and the quasi-split unitary group G^θ attached to $EM/(EM)^{(\theta)}$ then some automorphic representation π' nearly equivalent to π is U^σ distinguished. We prove this by exhibiting identities of the form

$$\sum_{\pi} \sum_{\phi \in \mathcal{B}(\pi)} \mathcal{P}_{U^\sigma}(\pi(\Phi^1)\phi) \overline{\mathcal{P}_{U^\sigma}(\phi)} = 2 \sum_{\Pi} \sum_{\phi_0 \in \mathcal{B}(\Pi)} \mathcal{P}_{G^\sigma}(\Pi(f^1)\phi_0) \overline{\mathcal{P}_{G^\theta}(\phi_0)}$$

for a sufficiently large set of functions $f \in C_c^\infty(G(\mathbb{A}_F))$ and $\Phi \in C_c^\infty(U(\mathbb{A}_F))$ that match in the sense of Definition 4.1. Here the sum on the left is over cuspidal automorphic representations of $U(\mathbb{A}_F)$, the sum on the right is over cuspidal automorphic representations of $G(\mathbb{A}_F)$, and $\mathcal{B}(\pi)$ is an orthonormal basis of the π -isotypic subspace of $L^2(U(F) \backslash U(\mathbb{A}_F))$ consisting of smooth vectors; $\mathcal{B}(\Pi)$ is defined similarly. We refer the reader to Proposition 9.2 for more details.

Second, the fact that Π is a base change implies that $\Pi^\tau \cong \Pi$. By work of Flicker and his collaborators, our assumption on the pole of the Asai L -function implies that Π is distinguished by $\text{Res}_{M/F} \text{GL}_n$ and moreover that $\Pi^\vee \cong \Pi^\sigma$. This implies that $\Pi^{\theta^\vee} \cong \Pi$, and hence by work of Jacquet, Lapid, and their collaborators this in turn implies that Π is distinguished by G^θ . This second step is contained in Section 10 and we will say no more about it in the introduction.

The proof of the first step is the heart of this paper and is based on a particular case of a general principle which we now explain. Let H be a connected reductive group, let $\langle \tau \rangle = \text{Gal}(M/F)$, and let

$$G := \text{Res}_{M/F} H.$$

Let σ be an automorphism of H of order 2 and let $H^\sigma \leq H$ and $G^\sigma \leq G$ be the subgroups fixed by σ . Let $\theta = \sigma \circ \tau$, viewed as an automorphism of G , and let $G^\theta \leq G$ be the subgroup fixed by θ . Let π be a cuspidal automorphic representation of $H(\mathbb{A}_F)$. Assume that π admits a weak base change Π to $G(\mathbb{A}_F)$. In favorable circumstances the machinery developed in this paper together with suitable fundamental lemmas should imply that if Π is G^σ -distinguished and G^θ -distinguished then a cuspidal automorphic representation π' nearly equivalent to π is H^σ -distinguished.

We mention two large obstacles to turning the general principle

Π distinguished by G^σ and $G^\theta \Rightarrow \exists H^\sigma$ -distinguished π' nearly equivalent to π

into a theorem in any given case. First is the current lack of a relative analogue of the Arthur-Selberg trace formula or a topological relative trace formula. Second is the current lack of a theory of relative endoscopy and the fundamental lemmas that should come along with the theory. If we had the tools of a relative Arthur-Selberg trace formula and the theory of relative endoscopy in hand, one could probably remove the annoying local restrictions in Theorem 1.1 and prove much more general versions of the principle indicated above. However, we caution that if one considers involutions σ that are not Galois involutions as we are primarily concerned with in this paper, there seem to be new phenomena lurking that we cannot explain here. These new phenomena may make the principle false as it is stated, though we believe something close to it is true.

We end this outline with some speculation regarding the undefined term “relative endoscopy.” The theory of (twisted) endoscopy, when complete, should yield an understanding of automorphic representations of classical groups in terms of those on general linear groups via a twisted trace formula (see [Ar2, Section 30], for example). We conjecture that a similar theory will relate distinguished representations on a classical group to representations on GL_n that are distinguished with respect to two subgroups. We believe that Theorem 1.1 provides an interesting implication of these relations in the special case where endoscopy reduces to base change.

Remark. We note that our approach to the relative trace formula is modeled on that exposed in [JL] and the introduction to [JLR]. More precisely, let E/F be a quadratic extension, suppose $H^\sigma = \mathrm{GL}_2$, $H = \mathrm{Res}_{E/F} \mathrm{GL}_2$, and let σ be induced by the nontrivial automorphism σ of $\mathrm{Gal}(E/F)$. Then the relatively elliptic part of the trace formula developed in [JL] is roughly the “ $\tau = 1$ ” case of our theory. If we instead let $E = F \oplus F/F$ be the “split” quadratic extension and again let σ be the automorphism of H induced by the nontrivial F -automorphism of E/F , then our trace formula comparing distinction on $G := \mathrm{Res}_{M/F} H$ to distinction on H is roughly equivalent to the “elliptic part” of the trace formula comparison used to establish quadratic base change in [L1] (though, of course, Langlands treats

general cyclic extensions). This justifies our claim in the abstract that our formulae simultaneously generalize the relative trace formula and the twisted trace formula (in the quadratic case). See Proposition 9.2 for a precise statement regarding trace formula comparisons.

We now give a synopsis of the various sections. We fix some notation in the next section. In Section 3.1, following [JL] we develop a notion of relative classes and relative τ -classes generalizing conjugacy classes and twisted conjugacy classes, respectively. A norm map is also defined and studied, and in Section 3.3 we prove that it has the properties one would expect, at least in the “biquadratic unitary” case (see Section 3.1).

Section 4 studies the corresponding notion of matching of stable local relative orbital integrals. We also prove the fundamental lemma for unit elements in the “biquadratic case” and the fundamental lemma for Hecke functions when various objects split (e.g. M/F). In Section 5 we prove that one can find some function in $C_c^\infty(H(F_v))$ matching a given function in $C_c^\infty(G(F_v))$ for nonarchimedean v . We define the notion of a relatively τ -regular admissible representation in Section 6; we require this notion due to our incomplete knowledge of matching functions.

Section 7 concerns globalization of stable relative orbital integrals. Here we follow Labesse’s formulation [La1] of the work of Kottwitz and Shelstad (following Langlands, see [KS, L2]) on the usual trace formula. In Section 8 we discuss the geometric expansion of the stable twisted relative trace formula. In Section 9 we use the main theorem of [H] together with all of the previous work to give a spectral expansion of the twisted relative trace formula on G and the relative trace formula on H , at least in the case relevant for the proof of Theorem 1.1. As noted above, Section 10 contains the final argument proving Theorem 1.1.

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2. Selected notation and conventions.

2.1. Algebraic groups. Let G be an algebraic group over a field F . We write Z_G for the center of G and G° for the identity component of G in the Zariski topology. If $\gamma \in G(F)$ and $G' \leq G$ is a subgroup we write $C_{\gamma, G'}$ for the centralizer

in G' of γ . If the element γ is semisimple then we say it is *elliptic* if $C_{\gamma,G}/Z_G$ is anisotropic. If τ is an automorphism of G and $\gamma \in G(R)$ for some commutative F -algebra R we write

$$\gamma^{-\tau} := (\gamma^{-1})^\tau.$$

We write $C_{\gamma,G}^\tau$ for the τ -centralizer of γ in G ; it is the reductive F -group whose points in an F -algebra R are given by

$$C_{\gamma,G}^\tau(R) := \{g \in G(R) : g^{-1}\gamma g^\tau = \gamma\}.$$

A torus $T \leq G$ is said to be τ -split if for any commutative F -algebra R we have $g^\tau = g^{-1}$ for all $g \in T(R)$. If θ is another automorphism of G , we say that T is (τ, θ) -split if it is both τ and θ -split.

2.2. Adèles. The adèles of a number field F are denoted by $\mathbb{A}_F$. For a set of places S of F we write $\mathbb{A}_{F,S} := \mathbb{A}_F \cap \prod_{v \in S} F_v$ and $\mathbb{A}_F^S := \mathbb{A}_F \cap \prod_{v \notin S} F_v$. If S is finite we often write $F_S := \mathbb{A}_{F,S}$. The set of infinite places of F will be denoted by ∞ . Thus $\mathbb{A}_{\mathbb{Q},\infty} = \mathbb{R}$ and $\mathbb{A}_{\mathbb{Q}}^\infty := \prod_{\substack{p \in \mathbb{Z}_{>0} \\ p \text{ prime}}} \mathbb{Q}_p$. For an affine F -variety G and a subset $W \leq G(\mathbb{A}_F)$ the notation W_S (resp. W^S) will denote the projection of W to $G(\mathbb{A}_{F,S})$ (resp. $G(\mathbb{A}_F^S)$). If W is replaced by an element of $G(\mathbb{A}_F)$, or if G is an algebraic group and W is replaced by a character of $G(\mathbb{A}_F)$ or a Haar measure on $G(\mathbb{A}_F)$, the same notation will be in force; e.g., if $\gamma \in G(\mathbb{A}_F)$ then γ_v is the projection of γ to $G(F_v)$.

2.3. Harish-Chandra subgroups. Let G be a connected reductive group over a number field F . We write $A_G \leq Z_G(F \otimes_{\mathbb{Q}} \mathbb{R})$ for the connected component of the real points of the largest $\mathbb{Q}$ -split torus in the center of $\text{Res}_{F/\mathbb{Q}} G$. Here when we say “connected component” we mean in the real topology. Write X for the group of $\mathbb{Q}$ -rational characters of G . There is a morphism

$$HC_G : G(\mathbb{A}_F) \longrightarrow \text{Hom}(X, \mathbb{R})$$

defined by

$$(2.3.1) \quad \langle HC_G(x), \chi \rangle = |\log(x^\chi)|$$

for $x \in G(\mathbb{A}_F)$ and $\chi \in X$. We write

$$(2.3.2) \quad {}^1G(\mathbb{A}_F) := \ker(HC_G)$$

and refer to it as the Harish-Chandra subgroup of $G(\mathbb{A}_F)$. Note that $G(F) \leq {}^1G(\mathbb{A}_F)$ and $G(\mathbb{A}_F)$ is the direct product of A_G and ${}^1G(\mathbb{A}_F)$.

3. Twisted relative classes and norm maps. In this section we recall the notion of relative classes in algebraic groups with involution and extend this notion to the “ τ -twisted” case. In Section 3.2 we define stable relative classes and a relative version of the usual norm map, which, in the original setting of the trace formula, relates τ -conjugacy classes and conjugacy classes. In Section 3.3 we use work of Borovoi to show that the norm map has particularly nice properties in the “biquadratic unitary case.”

3.1. Twisted relative classes. Let F be a number field and let M/F be either a quadratic or a trivial extension. Let τ be the generator of $\text{Gal}(M/F)$. Fix, once and for all, an algebraic closure $\overline{M}$ of M . We usually regard $\overline{M}$ as an algebraic closure of F as well, and denote it by $\overline{F}$. For all places v of F , choose once and for all an algebraic closure $\overline{F}_v$ of the completion F_v of F and a compatible system of embeddings $\overline{F} \hookrightarrow \overline{F}_v$.

Let H be a reductive F -group with (semisimple) automorphism σ of order two (defined over F). We let H^σ be the reductive F -group whose points in an F -algebra R are given by

$$H^\sigma(R) := \{g \in H(R) : g^\sigma = g\}.$$

We assume that:

- H^σ is connected.

This will always be the case if H is semisimple and simply connected [St1, Section 8]. Write

$$\begin{aligned} G &:= \text{Res}_{M/F} H \\ G^\sigma &:= \text{Res}_{M/F} H^\sigma. \end{aligned}$$

The automorphism σ of H induces an automorphism of G which we will also denote by σ . Set $\theta := \sigma \circ \tau$ (where σ and τ are viewed as automorphisms of G), and write $G^\theta \leq G$ for the subgroup whose points in an F -algebra R are given by

$$G^\theta(R) := \{g \in G(R) : g = g^\theta\}.$$

We note that θ , σ , and τ all commute.

Remark. Let E/F be a quadratic extension of number fields with ME/F biquadratic. The primary case of interest for us in this paper is the case where $H = \text{Res}_{E/F} H^\sigma$ and σ is the automorphism of H defined by the generator of $\text{Gal}(E/F)$ (which we will also denote by σ). We refer to this case as the *biquadratic case*.

We have left actions of $H^\sigma \times H^\sigma$ on H and $G^\sigma \times G^\theta$ on G given by

$$\begin{aligned}
 (3.1.1) \quad & (H^\sigma \times H^\sigma)(R) \times H(R) \longrightarrow H(R) \\
 & (g_1, g_2, g) \longmapsto (g_1 g g_2^{-1}) \\
 & (G^\sigma \times G^\theta)(R) \times G(R) \longrightarrow G(R) \\
 & (g_1, g_2, g) \longmapsto (g_1 g g_2^{-1})
 \end{aligned}$$

for F -algebras R . Our goal in this section is to compare the sets of double cosets $H^\sigma(R) \backslash H(R) / H^\sigma(R)$ and $G^\sigma(R) \backslash G(R) / G^\theta(R)$.

For this purpose, following [Ri, Lemma 2.4] we introduce the subschemes $Q := Q_H \subset H$ and $S := S_G \subset G$. They are defined to be the scheme theoretic images of the morphisms given on points in an F -algebra R by

$$\begin{aligned}
 (3.1.2) \quad & B_\sigma : H(R) \longrightarrow H(R) \\
 & g \longmapsto g g^{-\sigma} \\
 & B_\theta : G(R) \longrightarrow G(R) \\
 & g \longmapsto g g^{-\theta},
 \end{aligned}$$

respectively; they are closed affine F -subschemes of H and G , respectively (see [HW, Section 2.1]). Notice that $B_\sigma(g_1 g g_2^{-1}) = g_1 B_\sigma(g) g_1^{-1}$ and $B_\theta(g_1 g g_2^{-1}) = g_1 B_\theta(g) g_1^{-\tau}$ for $(g_1, g_2) \in H^\sigma(R)^2$ (resp. $(g_1, g_2) \in G^\sigma \times G^\theta(R)$). There are injections

$$\begin{aligned}
 (3.1.3) \quad & H^\sigma(R) \backslash H(R) / H^\sigma(R) \longrightarrow H^\sigma(R) \backslash Q(R) \\
 & g \longmapsto g g^{-\sigma} \\
 & G^\sigma(R) \backslash G(R) / G^\theta(R) \longrightarrow G^\sigma(R) \backslash S(R) \\
 & g \longmapsto g g^{-\theta}
 \end{aligned}$$

where H^σ acts by conjugation on Q and G^σ acts by τ -conjugation on S .

For any F -algebra k and any $\gamma \in H(k)$ (resp. $\delta \in G(k)$) write H_γ (resp. G_δ) for the k -group whose points in a commutative k -algebra R are given by

$$\begin{aligned}
 (3.1.4) \quad & H_\gamma(R) := \{(g_1, g_2) \in H^\sigma(R) \times H^\sigma(R) : g_1^{-1} \gamma g_2 = \gamma\} \\
 & G_\delta(R) := \{(g_1, g_2) \in G^\sigma(R) \times G^\theta(R) : g_1^{-1} \delta g_2 = \delta\}.
 \end{aligned}$$

Write C_{γ, H^σ} (resp. $C_{\delta, G^\sigma}^\tau$) for the centralizer of γ in H^σ (resp. τ -centralizer of δ in G^σ).

LEMMA 3.1. *There are isomorphisms $H_\gamma \xrightarrow{\sim} C_{\gamma\gamma^{-\sigma}, H^\sigma}$ and $G_\delta \xrightarrow{\sim} C_{\delta\delta^{-\theta}, G^\sigma}^\tau$ given on a k -algebra R by*

$$(3.1.5) \quad \begin{aligned} H_\gamma(R) &\xrightarrow{\sim} C_{\gamma\gamma^{-\sigma}, H^\sigma}(R) \\ (g_1, g_2) &\longmapsto g_1 \\ G_\delta(R) &\xrightarrow{\sim} C_{\delta\delta^{-\theta}, G^\sigma}^\tau(R) \\ (g_1, g_2) &\longmapsto g_1. \end{aligned}$$

Proof. Injectivity is clear. For surjectivity, note that if $g_1^{-1}\gamma\gamma^{-\sigma}g_1 = \gamma\gamma^{-\sigma}$ for $g_1 \in H^\sigma(R)$ then

$$\gamma^{-1}g_1^{-1}\gamma = \gamma^{-\sigma}g_1^{-1}\gamma^\sigma = (\gamma^{-1}g_1^{-1}\gamma)^\sigma,$$

so $(g_1, (\gamma^{-1}g_1^{-1}\gamma)^{-1}) \in H_\gamma(R)$. Similarly, if $g_1^{-1}\delta\delta^{-\theta}g_1^\tau = \delta\delta^{-\theta}$ for $g_1 \in G^\sigma(R)$, then

$$\delta^{-1}g_1^{-1}\delta = \delta^{-\theta}g_1^{-\tau}\delta^\theta = (\delta^{-1}g_1^{-\sigma}\delta)^\theta = (\delta^{-1}g_1^{-1}\delta)^\theta$$

so $(g_1, (\delta^{-1}g_1^{-1}\delta)^{-1}) \in G_\delta(R)$. $\square$

LEMMA 3.2. *Let k/F be a field, let $\gamma \in H(k)$ and $\delta \in G(k)$. If $\gamma\gamma^{-\sigma}$ is semisimple (resp. $\delta\delta^{-\theta}$ is τ -semisimple) then $C_{\gamma\gamma^{-\sigma}, H^\sigma}$ (resp. $C_{\delta\delta^{-\theta}, G^\sigma}^\tau$) is reductive.*

Proof. See [H, Section 2]. $\square$

We also often use the following lemma:

LEMMA 3.3. *Let k/F be a field, let $\gamma \in H(k)$ and $\delta \in G(k)$. Assume we are in the biquadratic case and that $(H^\sigma)^\text{der}$ is simply connected. If $\gamma\gamma^{-\sigma}$ is semisimple (resp. $\delta\delta^{-\theta}$ is τ -semisimple) then $C_{\gamma\gamma^{-\sigma}, H^\sigma}$ (resp. $C_{\delta\delta^{-\theta}, G^\sigma}^\tau$) is connected.*

Proof. Recall that for any semisimple $x \in H^\sigma(k)$ the group C_{x, H^σ} is connected by a theorem of Steinberg ([St1], [K1, Section 3]). Upon passing to the algebraic closure, it is easy to see that this fact implies the lemma. $\square$

For an F -algebra R write

$$(3.1.6) \quad \begin{aligned} \Gamma_r(R) &:= H^\sigma(R) \backslash H(R) / H^\sigma(R) \\ \Gamma_{r\tau}(R) &:= G^\sigma(R) \backslash G(R) / G^\theta(R). \end{aligned}$$

We refer to the elements of $\Gamma_r(R)$ (resp. $\Gamma_{r\tau}(R)$) as relative classes (resp. relative τ -classes). If two elements $\gamma, \gamma' \in H(R)$ (resp. $\delta, \delta' \in G(R)$) map to the same class in $\Gamma_r(R)$ (resp. $\Gamma_{r\tau}(R)$) we say that they are in the same relative class (resp. relative τ -class). In this paper we will loosely refer to any construction or object involving relative τ -classes as the twisted case.

The following definitions are adaptations of those appearing in [JL, Fl1]:

Definition 3.4. Let k/F be a field. An element $\gamma \in H(k)$ is *relatively semisimple* (resp. relatively elliptic, relatively regular) if $\gamma\gamma^{-\sigma}$ is semisimple (resp. elliptic, regular) as an element of $H(k)$.

Definition 3.5. Let k/F be a field. An element $\delta \in G(k)$ is *relatively τ -semisimple* (resp. relatively τ -elliptic, relatively τ -regular) if $\delta\delta^{-\theta}$ is τ -semisimple (resp. τ -elliptic, τ -regular) in the usual sense.

For brevity, we say that δ is relatively τ -regular semisimple if it is both relatively τ -regular and relatively τ -semisimple, with similar conventions regarding other combinations of regular, semisimple, and elliptic. It is easy to check that $\gamma \in H(R)$ is relatively semisimple (resp. elliptic, regular) if and only if every element in $H^\sigma(R)\gamma H^\sigma(R)$ is relatively semisimple (resp. elliptic, regular). A similar statement holds in the twisted case. For a field k/F we denote by

$$(3.1.7) \quad \begin{aligned} \Gamma_r^{ss}(k) &:= \{H^\sigma(k)\gamma H^\sigma(k) \in \Gamma_r(k) : \gamma \text{ is relatively semisimple}\} \\ \Gamma_{r\tau}^{ss}(k) &:= \{G^\sigma(k)\delta G^\theta(k) \in \Gamma_{r\tau}(k) : \delta \text{ is relatively } \tau\text{-semisimple}\}. \end{aligned}$$

Let $Q^{ss} \subset Q$ be the subscheme whose points in an F -algebra R are semisimple elements of $H(R)$.

LEMMA 3.6. *Assume that we are in the biquadratic case and H^{der} is simply connected. For k either F or F_v for some place v of F the natural morphism (3.1.2) induce a bijection*

$$\Gamma_r^{ss}(k) \longleftrightarrow H^\sigma(k) \backslash Q^{ss}(k).$$

Proof. We claim that every element of $Q^{ss}(k)$ is of the form $\gamma\gamma^{-\sigma}$ for some $\gamma \in H(k)$. To prove the claim, first let $\alpha \in Q^{ss}(k)$ and consider $C_{\alpha,H}(k)$. Since α is semisimple and H^{der} is simply connected, $C_{\alpha,H}$ is a (connected) reductive group [St1], [K1, Section 3]. Using the fact that $\alpha = \alpha^{-\sigma}$, it is trivial to check that $C_{\alpha,H}$ is σ -invariant, and it follows that $C_{\alpha,H} = \text{Res}_{E \otimes_F k/k} I$ for some reductive subgroup $I \leq H^\sigma$. Let $T' \leq I$ be a maximal torus and let $T = \text{Res}_{E \otimes_F k/k} T'$. Then α is in the centralizer of T in $C_{\alpha,H}$, and it follows that $\alpha \in T(k)$ [B, Section IV.11.12].

Denoting by $\langle \sigma \rangle$ the subgroup of the group of automorphisms of $T(k)$ generated by σ , we have

$$N_{\langle \sigma \rangle} T(k) / I_{\langle \sigma \rangle} T(k) = \widehat{H}^{-1}(\langle \sigma \rangle, T(k)) \cong H^1(\langle \sigma \rangle, T(k)) \cong 1$$

(see [S1, Sections VIII.1 and VIII.4] for notation and the existence of the first isomorphism). Here we have used Shapiro's lemma for the last isomorphism. By definition of $N_{\langle \sigma \rangle} T(k) / I_{\langle \sigma \rangle} T(k)$, it follows that $\alpha = \gamma\gamma^{-\sigma}$ for some $\gamma \in T(k)$. This completes the proof of our claim. We leave to the reader the straightforward task of deducing the lemma from the claim. $\square$

3.2. Norms of stable classes: definitions. In this section we define stable classes and a notion of norm. We assume throughout the subsection that k is either F or F_v for some place v of F . In this section and for the remainder of the paper, we will always make the following simplifying assumption:

- For all relatively semisimple $\gamma \in H(\overline{F})$ the group H_γ is connected.

By passing to an algebraic closure, it is easy to see that this assumption implies that G_δ is connected for all relatively τ -semisimple $\delta \in G(\overline{F})$. For a condition sufficient to ensure the assumption above holds, see Lemma 3.3.

Definition 3.7. Two relatively semisimple elements $\gamma, \gamma_0 \in H(k)$ are in the same *stable relative class* if and only if $x\gamma\gamma^{-\sigma}x^{-1} = \gamma_0\gamma_0^{-\sigma}$ for some $x \in H^\sigma(\overline{k})$.

Similarly, two elements $\delta, \delta_0 \in G(k)$ are in the same *stable relative τ -class* if and only if $y\delta\delta^{-\theta}y^{-\tau} = \delta_0\delta_0^{-\theta}$ for some $y \in G^\sigma(\overline{k})$.

For a pair of reductive k -groups I, H , define

$$\mathcal{D}(I, H; k) := \ker [H^1(k, I) \longrightarrow H^1(k, H)].$$

We note that the set of relative classes in the stable relative class of $\gamma_0 \in H(k)$ is in bijection with

$$(3.2.1) \quad \mathcal{D}(C_{\gamma_0\gamma_0^{-\sigma}, H^\sigma}, H^\sigma; k)$$

and the set of relative τ -classes in the stable relative τ -class of $\delta_0 \in G(k)$ is in bijection with

$$(3.2.2) \quad \mathcal{D}(C_{\delta_0\delta_0^{-\theta}, G^\sigma}^\tau, G^\sigma; k).$$

In particular, if $k = F_v$ for some place v of F , then both of these sets of stable classes are finite (compare [La1, Sections 1.8 and 2.3]).

Later, when defining orbital integrals, the following observation will be useful. Suppose that γ, γ_0 are in the same stable relative class; choose x as in Definition 3.7. Then we have an inner twist

$$(3.2.3) \quad \begin{array}{c} H_\gamma(\overline{k}) \xrightarrow{\sim} C_{\gamma\gamma^{-\sigma}, H^\sigma}(\overline{k}) \xrightarrow{\sim} C_{\gamma_0\gamma_0^{-\sigma}, H^\sigma}(\overline{k}) \xrightarrow{\sim} H_{\gamma_0}(\overline{k}) \\ g \longmapsto xgx^{-1} \end{array}$$

where the first and last isomorphism are those of (3.1.5) (compare [K1, Lemma 3.2]). Similarly, if δ, δ_0 are in the same stable relative τ -class, choose y as in Definition 3.7. Then there is an inner twisting

$$(3.2.4) \quad \begin{array}{c} G_\delta(\overline{k}) \xrightarrow{\sim} C_{\delta\delta^{-\theta}, G^\sigma}^\tau(\overline{k}) \xrightarrow{\sim} C_{\delta_0\delta_0^{-\theta}, G^\sigma}^\tau(\overline{k}) \xrightarrow{\sim} G_{\delta_0}(\overline{k}) \\ g \longmapsto ygy^{-1}. \end{array}$$

We have the following definition:

Definition 3.8. Suppose that $\gamma \in H(k)$ is relatively semisimple. We say that γ is a *norm* of $\delta \in G(k)$ if there is an $y \in G^\sigma(\bar{k})$ such that

$$y\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau y^{-1} = \gamma\gamma^{-\sigma}.$$

The following lemma together with Proposition 3.10 below can be used to show that norms exist in certain cases:

LEMMA 3.9. *Suppose that $\delta \in G(k)$ is relatively τ -semisimple. Let $O(\bar{k}) \subset Q(\bar{k})$ denote the subset of elements that are $G^\sigma(\bar{k})$ -conjugate to $\delta\delta^{-\theta}(\delta\delta^{-\theta})^{-\sigma} = \delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$. Then $O(\bar{k})$ is fixed by $\text{Gal}(\bar{k}/k)$ and is the set of $\bar{k}$ -points of a (nonempty) homogeneous space O for H_k^σ . Any $\gamma \in H(k)$ such that $\gamma\gamma^{-\sigma} \in O(k)$ is a norm of δ .*

Proof. Let $\tilde{Q} := \text{Res}_{M/F} Q$; it is the scheme theoretic image of the map $G \rightarrow G$ defined by

$$\begin{aligned} G(R) &\longrightarrow G(R) \\ g &\longmapsto gg^{-\sigma} \end{aligned}$$

for F -algebras R . The group scheme G^σ acts on $\tilde{Q}$ by conjugation. Note that $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau \in \tilde{Q}(k)$. We claim that a $G^\sigma(\bar{k})$ -conjugate $\bar{\gamma}$ of $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$ is contained in $Q(\bar{k})$. Assuming the claim for a moment, it is easy to check that

$$O(\bar{k}) := \{h^{-1}\bar{\gamma}h : h \in H^\sigma(\bar{k})\} = \{g^{-1}\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau g : g \in G^\sigma(\bar{k})\} \cap Q(\bar{k}).$$

Since the set on the right is fixed by $\text{Gal}(\bar{k}/k)$, the set on the left is as well. It follows that $O(\bar{k})$ is the set of $\bar{k}$ -points of a homogeneous space O for H_k^σ .

Choose an isomorphism

$$(3.2.5) \quad G_{\bar{k}} \xrightarrow{\sim} H_{\bar{k}} \times H_{\bar{k}}$$

equivariant with respect to σ and intertwining τ with $(x, y) \mapsto (y, x)$. Let (δ_1, δ_2) be the image of $\delta\delta^{-\theta}$ under this isomorphism. Then $\delta_2 = \delta_1^{-\sigma}$. Replacing $\delta\delta^{-\theta}$ by $g^{-1}\delta\delta^{-\theta}g^\tau$ for some $g \in G^\sigma(\bar{k})$ does not affect the $G^\sigma(\bar{k})$ -conjugacy class of $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$. Translating this statement to $H_{\bar{k}} \times H_{\bar{k}}$ using (3.2.5), we see that to prove the claim it suffices to exhibit $h_1, h_2 \in H^\sigma(\bar{k})$ such that $h_1^{-1}\delta_1 h_2 = h_2^{-1}\delta_2 h_1$. Since $h_2^{-1}\delta_2 h_1 = (h_1^{-1}\delta_1 h_2)^{-\sigma}$, this is equivalent to the statement that δ_1 is in the relative class of an element t satisfying $t^{-\sigma} = t$. Because $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$ is semisimple by assumption, δ_1 is relatively semisimple. Thus δ_1 is in the relative class of an element t of a σ -split torus by [Ri, Theorem 7.5]. This element satisfies $t = t^{-\sigma}$. $\square$

Suppose that $\gamma \in H(k)$ is a norm of $\delta \in G(k)$. We then have an inner twist

$$(3.2.6) \quad \begin{aligned} G_\delta(\bar{k}) &\xrightarrow{\sim} C_{\delta\delta^{-\theta}, G^\sigma}^\tau(\bar{k}) \xrightarrow{\sim} C_{\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau, H^\sigma}(\bar{k}) \xrightarrow{\sim} C_{\gamma\gamma^{-\sigma}, H^\sigma}(\bar{k}) \xrightarrow{\sim} H_\gamma(\bar{k}) \\ g &\longmapsto ygy^{-1} \end{aligned}$$

for y as in Definition 3.8. Here the first and last isomorphism are those of (3.1.5), and the second is the restriction to $C_{\delta\delta^{-\theta}, G^\sigma}^\tau$ of the projection of $G^\sigma(\bar{k}) \cong H^\sigma(\bar{k}) \times H^\sigma(\bar{k})$ onto the factor indexed by the identity.

3.3. Existence of norms. If we were in the standard trace formula setting, now would be the point where we would use the Kottwitz-Steinberg theorem, which states that a conjugacy class defined over F in a quasi-split reductive group with simply connected derived group contains an element [K1], and therefore (in this setting) norms exist. Unfortunately, no general analogue of this theorem is known in the relative setting. In fact, preliminary investigations suggest that it is not true.

In this section we provide a conditional relative analogue of the Kottwitz-Steinberg theorem using work of Borovoi. We say that H is a unitary group if there is an F -algebra D and an involution $\dagger : D \rightarrow D$ of the second kind such that $H(R)$ is equal to

$$\{g \in (D \otimes_F R)^\times : gg^\dagger = 1\}$$

for F -algebras R . We assume in addition that either D is simple or the center of D is $F \oplus F$ and $D = D_0 \times D_0$ for some simple algebra D_0 . Throughout this subsection we always assume that H is a unitary group and that we are in the biquadratic case. Thus $(H^\sigma)^{\text{der}}$ is simply connected. Moreover since we are in the biquadratic case we have $Z_{H^\sigma} \leq Z_H$, so the H^σ -orbit of any $\alpha \in Q(\bar{F})$ under conjugation is equal to its orbit under the subgroup $(H^\sigma)^{\text{der}}$. This fact is used in the proof of the following proposition:

PROPOSITION 3.10. *If $\delta \in G(F) = \text{Res}_{M/F} H(F)$ is relatively τ -regular semisimple, the element δ_{v_0} is relatively τ -elliptic for some place v_0 , and for all $v | \infty$ there is a $\gamma \in H(F_v)$ that is a norm of $\delta \in G(F_v)$, then there is a $\gamma \in H(F)$ such that γ is a norm of δ .*

Before proving Proposition 3.10 we will state and prove a converse to it:

LEMMA 3.11. *Let k be either F or F_v for some place v of F . If $\gamma \in H(k)$ is relatively semisimple, then it is a norm of some $\delta \in G(k)$. In fact, $\gamma\gamma^{-\sigma} = \delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$ for some $\delta \in G(k)$.*

Proof. By the proof of Lemma 3.6, for any relatively semisimple $\gamma \in H(k)$ we can choose a σ -stable torus $T \leq H_k$ such that $\gamma\gamma^{-\sigma}$ is in the image of the map

$$\begin{aligned} T(k) &\longrightarrow Q(k) \\ g &\longmapsto gg^{-\sigma}. \end{aligned}$$

Let $\tilde{T} := \text{Res}_{M \otimes k/k} T \leq G_k$. Then we have two composite maps

$$\begin{aligned} \psi_1 : \tilde{T}(k) &\xrightarrow{x \mapsto xx^\tau} T(k) \xrightarrow{y \mapsto yy^{-\sigma}} T \cap Q(k) \\ \psi_2 : \tilde{T}(k) &\xrightarrow{x \mapsto xx^{-\theta}} \tilde{T} \cap S(k) \xrightarrow{y \mapsto yy^\tau} T \cap Q(k). \end{aligned}$$

It suffices to show that ψ_2 is surjective. Since all of the groups in the definition of the ψ_i are commutative, $\psi_1 = \psi_2$, so it suffices to show that ψ_1 is surjective. The second map in the definition of ψ_1 is surjective by the argument in the proof of Lemma 3.6. The first map is surjective by [Ro2, Proposition 3.11.1(a)]. $\square$

We prove Proposition 3.10 using a cohomological obstruction to the existence of points in a homogeneous space developed in [Bo1]. In order to use Borovoi's results, we must develop a little notation. Set

$$(3.3.1) \quad \ker^j(F, \cdot) := \ker \left(H^j(F, \cdot) \longrightarrow \prod_v H^j(F_v, \cdot) \right)$$

where the product is over all places of v . Let H be an F -group and let X be a homogeneous space for H . Suppose for simplicity that the stabilizer $H_{\bar{x}}$ of a point $\bar{x} \in X(\bar{F})$ is connected and reductive. The stabilizer $H_{\bar{x}}$ is not in general the base change to $\bar{F}$ of an algebraic group over F . However, as explained in [Bo1, Section 1.7], the maximal toric quotient $H_{\bar{x}}/H_{\bar{x}}^{\text{der}}$ has a canonical F -form. We denote this F -form by $C_{\bar{x}, H}^m$. The obstruction $\text{Ob}(H, X)$ lies in $\ker^2(F, C_{\bar{x}, H}^m)$ [Bo1, Section 1.5].

We now prove Proposition 3.10:

Proof. Let $\alpha = \delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$. By Lemma 3.9 the set of elements of $Q(\bar{F})$ that are $G^\sigma(\bar{F})$ -conjugate to α defines a homogeneous space $O(\alpha)$ for H^σ and hence also for $(H^\sigma)^{\text{der}}$ since $Z_{H^\sigma} \leq Z_H$. By [Bo1, Theorem 1.6] it suffices to show that

$$\text{Ob}((H^\sigma)^{\text{der}}, O(\alpha)) \in \ker^2(F, C_{\alpha, (H^\sigma)^{\text{der}}}^m)$$

is trivial; to do this we will show that $(C_{\alpha, (H^\sigma)^{\text{der}}}^m)_{F_{v_0}}$ is anisotropic and hence $\ker^2(F, C_{\alpha, (H^\sigma)^{\text{der}}}^m) = 0$ by [Sa, Lemme 1.9]. One checks via [Bo1, Section 1.7] that there is a natural injection

$$C_{\alpha, (H^\sigma)^{\text{der}}}^m \longrightarrow C_{\alpha, H^\sigma}^m$$

and $Z_{H^\sigma} C_{\alpha, (H^\sigma)^\text{der}}^m = C_{\alpha, H^\sigma}^m$. Hence to prove that the torus $(C_{\alpha, (H^\sigma)^\text{der}}^m)_{F_{v_0}}$ is anisotropic it suffices to show that the torus $(C_{\alpha, H^\sigma}^m / Z_{H^\sigma})_{F_{v_0}}$ is anisotropic.

The $H(\overline{F})$ -conjugacy class of α is fixed by $\text{Gal}(\overline{F}/F)$. Denote this conjugacy class by O_H ; we view O_H as a homogeneous space for H . We have $(C_{\alpha, H^\sigma}^m)_{\overline{F}} = C_{\alpha, H^\sigma}$ and $(C_{\alpha, H}^m)_{\overline{F}} = C_{\alpha, H}$ since α is regular semisimple and H^der is simply connected. We therefore have a natural σ -equivariant embedding

$$(3.3.2) \quad C_{\alpha, H^\sigma} \rightarrow (C_{\alpha, H}^m)_{\overline{F}}$$

where we define $C_{\alpha, H}^m$ with respect to the homogeneous space O_H for H . Thus (3.3.2) defines an F -form of C_{α, H^σ} . By assumption, α^ξ is $H^\sigma(\overline{F})$ -conjugate to α for all $\xi \in \text{Gal}(\overline{F}/F)$. Using this fact, one checks that the F -form of C_{α, H^σ} defined by (3.3.2) is isomorphic to the F -torus C_{α, H^σ}^m defined with respect to $O(\alpha)$ (see [Bo1, Section 1.7]). Thus, by (3.3.2), we have that

$$C_{\alpha, H^\sigma}^m(R) = \{g \in C_{\alpha, H}^m(R) : g^\sigma = g\}$$

for commutative F -algebras R . By the theorem of Kottwitz-Steinberg [K1, Theorem 4.1] the $H(\overline{F})$ -conjugacy class of α contains an element $\alpha' \in H(F)$. We therefore have a natural isomorphism of F -groups $C_{\alpha, H}^m \cong C_{\alpha', H}^m \cong C_{\alpha', H}$ (see [Bo1, Section 1.7]). Since we assumed that δ_{v_0} was relatively τ -elliptic, we have that $(C_{\alpha', H} / Z_H)_{F_{v_0}}$ is anisotropic, and hence the same is true of $(C_{\alpha, H^\sigma}^m / Z_{H^\sigma})_{F_{v_0}}$, completing the proof of the proposition. $\square$

4. Matching of functions. In Section 3.2 we defined a notion of norm for relative classes. In this section we define a corresponding notion of matching functions. This necessitates the introduction of local orbital integrals and their stable analogues (these are defined in Section 4.1 and Section 4.2, respectively). We then define what we mean by matching of functions in Section 4.3. As in the usual trace formula, it is useful to have at our disposal local orbital integrals twisted by a character of a certain cohomology group; these relative κ -orbital integrals are defined in Section 4.4. They play a role in the prestabilization of (the regular elliptic part of) the twisted relative trace formula carried out in Sections 7 and 8 below. In Section 4.5 and Section 4.6 we prove various matching statements.

4.1. Local orbital integrals: definitions. Let v be a place of F . Let $\Phi \in C_c^\infty(H(F_v))$ and let $\gamma \in H(F_v)$ be a relatively semisimple element. The (local) *relative orbital integral* is defined by

$$(4.1.1) \quad \text{RO}_\gamma(\Phi) := \text{RO}_\gamma(\Phi, dt_\gamma) := \iint_{H_\gamma(F_v) \backslash H^\sigma(F_v)^2} \Phi(h_1^{-1} \gamma h_2) \frac{dh_1 dh_2}{dt_\gamma}$$

where dh_i and dt_γ are Haar measures on $H^\sigma(F_v)$ and $H_\gamma(F_v)$, respectively.

Similarly, let $f \in C_c^\infty(G(F_v))$ and let $\delta \in G(F_v)$ be a relatively τ -semisimple element. The (local) *twisted relative orbital integral* is defined by

$$\text{TRO}_\delta(f) := \text{TRO}_\delta(f, dt_\delta) := \iint_{G_\delta(F_v) \backslash G^\sigma \times G^\theta(F_v)} f(g_1^{-1} \delta g_2) \frac{dg_1 dg_2}{dt_\delta},$$

where dg_i , and dt_δ are Haar measures on $G^\sigma(F_v)$, $G^\theta(F_v)$ and $G_\delta(F_v)$, respectively. These relative orbital integrals and twisted relative orbital integrals are all absolutely convergent [H].

4.2. Stable local orbital integrals. We assume the notation of the previous subsection. For any relatively semisimple $\gamma_0 \in H(F_v)$ we define the *stable relative orbital integral* by

$$(4.2.1) \quad \text{SRO}_{\gamma_0}(\Phi) := \sum_{\gamma \sim \gamma_0} e(H_\gamma) \text{RO}_\gamma(\Phi, dt_\gamma),$$

where $e(H_\gamma)$ denotes the Kottwitz sign as in [K2] and the sum is over a set of representatives for the set of relative classes in the stable relative class of γ_0 .

Similarly, for a relatively τ -semisimple $\delta_0 \in G(F_v)$ we define the *stable twisted relative orbital integral* by

$$(4.2.2) \quad \text{STRO}_{\delta_0}(f) := \sum_{\delta \sim \delta_0} e(G_\delta) \text{TRO}_\delta(f, dt_\delta),$$

where the sum is over a set of representatives for the set of relative τ -classes in the stable relative τ -class of δ_0 . In both cases, we assume that the measures dt_γ (resp. dt_δ) are compatible with respect to the inner twists given by (3.2.3) and (3.2.4), respectively. For the definition of compatible, see, e.g., [K5, p. 631]. We note that the sums in (4.2.1) and (4.2.2) are finite since their cardinality is equal to the cardinality of the finite groups $\mathcal{D}(H_\gamma, H^\sigma \times H^\theta; F_v)$ and $\mathcal{D}(G_\delta, G^\sigma \times G^\theta; F_v)$, respectively (see Section 3.2).

4.3. Matching of functions: definition.

Definition 4.1. Two functions $\Phi \in C_c^\infty(H(F_v))$ and $f \in C_c^\infty(G(F_v))$ *match on the relatively regular set* (resp. *match*) if

$$\text{SRO}_\gamma(\Phi) = \text{STRO}_\delta(f)$$

whenever a relatively regular semisimple (resp. relatively semisimple) $\gamma \in H(F_v)$ is a norm of $\delta \in G(F_v)$,

$$\text{SRO}_\gamma(\Phi) = 0$$

whenever γ is relatively regular semisimple (resp. relatively semisimple) and not a norm of a $\delta \in G(F_v)$, and

$$\text{STRO}_\delta(f) = 0$$

whenever δ is relatively τ -regular semisimple (resp. relatively τ -semisimple) and does not admit a norm $\gamma \in H(F_v)$.

In the definition, we assume that the implicit Haar measures are compatible with respect to the inner twist of (3.2.6).

Remark. The inner twists given by (3.2.3), (3.2.4), and (3.2.6) are not canonical, since they depend on choices of elements in $H^\sigma(\bar{k})$ and $G^\sigma(\bar{k})$. However, if H_1 and H_2 are inner forms of the same connected reductive F -group, then the notion of a measure on H_1 being compatible with a measure on H_2 is independent of the particular inner twist realizing the two groups as inner forms of each other (see [K5, p. 631]). Thus the ambiguity in (3.2.3), (3.2.4), and (3.2.6) is irrelevant for our purposes.

4.4. Local relative κ -orbital integrals. In this section we define relative versions of the κ -orbital integrals that occur in Langland's and Kottwitz's prestabilization of the usual trace formula [K4]. We use Labesse's reformulation; see [La1] for notation and generalities regarding abelianized cohomology.

Let v be a place of F and let $I \leq H$ be a pair of connected reductive F_v -groups. Letting a superscript “ D ” denote the Pontryagin dual, we set notation for the finite abelian groups

$$(4.4.1) \quad \mathfrak{K}(I, H; F_v) := H_{\text{ab}}^0(F_v, I \backslash H)^D.$$

Let $\gamma \in H(F_v)$ be relatively semisimple and let $\delta \in G(F_v)$ be relatively τ -semisimple. For $\kappa_{Hv} \in \mathfrak{K}(H_\gamma, H^\sigma \times H^\sigma; F_v)$ and $\Phi_v \in C_c^\infty(H(F_v))$ (resp. $\kappa_v \in \mathfrak{K}(G_\delta, G^\sigma \times G^\theta; F_v)$ and $f_v \in C_c^\infty(G(F_v))$) we set

$$(4.4.2) \quad \begin{aligned} & \text{RO}_\gamma^{\kappa_{Hv}}(\Phi_v) \\ &= \int_{H^0(F_v, H_\gamma \backslash H^\sigma \times H^\sigma)} \langle \kappa_{Hv}, (\dot{h}_1, \dot{h}_2^{-1}) \rangle e((H_{h_1^{-1}\gamma h_2})_{F_v}) \Phi_v(h_1^{-1}\gamma h_2) d\dot{h}_1 d\dot{h}_2, \\ & \text{TRO}_\delta^{\kappa_v}(f_v) \\ &= \int_{H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)} \langle \kappa_v, (\dot{g}_1, \dot{g}_2^{-1}) \rangle e((G_{g_1^{-1}\delta g_2})_{F_v}) f_v(g_1^{-1}\delta g_2) d\dot{g}_1 d\dot{g}_2. \end{aligned}$$

Here we use a system of compatible measures as in Section 4.2 to define the measures $d\dot{h}_1 d\dot{h}_2$ on $H^0(F_v, H_\gamma \backslash H^\sigma \times H^\sigma)$ and $d\dot{g}_1 d\dot{g}_2$ on $H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)$, respectively (compare [La1, p. 42, 68]). Of course, the orbital integrals depend on this choice, but we will not encode it into our notation. In defining (4.4.2),

we choose for each class $(\dot{h}_1, \dot{h}_2) \in H^0(F_v, H_\gamma \backslash H^\sigma \times H^\sigma)$ an element $(h_1, h_2) \in H^\sigma \times H^\sigma(\overline{F}_v)$ whose image in $H^0(F_v, H_\gamma \backslash H^\sigma \times H^\sigma)$ is $(\dot{h}_1, \dot{h}_2)$. Similarly, we choose for each class $(\dot{g}_1, \dot{g}_2) \in H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)$ an element $(g_1, g_2) \in G^\sigma \times G^\theta(\overline{F}_v)$ whose image in $H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)$ is $(\dot{g}_1, \dot{g}_2)$. The integrals do not depend on these choices. The symbol $\langle \kappa_{Hv}, (\dot{h}_1, \dot{h}_2^{-1}) \rangle$ is the value of κ_v on the image of $(\dot{h}_1, \dot{h}_2^{-1})$ under the abelianization map, and similarly for $\langle \kappa_v, (\dot{g}_1, \dot{g}_2^{-1}) \rangle$. In the case that κ_{Hv} (resp. κ_v) is trivial, we have

$$\begin{aligned} \text{RO}_\gamma^{\kappa_{Hv}}(\Phi_v) &= \text{SRO}_\gamma(\Phi_v) \\ \text{TRO}_\delta^{\kappa_v}(f_v) &= \text{STRO}_\delta(f_v). \end{aligned}$$

4.5. Proof of the fundamental lemma: first case. Let v be a place of F . For this subsection we assume that there is an isomorphism

$$(4.5.1) \quad G_{F_v} \cong H_{F_v} \times H_{F_v}$$

equivariant with respect to σ and intertwining τ with $(x, y) \mapsto (y, x)$. For example, M/F could be split at v or (in the biquadratic case) ME/F could be split at v with E/F and M/F unramified and E/F nonsplit at v . In the latter case we have an isomorphism

$$(4.5.2) \quad E \otimes_F M \otimes_F F_v \cong E_v \oplus E_v$$

equivariant with respect to the natural action of σ on both sides and intertwining τ on the left with $(x, y) \mapsto (y, x)$ on the right.

Remark. By class field theory, at almost every place either E/F is split or E/F and M/F are unramified with E/F nonsplit at v .

Definition 4.2. An element $\gamma \in H(F_v)$ is a *split norm* of an element $\delta = (\delta_1, \delta_2) \in G(F_v)$ if γ is in the relative class of $\delta_1 \delta_2^{-\sigma}$.

The basic properties of this notion are summarized in the following lemma:

LEMMA 4.3. *If γ is a split norm of δ , then every element of the relative class of γ is a split norm of every element of the relative τ -class of δ . If γ and γ' are split norms of δ , then γ and γ' are in the same relative class. If δ and δ' both have split norm γ , then δ and δ' are in the same relative τ -class. Every $\gamma \in H(F_v)$ is a split norm and every $\delta \in G(F_v)$ has a split norm. If γ is relatively semisimple and γ is a split norm of δ , then it is a norm of δ .*

Proof. The first three assertions are easy to verify. For the fourth, note that $(\gamma, 1) \in G(F_v)$ has split norm γ for any $\gamma \in H(F_v)$. Conversely, $\delta = (\delta_1, \delta_2)$ has split norm $\delta_1 \delta_2^{-\sigma}$.

We now prove the final statement of the lemma. Suppose γ is a split norm of δ . Write

$$\delta\delta^{-\theta} = (\delta_1\delta_2^{-\sigma}, \delta_2\delta_1^{-\sigma}) = (x_\delta, x_\delta^{-\sigma})$$

for some $x_\delta \in H(F_v)$. At the expense of replacing γ with something in its relative class, we can and do assume that $\gamma = x_\delta$. Thus it suffices to show that we can choose $g \in H^\sigma(\overline{F}_v)$ such that $x_\delta x_\delta^{-\sigma} = g^{-1} x_\delta^{-\sigma} x_\delta g$, for then

$$(1, g)^{-1} \delta\delta^{-\theta} (\delta\delta^{-\theta})^\tau (1, g) = (1, g)^{-1} (x_\delta x_\delta^{-\sigma}, x_\delta^{-\sigma} x_\delta) (1, g) = (x_\delta x_\delta^{-\sigma}, x_\delta x_\delta^{-\sigma}),$$

which implies $\gamma = x_\delta$ is a norm of δ .

Since γ is relatively semisimple, there is a σ -split torus $T_\sigma \leq H_{\overline{F}_v}$ such that $x_\delta x_\delta^{-\sigma} \in T_\sigma(\overline{F}_v)$ [Ri, Theorem 7.5]. Since the map $x \mapsto x x^{-\sigma}$ coincides with $x \mapsto x^2$ on T_σ , we can and do choose $x' \in T_\sigma(\overline{F}_v)$ such that $x' x'^{-\sigma} = x_\delta x_\delta^{-\sigma}$. The fiber of the map $H \rightarrow Q$ of (3.1.2) over any point in $Q(\overline{F}_v)$ is a $H^\sigma(\overline{F})$ -torsor, so we have $x_\delta g' = x' \in T_\sigma(\overline{F})$ for some $g' \in H^\sigma(\overline{F}_v)$. Since T_σ is σ -stable, we have that $(x_\delta g')^{-\sigma}$ commutes with $x_\delta g'$. In other words,

$$x_\delta x_\delta^{-\sigma} = (x_\delta g')(x_\delta g')^{-\sigma} = (x_\delta g')^{-\sigma} (x_\delta g') = g^{-1} x_\delta^{-\sigma} x_\delta g. \quad \square$$

If $\delta = (\delta_1, \delta_2) \in G(F_v)$ then there is an isomorphism $G_{(\delta_1, \delta_2)} \xrightarrow{\sim} H_{\delta_1 \delta_2^{-\sigma}}$ given on F_v -algebras R by

$$(4.5.3) \quad \begin{aligned} G_{(\delta_1, \delta_2)}(R) &\longrightarrow H_{\delta_1 \delta_2^{-\sigma}}(R) \\ (g_1, g_2, g_3, g_3^\sigma) &\longmapsto (g_1, g_2) \end{aligned}$$

and thus isomorphisms $G_\delta \xrightarrow{\sim} H_\gamma$ for any split norm γ of δ . For the purposes of the following definition and lemma, fix a compact open subgroup $K_H \leq H(F_v)$ that is σ -stable and let $K = K_H \times K_H \leq G(F_v) \cong H(F_v) \times H(F_v)$.

Definition 4.4. Two functions $\Phi \in C_c^\infty(H(F_v))$ and $f \in C_c^\infty(G(F_v))$ *split match* if

$$\text{RO}_\gamma(\Phi, dt_\gamma) = \text{TRO}_\delta(f, dt_\delta)$$

whenever $\gamma \in H(F_v)$ is a split norm of $\delta \in G(F_v)$. Here the Haar measures dt_γ and dt_δ are normalized so that the isomorphism $G_\delta(F_v) \xrightarrow{\sim} H_\gamma(F_v)$ induced by (4.5.3) is measure-preserving and the Haar measure on $H(F_v)$ (resp. $G(F_v)$) gives K_H (resp. K) volume 1.

For $\Phi_i \in C_c^\infty(H(F_v))$ ($i \in \{1, 2\}$) we write $\Phi_1 \times \Phi_2 \in C_c^\infty(G(F_v)) = C_c^\infty(H \times H(F_v))$ for the product function using (4.5.1). We have the following lemma:

LEMMA 4.5. *Let $\Phi_1, \Phi_2 \in C_c^\infty(H(F_v)//K_H)$ and write $f = \Phi_1 \times \Phi_2 \in C_c^\infty(G(F_v)//K)$ for the product function. Write $\Phi_2^{-\sigma}(g) := \Phi_2(g^{-\sigma})$. The functions f and $\Phi_1 * \Phi_2^{-\sigma}$ split match, where the $*$ denotes convolution in $C_c^\infty(H(F_v)//K_H)$.*

Proof. Let $\delta = (\delta_1, \delta_2) \in G(F_v)$. We have

$$\begin{aligned}
 & \int_{G_\delta(F_v) \backslash G^\sigma \times G^\theta(F_v)} f(g^{-1}\delta h) dg dh \\
 (4.5.4) \quad &= \int_{H_\gamma(F_v) \backslash H^\sigma \times H^\sigma(F_v)} \left(\int_{H(F_v)} \Phi_1(g_1^{-1}\delta_1 h_1) \Phi_2(g_2^{-1}\delta_2 h_1^\sigma) dh_1 \right) dg_1 dg_2 \\
 &= \int_{H_\gamma(F_v) \backslash H^\sigma \times H^\sigma(F_v)} \Phi_1 * \Phi_2^{-\sigma}(g_1^{-1}\delta_1 \delta_2^{-\sigma} g_2) dg_1 dg_2.
 \end{aligned}$$

Here we have used (4.5.1) to write $g = (g_1, g_2) \in G^\sigma(F_v) \cong H^\sigma(F_v) \times H^\sigma(F_v)$ and $h = (h_1, h_1^\sigma) \in G^\theta(F_v)$ and we have used our assumption that the isomorphism $G_\delta(F_v) \xrightarrow{\sim} H_\gamma(F_v)$ given by (4.5.3) is measure preserving. With this in mind, the lemma follows from (4.5.4) and Lemma 4.3. $\square$

The inclusion $H \rightarrow G$ induces an L -map ${}^L H \rightarrow {}^L G$. Assume for the moment that H_{F_v} and G_{F_v} are both quasi-split, let $K \leq G(F_v)$ and $K_H \leq H(F_v)$ be hyperspecial subgroups and let

$$b : C_c^\infty(G(F_v)//K) \longrightarrow C_c^\infty(H(F_v)//K_H)$$

be the base change homomorphism induced by the L -map above. One can check that with respect to the isomorphism (4.5.1) this homomorphism is given by

$$b(f_1 \times f_2) = f_1 * f_2.$$

Thus in view of Lemma 4.3 and the remark at the end of Section 4.3 we have the following corollary:

COROLLARY 4.6. *Let $K_H \leq H(F_v)$ be a σ -stable hyperspecial subgroup and let $K = K_H \times K_H \leq G(F_v) \cong H(F_v) \times H(F_v)$. Then if $f_1 \in C_c^\infty(H(F_v)//K_H)$ the functions $f_1 \times \text{ch}_{K_H}$ and $b(f_1 \times \text{ch}_{K_H}) = f_1$ match.*

4.6. Matching at the E -split places. Assume for the remainder of this section that we are in the biquadratic case. Let v be a place of F . In this section we will always assume that E/F splits at v . Thus there is an isomorphism

$$(4.6.1) \quad E \otimes_F F_v \cong F_v \oplus F_v$$

intertwining σ with $(x, y) \mapsto (y, x)$ and an isomorphism

$$(4.6.2) \quad H_{F_v} \cong H_{F_v}^\sigma \times H_{F_v}^\sigma$$

intertwining σ with $(x, y) \mapsto (y, x)$. Using this one easily deduces the following lemma:

LEMMA 4.7. *Let $\gamma \in H(F_v)$ and $\delta \in G(F_v)$ be relatively semisimple and relatively τ -semisimple, respectively. Use (4.6.1) to write $\gamma\gamma^{-\sigma} = (y_\gamma, y_\gamma^{-1})$ and $\delta\delta^{-\theta} = (x_\delta, x_\delta^{-\tau})$. We have that γ is a norm of δ if and only if $x_\delta x_\delta^{-\tau}$ is $G^\sigma(\overline{F}_v)$ -conjugate to y_γ .*

If $\Phi_i \in C_c^\infty(H^\sigma(F_v))$ (resp. $f_i \in C_c^\infty(G^\sigma(F_v))$) for $i \in \{1, 2\}$, we let $\Phi_1 \times \Phi_2 \in C_c^\infty(H(F_v))$ (resp. $f_1 \times f_2 \in C_c^\infty(G(F_v))$) be the product functions defined using the isomorphism (4.6.2). For the definition of the $\mathfrak{K}_1$ groups in the following proposition we refer the reader to Section 7.2.

PROPOSITION 4.8. *Suppose that $\gamma \in H(F_v)$ is relatively semisimple and $\delta \in G(F_v)$ is relatively τ -semisimple. Let Φ_i and f_i be as above. Finally let $\kappa_H \in \mathfrak{K}(H_\gamma, H^\sigma \times H^\sigma; F_v)_1$ and $\kappa \in \mathfrak{K}(G_\delta, G^\sigma \times G^\theta; F_v)_1$. We then have*

$$\begin{aligned} \mathrm{RO}_\gamma^{\kappa_H}(\Phi_1 \times \Phi_2) &= \mathcal{O}_{y_\gamma}^{\kappa_H}(\Phi_1 * \Phi_2^{-1}) \\ \mathrm{TRO}_\delta^\kappa(f_1 \times f_2) &= \mathcal{O}_{x_\delta \rtimes \tau}^\kappa(f_1 * f_2^{-\tau}) \end{aligned}$$

where $\Phi_2^{-1}(h) := \Phi_2(h^{-1})$, $f_2^{-\tau}(h) := f_2(h^{-\tau})$, $\gamma\gamma^{-\sigma} = (y_\gamma, y_\gamma^{-1})$, and $\delta\delta^{-\theta} = (x_\delta, x_\delta^{-\tau})$.

In the proposition, we use notation as in [HL, Section 1.5] for the κ -orbital integrals $\mathcal{O}_{y_\gamma}^{\kappa_H}$ and $\mathcal{O}_{x_\delta \rtimes \tau}^\kappa$, and the $\mathfrak{K}_1$ -groups are defined as in [La1, Section 1.8] (see also Section 7.2). Strictly speaking, the κ_H in $\mathcal{O}_{y_\gamma}^{\kappa_H}$ (resp. κ in $\mathcal{O}_{x_\delta \rtimes \tau}^\kappa$) should actually be the image of κ_H (resp. κ) under

$$\begin{aligned} (4.6.3) \quad \mathfrak{K}(H_\gamma, H^\sigma \times H^\sigma; F_v)_1 &\cong \mathfrak{K}(C_{\gamma\gamma^{-\sigma}, H^\sigma}, H^\sigma \times H^\sigma; F_v)_1 \cong \mathfrak{K}(C_{y_\gamma, H^\sigma}, H^\sigma; F_v)_1 \\ \mathfrak{K}(G_\delta, G^\sigma \times G^\theta; F_v)_1 &\cong \mathfrak{K}(C_{\delta\delta^{-\theta}, G^\sigma}^\tau, G^\sigma \times G^\theta; F_v)_1 \cong \mathfrak{K}(C_{x_\delta, G^\sigma}^\tau, G^\sigma; F_v)_1 \end{aligned}$$

where the left isomorphisms are induced by Lemma 3.1 and the right isomorphisms are due to the isomorphisms $C_{\gamma\gamma^{-\sigma}, H^\sigma} \xrightarrow{\sim} C_{y_\gamma, H^\sigma}$ and $C_{\delta\delta^{-\theta}, G^\sigma}^\tau \xrightarrow{\sim} C_{x_\delta, G^\sigma}^\tau$ induced by the projection of $H \cong H^\sigma \times H^\sigma$ (resp. $G \cong G^\sigma \times G^\sigma$) onto the first factor. However, we won't burden the notation by indicating this. Also, to make the κ_H -orbital integrals well-defined we need to specify choices of Haar measures on H^σ , G^σ , and various centralizers, but, again, we will not incorporate this into the notation.

Proof. The statement involving relative orbital integrals can be recovered from the statement involving twisted relative orbital integrals by taking τ to be trivial. Therefore, it suffices to prove the statement regarding twisted relative orbital integrals.

Note that there is an isomorphism of F_v -group schemes $G_\delta \rightarrow C_{x_\delta, G^\sigma}^\tau$ given by

$$(4.6.4) \quad \begin{aligned} G_\delta(R) &\longrightarrow C_{x_\delta, G^\sigma}^\tau(R) \\ ((u, u), (h, h^\tau)) &\longmapsto u \end{aligned}$$

for F_v -algebras R (compare Lemma 3.1). Moreover, for a given $\delta = (\delta_1, \delta_2) \in G(F_v)$ there is an isomorphism of affine F_v -schemes $G^\sigma \times G^\theta \rightarrow (G^\sigma)^2$ given by

$$(4.6.5) \quad \begin{aligned} G^\sigma \times G^\theta(R) &\longrightarrow G^\sigma \times G^\sigma(R) \\ ((u, u), (h, h^\tau)) &\longmapsto ((u, u), (u^{-1}\delta_2 h^\tau, u^{-1}\delta_2 h^\tau)) \end{aligned}$$

for F_v -algebras R . This maps G_δ onto $C_{x_\delta, G^\sigma}^\tau \times \{(\delta_2, \delta_2)\}$, and thus a domain of integration for $H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)$ onto $H^0(F_v, C_{x_\delta, G^\sigma}^\tau \backslash G^\sigma) \times G^\sigma(F_v)$. This observation is used in one change of variables in the following computation:

$$(4.6.6) \quad \begin{aligned} \text{TRO}_\delta^\kappa(f_1 \times f_2) &= \int_{H^0(F_v, G_\delta \backslash G^\sigma \times G^\theta)} \langle \kappa, (\dot{g}_1, \dot{g}_2) \rangle e(G_{g_1^{-1}\delta g_2}) f_1 \times f_2(g_1^{-1}\delta g_2) d\dot{g}_1 d\dot{g}_2 \\ &= \int_{H^0(F_v, C_{x_\delta, G^\sigma}^\tau \backslash G^\sigma)} \int_{G^\sigma(F_v)} \langle \kappa, (\dot{h}_1, \dot{\delta}_2^{-\tau} \dot{h}_1^\tau \dot{h}_2) \rangle e(C_{h_1^{-1}x_\delta h_1^\tau, G^\sigma}^\tau) \\ &\quad \times f_1(h_1^{-1}\delta_1 \delta_2^{-\tau} h_1^\tau h_2) f_2(h_2^\tau) dh_2 d\dot{h}_1. \end{aligned}$$

In the last equality we used the fact that there is an isomorphism of F_v -groups $G_{g_1^{-1}\delta g_2} \cong C_{g_1^{-1}x_\delta g_1^\tau, G^\sigma}^\tau$ for $(g_1, g_2) \in G^\sigma \times G^\theta(F_v)$ (compare (4.6.4)). If we temporarily denote the image of κ under (4.6.3) by $\bar{\kappa}$, then

$$(4.6.7) \quad \langle \kappa, ((\dot{h}_1, \dot{h}_1), (\dot{\delta}_2^{-\tau} \dot{h}_1^\tau \dot{h}_2^\tau, \dot{\delta}_2^{-1} \dot{h}_1 \dot{h}_2)) \rangle = \langle \bar{\kappa}, \dot{h}_1 \rangle$$

holds. From now on, we will omit the bar and just write κ . With (4.6.7) in mind, the expression in (4.6.6) is equal to

$$\begin{aligned} &\int_{H^0(F_v, C_{x_\delta, G^\sigma}^\tau \backslash G^\sigma)} \int_{G^\sigma(F_v)} \langle \kappa, \dot{h}_1 \rangle e(C_{h_1^{-1}x_\delta h_1^\tau, G^\sigma}^\tau) \\ &\quad \times f_1(h_1^{-1}\delta_1 \delta_2^{-\tau} h_1^\tau h_2) f_2(h_2^\tau) dh_2 d\dot{h}_1 \\ &= \mathcal{O}_{x_\delta \rtimes \tau}^\kappa(f_1 * f_2^{-\tau}). \end{aligned} \quad \square$$

Assume now that v is nonarchimedean. If $G_{F_v}^\sigma$ and $H_{F_v}^\sigma$ are both unramified, let $K' \leq G^\sigma(F_v)$ and $K'_H \leq H^\sigma(F_v)$ be hyperspecial subgroups and let

$$b : C_c^\infty(G^\sigma(F_v)/K') \longrightarrow C_c^\infty(H^\sigma(F_v)/K'_H)$$

be the homomorphism induced by the base change homomorphism ${}^L H^\sigma \rightarrow {}^L G^\sigma$.

COROLLARY 4.9. *If v is nonarchimedean, then for every $\Phi \in C_c^\infty(H(F_v))$ there is a matching $f \in C_c^\infty(G(F_v))$ and conversely. If $M \otimes_F F_v/F_v$ is unramified, $G_{F_v}^\sigma$ and $H_{F_v}^\sigma$ are unramified with hyperspecial subgroups K' , K'_H as above, and $\Phi_1 \in C_c^\infty(G^\sigma(F_v)/K')$ then $\Phi_1 \times \text{ch}_{K'}$ matches $b(\Phi_1) \times \text{ch}_{K'_H}$.*

Proof. This follows immediately from Proposition 4.8, Lemma 4.7, [La1, Théorème 3.3.1] and [La1, Proposition 3.7.3]. $\square$

5. Matching at the ramified places. We assume the notation of Section 4 regarding the definition of stable twisted relative orbital integrals and matching of functions. In Section 4 we provided a supply of matching functions for places of F where various data were unramified. We now prove a weaker matching statement for the ramified places. It is contained in the following theorem, the main theorem of this section:

THEOREM 5.1. *Let v be a place of F . If $f \in C_c^\infty(G(F_v))$ is supported in a sufficiently small neighborhood of a relatively τ -regular semisimple $\delta \in G(F_v)$ admitting a norm $\gamma \in H(F_v)$ then there is a $\Phi \in C_c^\infty(H(F_v))$ that matches f on the relatively regular set.*

The proof will occupy the remainder of the section. Our approach is an adaptation of that in [La1, Section 3.1]. We also found the work of Rader-Rallis [RR] and Hakim [Hak2] useful.

5.1. Local constancy of the relative orbital integrals. Let v be a place of F and let $\delta \in G(F_v)$ be a relatively τ -regular semisimple element. In this section we show that the relative orbital integrals of functions with support in a small neighborhood of δ can be viewed as a locally constant functions on a torus related to δ . To ease notation, throughout this section we abbreviate $H = H_{F_v}$, $S = S_{F_v}$, etc.

We begin with the following lemma:

LEMMA 5.2. *If $\gamma \in H(F_v)$ is relatively regular semisimple, then the maximal σ -split subtorus of $C_{\gamma\gamma^{-\sigma},H}$ is a maximal σ -split torus of H . If δ is relatively τ -regular semisimple then the maximal σ -split subtorus of $C_{\delta\delta^{-\theta},G}^\tau$ has the same dimension as a maximal σ -split torus of H .*

We write T_γ for the maximal σ -split subtorus of $C_{\gamma\gamma^{-\sigma},H}$ and $\widetilde{T}_\delta$ for the maximal σ -split subtorus of $C_{\delta\delta^{-\theta},G}^\tau$.

Proof. The element $\gamma\gamma^{-\sigma}$ is contained in a maximal σ -split torus T_σ of $H_{\overline{F}_v}$ [Ri, Theorem 7.5] which in turn is contained in a maximal σ -stable torus T [He, Proposition 1.4]. Moreover, T_σ is the unique maximal σ -split torus of T (this follows from [Ri, Theorem 7.5]). Since $\gamma\gamma^{-\sigma}$ is regular semisimple, it is contained in a unique maximal torus, so $T = (C_{\gamma\gamma^{-\sigma},H})_{\overline{F}_v}$, and it follows that $T_\sigma = T_{\gamma\overline{F}_v}$. Here

we are implicitly using the fact that T_σ is in fact defined over F_v , being the σ -split component of the F_v -torus $C_{\gamma\gamma^{-\sigma}, H}$. Thus T_γ is a maximal σ -split torus.

We now prove the second claim. Choose an isomorphism $G_{\overline{F}_v} \cong H_{\overline{F}_v} \times H_{\overline{F}_v}$ equivariant with respect to σ and intertwining τ with $(x, y) \mapsto (y, x)$. Using this isomorphism, write $\delta\delta^{-\theta} = (x_\delta, x_\delta^{-\sigma})$ for some $x_\delta \in H(\overline{F}_v)$. Our assumption that δ is relatively τ -regular semisimple implies that x_δ is relatively semisimple. By the argument above, we see that there is a (unique) maximal σ -split subtorus of $C_{x_\delta x_\delta^{-\sigma}, H}$ which is moreover a maximal σ -split subtorus of H . On the other hand, there is a σ -equivariant isomorphism

$$(C_{\delta\delta^{-\theta}, G}^\tau)_{\overline{F}_v} \xrightarrow{\sim} C_{x_\delta x_\delta^{-\sigma}, H}$$

induced by the projection of $G_{\overline{F}_v} \cong H_{\overline{F}_v} \times H_{\overline{F}_v}$ onto the first factor. It follows that the maximal σ -split subtorus of $(C_{\delta\delta^{-\theta}, H}^\tau)_{\overline{F}_v}$ has the same dimension as a maximal σ -split torus of $H_{\overline{F}_v}$. The maximal σ -split torus of $(C_{\delta\delta^{-\theta}, G}^\tau)_{\overline{F}_v}$ is in fact defined over F_v (compare [He, Section 1.3]), and the lemma follows. $\square$

Our first step to proving Theorem 5.1 is the following proposition:

PROPOSITION 5.3. *Suppose that $\delta_0 \in G(F_v)$ is relatively τ -regular semisimple. Let $\mathcal{V}$ be a neighborhood of 1 in $\tilde{T}_{\delta_0}(F_v)$ and $\mathcal{W}$ be a neighborhood of δ_0 in $G(F_v)$.*

(1) *Suppose that $f \in C_c^\infty(G(F_v))$ has support in $\mathcal{W}$. If $\mathcal{V}$ and $\mathcal{W}$ are sufficiently small, then there is a function $\psi \in C_c^\infty(\tilde{T}_{\delta_0}(F_v))$ with support in $\mathcal{V}$ such that if $\delta \in G(F_v)$ is τ -regular semisimple and its $G^\sigma \times G^\theta(F_v)$ -orbit meets $\mathcal{W}$, then*

$$\delta = g_1^{-1} t \delta_0 g_2$$

with $t \in \tilde{T}_{\delta_0}(F_v)$, $(g_1, g_2) \in G^\sigma \times G^\theta(F_v)$, and

$$\text{TRO}_\delta(f) = \psi(t).$$

Moreover, if $t \in \tilde{T}_{\delta_0}(F_v)$ and $t\delta_0$ is in the stable relative τ -class of δ , then

$$\text{STRO}_\delta(f) = \psi(t).$$

(2) *Conversely, if $\psi \in C_c^\infty(\tilde{T}_{\delta_0}(F_v))$ has support in $\mathcal{V}$ and the neighborhoods $\mathcal{V}$ and $\mathcal{W}$ are sufficiently small, then there is an $f \in C_c^\infty(G(F_v))$ satisfying the identities of (1).*

In [La1, Section 3.1], Labesse proves the analogue of Proposition 5.3 in the context of the usual trace formula. Our proof follows his closely. We require some preparatory lemmas:

LEMMA 5.4. *Assume δ_0 is relatively τ -regular semisimple. There is an analytic subvariety $Y \subset G^\sigma \times G^\theta(F_v)$ that is symmetric (i.e. $(x, y) \in Y$ if and only if*

$(x^{-1}, y^{-1}) \in Y$) and a neighborhood $\mathcal{V}$ of 1 in $\tilde{T}_{\delta_0}(F_v)$ such that the following are true:

(i) The map

$$\begin{aligned} Y \times \mathcal{V} &\longrightarrow G(F_v) \\ (x, y, t) &\longmapsto x^{-1}t\delta_0 y \end{aligned}$$

is a diffeomorphism from $Y \times \mathcal{V}$ to a neighborhood $\mathcal{W} = \mathcal{W}(Y, \mathcal{V})$ of $\delta_0 \in G(F_v)$.

(ii) If $(x, y) \in Y$, then $x^{-1}t\delta_0 y \in \mathcal{W}$ and $t \in \tilde{T}_{\delta_0}(F_v)$ imply $t \in \mathcal{V}$. Thus

$$\tilde{T}_{\delta_0}(F_v)\delta_0 \cap \mathcal{W} = \mathcal{V}\delta_0.$$

Proof. Choose a complement $\mathfrak{n}$ of $\mathfrak{c} = \text{Lie } C_{\delta_0\delta_0^{-\theta}, G^\sigma}^\tau(F_v)$ in $\mathfrak{g}^\sigma := \text{Lie } G^\sigma(F_v)$:

$$\mathfrak{g}^\sigma = \mathfrak{n} \oplus \mathfrak{c}.$$

Let $\mathfrak{g}^\theta := \text{Lie } G^\theta(F_v)$, let $\mathcal{O}$ be a neighborhood of 0 in $\mathfrak{n} \oplus \mathfrak{g}^\theta$, and let $Y = \text{Exp } \mathcal{O}$ be its image under the exponential map. Thus $Y \subset G^\sigma \times G^\theta(F_v)$ is an analytic subvariety. We claim that the analytic morphism

$$\begin{aligned} (5.1.1) \quad Y \times \tilde{T}_{\delta_0}(F_v) &\longrightarrow G(F_v) \\ (g_1, g_2, t) &\longmapsto (g_1^{-1}t\delta_0 g_2) \end{aligned}$$

is locally an isomorphism at $(1, 1, 1)$.

In order to show this, we first explain how the work of Rader and Rallis implies that the analytic morphism

$$\begin{aligned} (5.1.2) \quad G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v) &\longrightarrow G(F_v) \\ (g_1, g_2, t) &\longmapsto g_1^{-1}t\delta_0 g_2 \end{aligned}$$

is a submersion whose fibers are of dimension $\dim_{F_v} C_{\delta_0\delta_0^{-\theta}, G^\sigma}^\tau$. If τ is trivial, then this follows from [RR, Theorem 3.4(1)]. Assume that τ is not trivial. Note that (5.1.2) fits into a commutative diagram

$$(5.1.3) \quad \begin{array}{ccc} G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v) & \xrightarrow{\hspace{2cm}} & G(F_v) \\ \downarrow & & \downarrow g \mapsto gg^{-\theta} \\ G^\sigma(F_v) \times \tilde{T}_{\delta_0}(F_v) & \xrightarrow{(g_1, t) \mapsto g_1^{-1}t^2\delta_0\delta_0^{-\theta}g_1^\tau} & S(F_v) \end{array}$$

where the left arrow is the canonical projection and the top is (5.1.2). The vertical arrows are induced by smooth surjective maps of affine varieties both of relative dimension $\dim_{F_v} G^\theta$. Thus if we show that the bottom map is a submersion at

$(1, 1)$, it follows that the top is a submersion at $(1, 1, 1)$. Since the map $t \mapsto t^2$ on $\tilde{T}_{\delta_0}$ is an isogeny, in order to show that the bottom map of (5.1.3) is a submersion at $(1, 1)$ it suffices to show that the map

$$(5.1.4) \quad \begin{aligned} G^\sigma(F_v) \times \tilde{T}_{\delta_0}(F_v) &\longrightarrow S(F_v) \\ (g_1, t) &\longmapsto g_1^{-1} t \delta_0 \delta_0^{-\theta} g_1^\tau \end{aligned}$$

is a submersion at $(1, 1)$.

To see this, let M_v be F_v if v splits in M/F and $M \otimes_F F_v$ otherwise. To prove that (5.1.4) is a submersion whose fibers are of dimension $\dim_{F_v} C_{\delta_0 \delta_0^{-\theta}, G^\sigma}^\tau$, it suffices to show that

$$(5.1.5) \quad \begin{aligned} G^\sigma(M_v) \times \tilde{T}_{\delta_0}(M_v) &\longrightarrow S(M_v) \\ (g_1, t) &\longmapsto g_1^{-1} t \delta_0 \delta_0^{-\theta} g_1^\tau \end{aligned}$$

is a submersion at $(1, 1)$ of relative dimension $\dim_{F_v} C_{\delta_0 \delta_0^{-\theta}, G^\sigma}^\tau$. Choose an isomorphism $G_{M_v} \cong H_{M_v} \times H_{M_v}$ equivariant with respect to σ and intertwining τ with $(x, y) \mapsto (y, x)$. Using this isomorphism, write $\delta_0 \delta_0^{-\theta} = (x_0, x_0^{-\sigma})$ for some $x_0 \in H(M_v)$ and let T_{x_0} be the largest σ -split torus in $C_{x_0 x_0^{-\sigma}, H}$. There is a commutative diagram of analytic morphisms

$$(5.1.6) \quad \begin{array}{ccc} G^\sigma(M_v) \times \tilde{T}_{\delta_0}(M_v) & \xrightarrow{\hspace{2cm}} & S(M_v) \\ \downarrow (u_1, u_2, t_1, t_2) \mapsto (u_1, u_2, t_1) & & \downarrow (u_1, u_2) \mapsto u_1 \\ H^\sigma(M_v) \times H^\sigma(M_v) \times T_{x_0}(M_v) & \xrightarrow{(u_1, u_2, t) \mapsto u_1^{-1} t x_0 u_2} & H(M_v) \end{array}$$

where the top arrow is (5.1.5) and the vertical arrows are analytic isomorphisms induced by the isomorphism $G(M_v) \cong H(M_v) \times H(M_v)$. Using Lemma 5.2 and [RR, Theorem 3.4(1)] (and the reference therein for the real case), we see that the bottom map is a submersion at $(1, 1, 1)$ of relative dimension $\dim_{F_v} C_{x_0 x_0^{-\sigma}, H^\sigma}$. It follows that the top vertical map of (5.1.6) is a submersion of relative dimension $\dim_{F_v} C_{x_0 x_0^{-\sigma}, H}$. In view of the σ -equivariant isomorphism $C_{\delta_0 \delta_0^{-\theta}, G}^\tau \xrightarrow{\sim} C_{x_0 x_0^{-\sigma}, H}$ induced by the projection of $G_{M_v} \cong H_{M_v} \times H_{M_v}$ onto the first factor, this together with our comments above implies that (5.1.2) is a submersion of relative dimension $\dim_{F_v} C_{\delta_0 \delta_0^{-\theta}, G^\sigma}^\tau$, as claimed.

The analytic subvariety

$$Y \times \tilde{T}_{\delta_0}(F_v) \subset G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v)$$

is transverse to the subvariety

$$X := \{(g_1, g_2, 1) : (g_1, g_2) \in G_{\delta_0}(F_v)\} \subset G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v)$$

at $(1, 1, 1)$. The submersion (5.1.2) maps X identically to δ_0 . Our claim that (5.1.1) is a local isomorphism follows. Thus we can and do choose $\mathcal{O}$ and $\mathcal{V}$ small enough so that the map

$$Y \times \mathcal{V} \longrightarrow G(F_v)$$

induced by restricting (5.1.1) is an isomorphism onto an open neighborhood $\mathcal{W}(Y, \mathcal{V})$ of δ_0 . This proves (i).

Fix a neighborhood $\mathcal{V}_1$ of 1 in $\tilde{T}_{\delta_0}(F_v)$ and an analytic subvariety $Y_1 \subset G^\sigma \times G^\theta(F_v)$ small enough that (i) holds. For an analytic subvariety $Y \subset Y_1$ and a neighborhood $\mathcal{V} \subset \mathcal{V}_1$ of 1, let

$$\mathcal{W} := \mathcal{W}(Y, \mathcal{V})$$

be the image of $Y \times \mathcal{V}$ under (5.1.1). Since $\tilde{T}_{\delta_0}(F_v)$ is a closed subgroup of $G(F_v)$, we can choose Y and $\mathcal{V}$ small enough so that

$$\tilde{T}_{\delta_0}(F_v)\delta_0 \cap g_1^{-1}\mathcal{W}g_2 \subset \mathcal{V}_1\delta_0$$

for each $(g_1, g_2) \in Y$. Thus $g_1 t \delta_0 g_2^{-1} \in \mathcal{W}$ with $t \in \tilde{T}_{\delta_0}(F_v)$ and $(g_1, g_2) \in Y$ implies that $t \in \mathcal{V}_1$. By the proof of part (i), any $x \in \mathcal{W}(Y_1, \mathcal{V}_1)$ can be written in a unique manner as $x = g_1^{-1} t \delta_0 g_2$ with $(g_1, g_2) \in Y_1$ and $t \in \mathcal{V}_1$. We conclude that if $g_1^{-1} t \delta_0 g_2 \in \mathcal{W}(Y, \mathcal{V})$ with $t \in \tilde{T}_{\delta_0}(F_v)$ and $(g_1, g_2) \in Y$ then $t \in \mathcal{V}$. $\square$

LEMMA 5.5. *Suppose that $\mathcal{W} \subset G(F_v)$ is a relatively compact neighborhood of δ_0 and $\mathcal{V}$ is a neighborhood of 1 in $\tilde{T}_{\delta_0}(F_v)$. If $\mathcal{V}$ is sufficiently small, then there is a compact subset $\Omega \subset G^\sigma \times G^\theta(F_v)$ such that if*

$$g_1^{-1} t \delta_0 g_2 \in \mathcal{W}$$

with $t \in \mathcal{V}$ then $(g_1, g_2) \in G_{\delta_0}(F_v)\Omega$.

Proof. Consider the continuous map

$$\begin{aligned} A : G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v) &\longrightarrow G(F_v) \\ (g_1, g_2, t) &\longmapsto g_1^{-1} t \delta_0 g_2 \end{aligned}$$

and let

$$P : G^\sigma \times G^\theta(F_v) \times \tilde{T}_{\delta_0}(F_v) \longrightarrow G^\sigma \times G^\theta(F_v)$$

be the canonical projection. The lemma is equivalent to the statement that if $\mathcal{V}$ and $\mathcal{W}$ are sufficiently small (with $\mathcal{W}$ relatively compact) then there exists a compact

set $\Omega \subset G^\sigma \times G^\theta(F_v)$ such that $P \circ A^{-1}\mathcal{W} \subset G_{\delta_0}(F_v)\Omega$. This latter statement is a consequence of [RR, Proposition 2.5] (and the reference therein in the real case); in loc. cit. one takes $M = G^\sigma \times G^\theta(F_v)$ and $H = G_{\delta_0}(F_v)$. $\square$

LEMMA 5.6. *Let $\mathcal{V}$ be an open neighborhood of 1 in $\tilde{T}_{\delta_0}(F_v)$ and suppose $m, m' \in \mathcal{V}$. If $\mathcal{V}$ is sufficiently small and there is a $(g_1, g_2) \in G^\sigma \times G^\theta(F_v)$ such that $g_1^{-1}m\delta_0g_2 = m'\delta_0$, then $(g_1, g_2) \in G_{\delta_0}(F_v)$.*

Proof. The proof of [La1, Lemme 3.1.4] easily adapts to our situation to prove the lemma. We only note that the analogue of [La1, Lemme 3.1.2] is trivial in our situation because δ_0 is relatively τ -regular semisimple, and the analogues of [La1, Lemme 3.1.1 and Lemme 3.1.3] are given by Lemma 5.4 and Lemma 5.5, respectively. $\square$

LEMMA 5.7. *Under the assumptions of Lemma 5.6, if $m'\delta_0$ and $m\delta_0$ are in the same stable relative τ -class, then*

$$\bar{g}_1^{-1}m\bar{g}_1 = m'$$

for some $\bar{g}_1 \in C_{\delta_0\delta_0^{-\theta}}^\tau(\bar{F}_v)$. Hence $m = m'$.

Proof. Let F_1/F_v be a finite field extension such that $C_{\delta_0\delta_0^{-\theta}F_1}^\tau$ is split. Then if m, m' have the property that there exists $(\bar{g}_1, \bar{g}_2) \in G^\sigma \times G^\theta(\bar{F}_v)$ satisfying $\bar{g}_1^{-1}m\delta_0\bar{g}_2 = m'\delta_0$, we can and do assume that $(\bar{g}_1, \bar{g}_2) \in G^\sigma \times G^\theta(F_1)$ (compare (3.2.2)). Applying Lemma 5.6 “over F_1 ” we have that $(\bar{g}_1, \bar{g}_2) \in G_{\delta_0}(F_1)$. Since $C_{\delta_0\delta_0^{-\theta}}^\tau$ is a torus, $\bar{g}_1$ commutes with m , which proves the last statement. $\square$

LEMMA 5.8. *For $Y, \mathcal{V}$, and $\mathcal{W}(Y, \mathcal{V})$ as in Lemma 5.4, suppose $Y \subset \mathcal{U}$ where $\mathcal{U}$ is an open neighborhood of 1 in $G^\sigma \times G^\theta(F_v)$. If $\mathcal{U}$ and $\mathcal{V}$ are sufficiently small, then each*

$$g_1^{-1}t\delta_0g_2 \in \mathcal{W}(Y, \mathcal{V})$$

with $(g_1, g_2) \in G^\sigma \times G^\theta(F_v)$ and $t \in \mathcal{V}$ satisfies $(g_1, g_2) \in G_{\delta_0}(F_v)Y$.

Proof. By our hypothesis and Lemma 5.4, we have

$$g_1^{-1}t\delta_0g_2 = y_1^{-1}t'\delta_0y_2$$

for some $(y_1, y_2) \in Y$ and $t' \in \mathcal{V}$. By Lemma 5.6, if $\mathcal{U}$ and $\mathcal{V}$ are sufficiently small we have that $(g_1y_1^{-1}, g_2y_2^{-1}) \in G_{\delta_0}(F_v)$. $\square$

With these lemmas in place, we can now prove Proposition 5.3:

Proof of Proposition 5.3. Choose a neighborhood $\mathcal{V}$ of 1 in $\tilde{T}_{\delta_0}(F_v)$ and an analytic subvariety $Y \subset G^\sigma \times G^\theta(F_v)$ satisfying the conclusion of Lemma 5.4. Let

dg_1, dg_2 , and dt_{δ_0} be Haar measures on $G^\sigma(F_v)$, $G^\theta(F_v)$, and $G_{\delta_0}(F_v)$, respectively. In this proof we use these measures to define $\text{TRO}_{\delta_0}(f) = \text{TRO}_{\delta_0}(f, dt_{\delta_0})$. There is a natural map

$$Y \longrightarrow G_{\delta_0}(F_v) \backslash G^\sigma \times G^\theta(F_v);$$

let $d\mu$ be the measure on Y that is the pullback of the measure $\frac{dg_1 dg_2}{dt_{\delta_0}}$ with respect to this map. Suppose that $f \in C_c^\infty(G(F_v))$ has support in $\mathcal{W} := \mathcal{W}(Y, \mathcal{V})$ and that Y and $\mathcal{V}$ are sufficiently small. For each $t \in \tilde{T}_{\delta_0}(F_v)$, define

$$\psi(t) := \int_Y f(g_1^{-1} t \delta_0 g_2) d\mu(g_1, g_2).$$

By Lemma 5.4 the function ψ on $\tilde{T}_{\delta_0}(F_v)$ is smooth and compactly supported (with support in $\mathcal{V}$). Since $G_{\delta_0} = G_\delta$ we have

$$\text{TRO}_\delta(f) = \int_{G_{\delta_0}(F_v) \backslash G^\sigma \times G^\theta(F_v)} f(g_1^{-1} t \delta_0 g_2) \frac{dg_1 dg_2}{dt_{\delta_0}}$$

for some $t \in \mathcal{V}$. In view of Lemma 5.8 this implies

$$\text{TRO}_\delta(f) = \int_Y f(g_1^{-1} t \delta_0 g_2) d\mu(g_1, g_2)$$

and thus

$$\text{TRO}_\delta(f) = \psi(t).$$

The assertion involving stable twisted relative orbital integrals follows from Lemma 5.7, and this completes the proof of (1).

For the proof of (2), suppose that we are given ψ with support in $\mathcal{V}$. Each $\delta \in \mathcal{W}$ can be written in a unique fashion as

$$\delta = g_1^{-1} t \delta_0 g_2$$

with $(g_1, g_2) \in Y$ and $t \in \mathcal{V}$. Define

$$(J^\beta \psi)(\delta) = \beta(g_1, g_2) \psi(t)$$

where $\beta \in C_c^\infty(Y)$ is chosen so that

$$\int_Y \beta(g_1, g_2) d\mu(g_1, g_2) = 1.$$

Setting $f = J^\beta \psi$, statement (2) follows from the observation that

$$\int_Y J^\beta \psi = \psi.$$

□

5.2. Proof of Theorem 5.1. For this entire subsection we assume the hypotheses of Theorem 5.1. We assume f is supported in a neighborhood $\mathcal{W} = \mathcal{W}(Y, \mathcal{V})$ of a relatively τ -regular semisimple element δ_0 , with Y and $\mathcal{V}$ as in Lemma 5.4. With notation and assumptions as in Proposition 5.3, we have that

$$\text{STRO}_\delta(f) = 0$$

if the stable class of δ does not meet $\tilde{T}_{\delta_0}(F_v)\delta_0$, and

$$\text{STRO}_\delta(f) = \psi(t)$$

if δ is in the stable class of $t\delta_0$ with $t \in \tilde{T}_{\delta_0}(F_v)$.

In a sufficiently small neighborhood of the identity in $\tilde{T}_{\delta_0}(F_v)$, the map $t \mapsto t^2$ is an isomorphism that preserves conjugacy classes. Thus we can and do choose a function $\psi_1 \in C_c^\infty(\tilde{T}_{\delta_0}(F_v))$ with support in a small neighborhood $\mathcal{V}_1$ of the identity such that

$$\psi(t) = \psi_1(t^2).$$

By shrinking $\mathcal{V}$ if necessary, we can make $\mathcal{V}_1$ as small as we wish.

Assume that δ_0 has norm $\gamma_0 \in H(F_v)$. Thus $C_{\delta_0\delta_0^{-\theta}, G}^\tau$ and $C_{\gamma_0\gamma_0^{-\sigma}, H}$ are inner twists of each other, and the inner twist can be defined by an element of $H^\sigma(\overline{F}_v)$ (compare (3.2.6)). Since δ_0 is relatively τ -regular, $C_{\delta_0\delta_0^{-\theta}}^\tau$ is a torus, and thus the inner twist induces a σ -equivariant isomorphism

$$(5.2.1) \quad B : C_{\delta_0\delta_0^{-\theta}, G}^\tau \longrightarrow C_{\gamma_0\gamma_0^{-\sigma}, H}.$$

Note that $\tilde{T}_{\delta_0}$ is a maximal σ -split torus of $C_{\delta_0\delta_0^{-\theta}, G}^\tau$. Since T_{γ_0} and $\tilde{T}_{\delta_0}$ have the same dimension by Lemma 5.2, the isomorphism (5.2.1) induces another isomorphism

$$B : \tilde{T}_{\delta_0} \longrightarrow T_{\gamma_0}$$

such that for F_v -algebras R one has $B(g_1^{-1}tg_1) = B(g_1)^{-1}B(t)B(g_1)$ for $g_1 \in C_{\delta_0\delta_0^{-\theta}, G}^\tau(R)$ and $t \in \tilde{T}_{\delta_0}(R)$. Set $\psi_2 := \psi_1 \circ B^{-1} \in C_c^\infty(T_{\gamma_0}(F_v))$. Then ψ_2 has support in $B(\mathcal{V}_1)$, and

$$(5.2.2) \quad \psi_1(t^2) = \psi_2(B(t^2)).$$

We note that $B(t^2)\gamma_0$ is a norm of $t\delta_0$ (this is the reason for employing the squaring map above).

Invoking Proposition 5.3(2), we shrink $\mathcal{V}$ and $\mathcal{V}_1$ if necessary and choose a function $\Phi \in C_c^\infty(H(F_v))$ such that

$$\text{SRO}_\gamma(\Phi) = 0$$

if the stable class of γ does not meet $T_{\gamma_0}(F_v)\gamma_0$, and

$$\text{SRO}_\gamma(\Phi) = \psi_2(t_2)$$

if $t_2\gamma_0$ and γ are in the same stable class. This implies the assertion of the theorem. $\square$

6. Spherical characters. In this section we introduce the notion of a relatively τ -regular and relatively τ -elliptic representation of a reductive group over a local field. The first of these notions was used in the statement of Theorem 1.1, and we believe the second to be of interest as well. Let Π_v be an irreducible admissible representation of $G(F_v)$ with space V_{Π_v} and choose

$$\Lambda = \sum_i \lambda_i \otimes \lambda_i^\vee \in \text{Hom}_{G^\sigma(F_v)}(V_{\Pi_v}, \mathbb{C}) \otimes \text{Hom}_{G^\theta(F_v)}(V_{\Pi_v^\vee}, \mathbb{C});$$

here $\mathbb{C}$ is the trivial representation and $\Pi_v^\vee$ is the contragredient representation acting on $V_{\Pi_v^\vee}$. The linear form Λ defines a distribution (i.e. a linear map)

$$(6.0.3) \quad \begin{aligned} \Theta_\Lambda : C_c^\infty(G(F_v)) &\longrightarrow \mathbb{C} \\ f_v &\longmapsto \sum_i \langle \Pi_v(f_v) \lambda_i, \lambda_i^\vee \rangle. \end{aligned}$$

A *spherical matrix coefficient* (of Π_v) is a distribution attached to Λ in this manner (compare [Hak2]). If the extension M/F is trivial (so $H = G$ and $G^\sigma = G^\theta = H^\sigma$) then the spherical matrix coefficient Θ_Λ is representable by a locally constant function on the relatively regular subset of $G(F_v) = H(F_v)$ by [Hak2, Lemma 6]. We denote this function by Θ_Λ as well. The authors suspect that the same result is true when τ is nontrivial, but we will not prove this.

Definition 6.1. An admissible representation Π_v of $G(F_v)$ is *relatively Λ -regular* if there is a function $f \in C_c^\infty(G(F_v))$ supported in the set of relatively τ -regular elements of $G(F_v)$ admitting norms in $H(F_v)$ such that

$$\Theta_\Lambda(f) \neq 0.$$

It is *relatively τ -regular* if it is relatively Λ -regular for all nonzero

$$\Lambda \in \text{Hom}_{G^\sigma(F_v)}(V_{\Pi_v}, \mathbb{C}) \otimes \text{Hom}_{G^\theta(F_v)}(V_{\Pi_v^\vee}, \mathbb{C}).$$

Definition 6.2. An admissible representation Π_v of $G(F_v)$ is *relatively Λ -elliptic* if there is a function $f \in C_c^\infty(G(F_v))$ supported in the set of relatively τ -elliptic semisimple elements of $G(F_v)$ admitting norms in $H(F_v)$ such that

$$\Theta_\Lambda(f) \neq 0.$$

It is *relatively τ -elliptic* if it is relatively Λ -elliptic for all nonzero

$$\Lambda \in \mathrm{Hom}_{G^\sigma(F_v)}(V_{\Pi_v}, \mathbb{C}) \otimes \mathrm{Hom}_{G^\theta(F_v)}(V_{\Pi_v^\vee}, \mathbb{C}).$$

If τ is trivial then we omit it from notation, writing relatively regular for relatively τ -regular. The analogous convention with regular replaced by elliptic will also be in force. If τ is trivial the norm map is the identity map so the condition on elements admitting norms can be omitted.

Remark. In certain circumstances one can use the simple twisted relative trace formula of [H] to prove the existence of relatively τ -regular (resp. τ -semisimple) representations.

It is well known that the character of an irreducible admissible representation does not vanish on the regular semisimple set. Thus if τ is trivial, $G := H = H^\sigma \times H^\sigma$ and $\sigma : H \rightarrow H$ is the automorphism switching the two factors every irreducible admissible representation is relatively regular. We do not know if the same statement is true in general, and one has to be cautious given [RR, Section 4]. However, it seems likely that in the settings of interest to this paper every irreducible admissible representation arising as a local factor of a cuspidal automorphic representation that is both G^σ and G^θ -distinguished is relatively τ -regular.

7. Prestabilization of a single stable relative orbital integral. In this section, following work of Langlands, Kottwitz, Shelstad, and Labesse, we define stable relative and stable twisted relative global orbital integrals and show how they decompose into a sum of global relative κ -orbital integrals. The main result is Proposition 7.2. As anyone familiar with the usual (not relative) stable trace formula could guess, the reason for introducing the relative κ -orbital integrals is that they factor into local κ -orbital integrals. This makes it possible to apply the local matching theory developed in Section 3 and Section 4. Our treatment follows [La1], and we refer to loc. cit. for notation involving abelianized cohomology of reductive groups and quotients. We emphasize that we do *not* attempt to write the κ -orbital integrals given below in terms of stable relative orbital integrals on other groups; this is why the section is entitled “Prestabilization...” instead of “Stabilization...”. In Section 8 below, we show how to collect the relatively elliptic terms of the relative trace formula together.

We should note that in the biquadratic, non-twisted case, a stabilization dependent on various conjectural fundamental lemmas was given by Flicker in [Fl4], together with a conjectural definition of relative transfer factors.

7.1. Global relative orbital integrals. Let $\Phi \in C_c^\infty(H(\mathbb{A}_F))$, $f \in C_c^\infty(G(\mathbb{A}_F))$. Moreover let $\gamma \in H(F)$ (resp. $\delta \in G(F)$) be a relatively semisimple element (resp. relatively τ -semisimple element). We define the (global) *relative orbital integral*

$$(7.1.1) \quad \text{RO}_\gamma(\Phi) := \text{RO}_\gamma(\Phi, dt_\gamma) := \iint_{H_\gamma(\mathbb{A}_F) \backslash H^\sigma(\mathbb{A}_F)^2} \Phi(h_1^{-1} \gamma h_2) \frac{dh_1 dh_2}{dt_\gamma}$$

where $dh_i = \otimes'_v dh_{i,v}$ and $dt_\gamma = \otimes'_v dt_{\gamma,v}$ are Haar measures on $H^\sigma(\mathbb{A}_F)$ and $H_\gamma(\mathbb{A}_F)$, respectively. Similarly, we define the (global) *twisted relative orbital integral*

$$(7.1.2) \quad \text{TRO}_\delta(f) := \text{TRO}_\delta(f, dt_\delta) := \iint_{G_\delta(\mathbb{A}_F) \backslash G^\sigma \times G^\theta(\mathbb{A}_F)} f(g_1^{-1} \delta g_2) \frac{dg_1 dg_2}{dt_\delta}$$

where $dg_i = \otimes'_v dg_{i,v}$ and $dt_\delta := \otimes'_v dt_{\delta,v}$ are Haar measures on $G^\sigma(\mathbb{A}_F)$ and $G_\delta(\mathbb{A}_F)$, respectively. If $\Phi = \otimes'_v \Phi_v$ is factorable then

$$(7.1.3) \quad \text{RO}_\gamma(\Phi, dt_\gamma) = \prod_v \text{RO}_{\gamma_v}(\Phi_v, dt_{\gamma,v}).$$

If $f = \otimes'_v f_v$ is factorable then

$$(7.1.4) \quad \text{TRO}_\delta(f, dt_\delta) = \prod_v \text{TRO}_{\delta_v}(f_v, dt_{\delta,v}).$$

Let $\gamma_0 \in H(F)$ and $\delta_0 \in G(F)$ be relatively semisimple and relatively τ -semisimple, respectively. The (global) *stable relative orbital integral* is

$$(7.1.5) \quad \text{SRO}_{\gamma_0}(\Phi) := \sum_{\gamma \sim \gamma_0} \text{RO}_\gamma(\Phi, dt_\gamma)$$

where the sum is over relatively semisimple $\gamma \in H(F)$ in the same stable relative class as γ_0 . The (global) *stable twisted relative orbital integral* is

$$(7.1.6) \quad \text{STRO}_{\delta_0}(f) := \sum_{\delta \sim \delta_0} \text{TRO}_\delta(f, dt_\delta)$$

where the sum is over relatively τ -semisimple $\delta \in G(F)$ in the same stable relative τ -class as δ_0 . Here we assume that the $dt_\gamma = \otimes'_v dt_{\gamma,v}$ (resp. $dt_\delta = \otimes'_v dt_{\delta,v}$) are compatible in the sense that for each v the $dt_{\gamma,v}$ (resp. $dt_{\delta,v}$) are compatible as in Section 4.3.

In analogy with (4.4.1), for a pair of (connected) reductive F -groups $I \leq H$ we set notation for the abelian group

$$\mathfrak{R}(I, H; F) := H_{\text{ab}}^0(\mathbb{A}_F/F, I \backslash H)^D.$$

For each place v there is a localization map

$$(7.1.7) \quad \begin{aligned} \mathfrak{K}(I, H; F) &\longrightarrow \mathfrak{K}(I, H; F_v) \\ \kappa_H &\longmapsto \kappa_{Hv} \end{aligned}$$

defined as the dual of

$$H_{\text{ab}}^0(F_v, I \setminus H) \longrightarrow H_{\text{ab}}^0(\mathbb{A}_F, I \setminus H) \longrightarrow H_{\text{ab}}^0(\mathbb{A}_F/F, I \setminus H).$$

Let $\Phi = \otimes'_v \Phi_v \in C_c^\infty(H(\mathbb{A}_F))$ and $f = \otimes'_v f_v \in C_c^\infty(G(\mathbb{A}_F))$ be factorable, and let $\gamma \in H(F)$ (resp. $\delta \in G(F)$) be relatively semisimple (resp. relatively τ -semisimple). Finally, let $\kappa_H \in \mathfrak{K}(H_\gamma^\sigma, H^\sigma \times H^\sigma; F)$ and $\kappa \in \mathfrak{K}(G_\delta, G^\sigma \times G^\sigma; F)$. We then set

$$(7.1.8) \quad \begin{aligned} \text{RO}_\gamma^{\kappa_H}(\Phi) &:= \prod_v \text{RO}_{\gamma_v}^{\kappa_{Hv}}(\Phi_v) \\ \text{TRO}_\delta^\kappa(f) &:= \prod_v \text{TRO}_{\delta_v}^{\kappa_v}(f_v), \end{aligned}$$

whenever this product is well-defined (i.e. convergent). The following proposition ensures convergence:

PROPOSITION 7.1. *Let $\gamma_0 \in H(F)$ (resp. $\delta_0 \in G(F)$) be relatively regular semisimple (resp. relatively τ -regular semisimple) and let $\kappa_H \in \mathfrak{K}(H_{\gamma_0}, H^\sigma \times H^\sigma; F)$ (resp. $\kappa \in \mathfrak{K}(G_{\delta_0}, G^\sigma \times G^\sigma; F)$). Moreover, let $\Phi = \otimes'_v \Phi_v \in C_c^\infty(H(\mathbb{A}_F))$ and $f = \otimes'_v f_v \in C_c^\infty(G(\mathbb{A}_F))$. There is a finite set S of places of F such that if $v \notin S$ then $\text{RO}_{\gamma_{0v}}^{\kappa_{Hv}}(\Phi_v) = 1$ (resp. $\text{TRO}_{\delta_{0v}}^{\kappa_v}(f_v) = 1$).*

We note that this is a weak relative analogue of the results of [K4, Section 7].

Proof. Note that $(G_\delta)_{F_v} = G_{\delta F_v}$ is quasi-split for almost all v , and hence $e(G_{\delta F_v}) = 1$ for almost all v , and the character κ_v is trivial on the intersection of the $G^\sigma \times G^\theta$ -orbit of δ_v and the support of f_v for almost all v by the definition of the (restricted direct) topology on $H_{\text{ab}}^0(\mathbb{A}_F, G_\gamma \setminus G^\sigma \times G^\theta)$ [La1, Section 1.4-1.8] and the proof of [H, Proposition 3.4]. With this in mind, the proposition follows immediately from [H, Proposition 3.2] and the proof of [H, Proposition 3.4]. $\square$

7.2. Some cohomology groups. In this section we collect notation for some Galois cohomology groups that will be used in the following subsections. Let v be a place of F and let $I \leq H$ be a pair of connected reductive F_v -groups. Set

$$(7.2.1) \quad \mathfrak{E}(I, H; F_v) := \ker [H_{\text{ab}}^1(F_v, I) \rightarrow H_{\text{ab}}^1(F_v, H)].$$

There is a natural map

$$H_{\text{ab}}^0(F_v, I \setminus H) \longrightarrow \mathfrak{E}(I, H; F_v)$$

induced by the long exact sequence attached to the crossed $\text{Gal}(\overline{F}_v/\overline{F}_v)$ -module $I(\overline{F}_v) \rightarrow H(\overline{F}_v)$ [La1, p. 20]. The image of the dual map

$$\mathfrak{E}(I, H; F_v)^D \longrightarrow \mathfrak{K}(I, H; F_v)$$

is denoted $\mathfrak{K}(I, H; F_v)_1$. Similarly, if $I \leq H$ is a pair of connected reductive F -groups, write

$$\mathfrak{E}(I, H; \mathbb{A}_F/F) := \text{coker} [H_{\text{ab}}^0(\mathbb{A}_F, H) \longrightarrow H_{\text{ab}}^0(\mathbb{A}_F/F, I \setminus H)].$$

There is a natural quotient map

$$H_{\text{ab}}^0(\mathbb{A}_F/F, I \setminus H) \longrightarrow \mathfrak{E}(I, H; \mathbb{A}_F/F).$$

The image of the dual map

$$\mathfrak{E}(I, H; \mathbb{A}_F/F)^D \longrightarrow \mathfrak{K}(I, H; F)$$

is denoted $\mathfrak{K}(I, H; F)_1$. The localization map (7.1.7) induces a homomorphism

$$(7.2.2) \quad \mathfrak{K}(I, H; F)_1 \longrightarrow \prod_v \mathfrak{K}(I, H; F_v)_1$$

(compare [La1, p. 43]). The kernel of this map is denoted $\mathfrak{K}(I, H; F)_0$.

7.3. Prestabilization of a single relatively elliptic term. For a reductive F -group H , write $\tau(H)$ for the Tamagawa number of H (this τ should not be confused with the Galois automorphism τ from above). Finally, for a pair of reductive F -groups I and H , write

$$(7.3.1) \quad d(I, H) := \# \text{coker} [H_{\text{ab}}^1(\mathbb{A}_F/F, I) \rightarrow H_{\text{ab}}^1(\mathbb{A}_F/F, H)].$$

The main result of this section is the following adaptation of the work of Langlands, Kottwitz, and Labesse [L2, K3, K4, La1] to our situation:

PROPOSITION 7.2. *Let $\Phi = \otimes'_v \Phi_v \in C_c^\infty(H(\mathbb{A}_F))$ and $f = \otimes'_v f_v \in C_c^\infty(G(\mathbb{A}_F))$ be factorable. If $\gamma \in H(F)$ is relative regular and relatively elliptic then*

$$\begin{aligned} \text{SRO}_\gamma(\Phi) &= \frac{\tau(H^\sigma \times H^\sigma)}{\tau(H_\gamma)d(H_\gamma, H^\sigma \times H^\sigma)} \sum_{\kappa_H} \text{RO}_\gamma^{\kappa_H}(\Phi) \\ &= \frac{\tau(H^\sigma \times H^\sigma)}{\tau(H_\gamma)d(H_\gamma, H^\sigma \times H^\sigma)} \sum_{\kappa_H} \prod_v \text{RO}_{\gamma_v}^{\kappa_{Hv}}(\Phi_v) \end{aligned}$$

where the sum is over $\kappa_H \in \mathfrak{K}(H_\gamma, H^\sigma \times H^\sigma; F)_1$.

Similarly, if $\delta \in G(F)$ is relatively τ -regular and relatively elliptic then

$$\begin{aligned} \text{STRO}_\delta(f) &= \frac{\tau(G^\sigma \times G^\theta)}{\tau(G_\gamma)d(G_\delta, G^\sigma \times G^\sigma)} \sum_{\kappa} \text{TRO}_\delta^\kappa(f) \\ &= \frac{\tau(G^\sigma \times G^\theta)}{\tau(G_\delta)d(G_\delta, G^\sigma \times G^\theta)} \sum_{\kappa} \prod_v \text{TRO}_{\gamma_v}^{\kappa_v}(f_v) \end{aligned}$$

where the sum is over $\kappa \in \mathfrak{K}(G_\gamma, G^\sigma \times G^\theta; F)_1$.

Proof. We begin by recalling that H_γ and G_δ are connected, and hence the groups

$$\begin{aligned} \mathfrak{K}(H_\gamma, H^\sigma \times H^\sigma; F)_1 &= \mathfrak{E}(H_\gamma, H^\sigma \times H^\sigma; \mathbb{A}_F/F)^D \\ \mathfrak{K}(G_\delta, G^\sigma \times G^\theta; F)_1 &= \mathfrak{E}(G_\delta, G^\sigma \times G^\theta; \mathbb{A}_F/F)^D \end{aligned}$$

are finite [La1, Proposition 1.8.4]. Moreover, the κ -orbital integrals converge by Proposition 7.1. With these observations in mind, the proof is a standard consequence of the Fourier transform on a finite group. In more detail, one combines the first exact sequence on [La1, p. 42] with [La1, Propositions 1.8.5 and 1.8.6] (compare the proof of [La1, Proposition 4.2.1]). $\square$

8. Grouping relatively elliptic terms. In this section, we group together the relatively elliptic and relatively τ -elliptic portions of the relative trace formula and twisted relative trace formula, respectively.

8.1. Haar measures. Recall the Harish-Chandra subgroups ${}^1H(\mathbb{A}_F)$ and the central subgroups $A_H \leq H(F \otimes_{\mathbb{Q}} \mathbb{R})$ of Section 2.3. For every $\gamma \in H(\mathbb{A}_F)$ and $\delta \in G(\mathbb{A}_F)$ write

$$\begin{aligned} {}^2H^\sigma(\mathbb{A}_F) &:= H^\sigma(\mathbb{A}_F) \cap {}^1H(\mathbb{A}_F) \\ {}^2G^\sigma(\mathbb{A}_F) &:= G^\sigma(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F) \\ {}^2G^\theta(\mathbb{A}_F) &:= G^\theta(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F) \\ {}^2H_\gamma(\mathbb{A}_F) &:= H_\gamma(\mathbb{A}_F) \cap {}^1H(\mathbb{A}_F) \times {}^1H(\mathbb{A}_F) \\ {}^2G_\delta(\mathbb{A}_F) &:= G_\delta(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F) \times {}^1G(\mathbb{A}_F). \end{aligned}$$

Moreover, write

$$\begin{aligned} A_H^\sigma &:= H^\sigma(F \otimes_{\mathbb{Q}} \mathbb{R}) \cap A_H \\ A_G^\sigma &:= G^\sigma(F \otimes_{\mathbb{Q}} \mathbb{R}) \cap A_G \\ A_G^\theta &:= G^\theta(F \otimes_{\mathbb{Q}} \mathbb{R}) \cap A_G \end{aligned}$$

and

$$\begin{aligned} A &:= \{(x, x) \in A_H^\sigma \times A_H^\sigma\} \\ \tilde{A} &:= \{(z, z) \in A_G^\sigma \times A_G^\theta : z \in A_G^\sigma \cap A_G^\theta\}. \end{aligned}$$

We then have decompositions

$$\begin{aligned} (8.1.1) \quad & H_\gamma(\mathbb{A}_F) \backslash H^\sigma(\mathbb{A}_F) \times H^\sigma(\mathbb{A}_F) \\ & \cong (A \backslash A_H^\sigma \times A_H^\sigma) \times ({}^2H_\gamma(\mathbb{A}_F) \backslash {}^2H^\sigma(\mathbb{A}_F) \times {}^2H^\sigma(\mathbb{A}_F)) \\ & G_\delta(\mathbb{A}_F) \backslash G^\sigma(\mathbb{A}_F) \times G^\theta(\mathbb{A}_F) \\ & = (\tilde{A} \backslash A_G^\sigma \times A_G^\theta) \times ({}^2G_\delta(\mathbb{A}_F) \backslash {}^2G^\sigma(\mathbb{A}_F) \times {}^2G^\theta(\mathbb{A}_F)). \end{aligned}$$

We now specify a choice of Haar measure on all of the groups appearing in (8.1.1), at least if γ is relatively elliptic (resp. δ is relatively τ -elliptic). Whenever a reductive F -group H appears, we give $H(\mathbb{A}_F)$, A_H , and ${}^1H(\mathbb{A}_F)$ the measures that are used in the definition of the Tamagawa number $\tau(H)$. We then fix, once and for all, measures $dz_1 = dz_2$ for A_H^σ , $d\tilde{z}$ for A_G^σ and $d\tilde{z}_\theta$ for A_G^θ , and stipulate that the isomorphisms

$$\begin{aligned} {}^2H^\sigma(\mathbb{A}_F) &\cong A_{H^\sigma} / A_H^\sigma \times {}^1H^\sigma(\mathbb{A}_F) \\ {}^2G^\sigma(\mathbb{A}_F) &\cong A_{G^\sigma} / A_G^\sigma \times {}^1G^\sigma(\mathbb{A}_F) \\ {}^2G^\theta(\mathbb{A}_F) &\cong A_{G^\theta} / A_G^\theta \times {}^1G^\theta(\mathbb{A}_F) \end{aligned}$$

are measure preserving. Notice that if γ is relatively elliptic (resp. δ is relatively τ -elliptic) then

$$\begin{aligned} {}^2H_\gamma(\mathbb{A}_F) &= {}^1H_\gamma(\mathbb{A}_F) \\ {}^2G_\delta(\mathbb{A}_F) &= {}^1G_\delta(\mathbb{A}_F). \end{aligned}$$

By our earlier convention, we have already endowed ${}^2H_\gamma(\mathbb{A}_F)$, ${}^2G_\delta(\mathbb{A}_F)$, $A = A_{H_\gamma}$ and $\tilde{A} = A_{G_\delta}$ with measures. Altogether, this endows all of the groups occurring in (8.1.1) with measures. Thus ${}^2H_\gamma(\mathbb{A}_F)$ and ${}^2G_\delta(\mathbb{A}_F)$ are given the unique Haar measures such that if γ is relatively elliptic and δ is relatively τ -elliptic then

$$\begin{aligned} \text{vol}(H_\gamma(F) \backslash {}^2H_\gamma(\mathbb{A}_F)) &= \tau(H_\gamma) \\ \text{vol}(G_\delta(F) \backslash {}^2G_\delta(\mathbb{A}_F)) &= \tau(G_\delta). \end{aligned}$$

For the rest of this paper, we use these choices of measures when we form relative orbital integrals and twisted relative orbital integrals. These measures will be compatible for the same reason that the Tamagawa measures in the usual trace formula are compatible, namely that the Tamagawa numbers of two inner forms of the same quasi-split reductive group are equal [K5, Ch].

For $\Phi \in C_c^\infty(H(\mathbb{A}_F))$ and $f \in C_c^\infty(G(\mathbb{A}_F))$ we define

$$(8.1.2) \quad \begin{aligned} \Phi^1(x) &:= \int_{A \backslash A_H^\sigma \times A_H^\sigma} \Phi(z_1^{-1} z_2 x) \frac{dz_1 dz_2}{du} \\ f^1(x) &:= \int_{\tilde{A} \backslash A_G^\sigma \times A_G^\theta} f(\tilde{z}^{-1} \tilde{z}_\theta x) \frac{d\tilde{z} d\tilde{z}_\theta}{dt}. \end{aligned}$$

8.2. Elliptic kernels. Let $\Phi \in C_c^\infty(H(\mathbb{A}_F))$, $f \in C_c^\infty(G(\mathbb{A}_F))$ and consider the kernel functions

$$\begin{aligned} K_\Phi(x, y) &:= \sum_{\gamma} \Phi^1(x^{-1} \gamma y) : {}^1H(\mathbb{A}_F) \times {}^1H(\mathbb{A}_F) \longrightarrow \mathbb{C} \\ K_f(x, y) &:= \sum_{\delta} f^1(x^{-1} \delta y) : {}^1G(\mathbb{A}_F) \times {}^1G(\mathbb{A}_F) \longrightarrow \mathbb{C} \end{aligned}$$

where the first sum is over relatively regular elliptic $\gamma \in H(F)$ and the second is over relatively τ -regular elliptic $\delta \in G(F)$. Define the integrals

$$\begin{aligned} \text{RT}_e(\Phi) &:= \iint_{(H^\sigma(F) \backslash {}^2H^\sigma(\mathbb{A}_F))^2} K_\Phi(h_1, h_2) dh_1 dh_2 \\ \text{TRT}_e(f) &:= \iint_{G^\sigma(F) \backslash {}^2G^\sigma(\mathbb{A}_F) \times G^\theta(F) \backslash {}^1G^\theta(\mathbb{A}_F)} K_f(g_1, g_2) dg_1 dg_2. \end{aligned}$$

Here the dh_i are both induced by the measure on ${}^2H^\sigma(\mathbb{A}_F)$ fixed above and dg_1, dg_2 are induced by the measures on ${}^2G^\sigma(\mathbb{A}_F)$ and ${}^2G^\theta(\mathbb{A}_F)$ fixed above, respectively. These integrals are absolutely convergent (see the proof of [H, Theorem 4.1]).

8.3. Stable geometric expansions. Using standard manipulations (compare [H]), we rewrite

$$(8.3.1) \quad \begin{aligned} \text{RT}_e(\Phi) &= \sum_{\gamma_0} \sum_{\gamma \sim \gamma_0} a(\gamma) \text{RO}_\gamma(\Phi) \\ \text{TRT}_e(f) &= \sum_{\delta_0} \sum_{\delta \sim \delta_0} a^\tau(\delta) \text{TRO}_\delta(f) \end{aligned}$$

where the exterior sums are over a set of representatives for the stable relatively regular elliptic classes (resp. stable relatively τ -regular elliptic classes) and the interior sums are over a set of representatives for the relative classes (resp. relative τ -classes) in the stable relative class of γ_0 (resp. stable relative τ -class of δ_0). Here

$$(8.3.2) \quad \begin{aligned} a(\gamma) &:= \text{vol}(H_\gamma(F) \backslash {}^2H_\gamma(\mathbb{A}_F)) = \text{vol}(H_\gamma(F) \backslash {}^1H_\gamma(\mathbb{A}_F)) = \tau(H_\gamma) \\ a^\tau(\delta) &:= \text{vol}(G_\delta(F) \backslash {}^2G_\delta(\mathbb{A}_F)) = \text{vol}(G_\delta(F) \backslash {}^1G_\delta(\mathbb{A}_F)) = \tau(G_\delta), \end{aligned}$$

where the volumes are taken with respect to the measures fixed in Section 8.1. Here, as in Section 8.1, we are using the fact that γ and δ are relatively elliptic and relatively τ -elliptic, respectively, to conclude that ${}^2H_\gamma(\mathbb{A}_F) = {}^1H_\gamma(\mathbb{A}_F)$ and ${}^2G_\delta(\mathbb{A}_F) = {}^1G_\delta(\mathbb{A}_F)$. Note that with the choice of Haar measure fixed in Section 8.1, the measures occurring in the summands corresponding to γ in the same stable relative class as a given γ_0 are compatible with respect to (3.2.3) above. Similarly, the measures occurring in the summands corresponding to δ in the same stable relative τ -class as a given δ_0 are compatible with respect to (3.2.3). This follows from the well-known fact that the Tamagawa numbers of two inner forms of the same quasi-split reductive group are equal [K5, Ch]. With this in mind, we group stable classes in (8.3.1) and obtain

$$(8.3.3) \quad \begin{aligned} \mathrm{RT}_e(\Phi) &= \sum_{\gamma_0} \tau(H_{\gamma_0}) \mathrm{SRO}_{\gamma_0}(\Phi) \\ \mathrm{TRT}_e(f) &= \sum_{\delta_0} \tau(G_{\delta_0}) \mathrm{STRO}_{\delta_0}(f) \end{aligned}$$

where the first sum is over a set $\{\gamma_0\}$ of representatives for the stable relative classes in $H(F)$ that consist of relatively regular elliptic elements and the second sum is over a set $\{\delta_0\}$ of representatives for the stable relative τ -classes in $G(F)$ that consist of relatively τ -regular semisimple elements. The measures inherent in the definition of $\mathrm{SRO}_{\gamma_0}(\Phi)$ and $\mathrm{STRO}_{\delta_0}(f)$ are specified as in Section 8.1.

Applying Proposition 7.2 we can rewrite (8.3.3) as

$$(8.3.4) \quad \begin{aligned} \mathrm{RT}_e(\Phi) &= \sum_{\gamma_0} \frac{\tau(H^\sigma \times H^\sigma)}{d(H_{\gamma_0}, H^\sigma \times H^\sigma)} \sum_{\kappa_H} \mathrm{RO}_{\gamma_0}^{\kappa_H}(\Phi) \\ \mathrm{TRT}_e(f) &= \sum_{\delta_0} \frac{\tau(G^\sigma \times G^\theta)}{d(G_{\delta_0}, G^\sigma \times G^\theta)} \sum_{\kappa} \mathrm{TRO}_{\delta_0}^{\kappa}(f). \end{aligned}$$

Here the interior sum indexed by γ_0 is over $\kappa_H \in \mathfrak{K}(H_{\gamma_0}, H^\sigma \times H^\sigma; F)_1$ and the interior sum indexed by δ_0 is over $\kappa \in \mathfrak{K}(G_{\delta_0}, G^\sigma \times G^\theta; F)_1$ (compare Proposition 7.2).

8.4. The unitary case. We now specialize our notation to our primary case of interest. Thus assume that $H^\sigma = U^\sigma$ is a unitary group with respect to a quadratic extension of fields M/F with M a CM field and F a totally real field. In other words, we assume that there is a simple algebra D over F with center M and an involution $\dagger$ of D such that the fixed field of $\dagger$ acting on M is F and such that if R is an F -algebra then

$$(8.4.1) \quad U^\sigma(R) := \{g \in (D \otimes_F R)^\times : gg^\dagger = 1\}.$$

We let E/F be a quadratic extension of fields with E totally real and assume moreover that we are in the biquadratic situation, thus $U := \text{Res}_{E/F} U^\sigma$ and σ is the automorphism induced by the generator of $\text{Gal}(E/F)$ which we will also denote by σ . We let τ be the generator of M/F and $G^\sigma := \text{Res}_{M/F} U^\sigma$, $G := \text{Res}_{M/F} U$. The automorphism τ defines an automorphism $\tau : G \rightarrow G$ such that the subgroup of G fixed by τ is U . Finally, choose a (finite-dimensional) representation

$$U_{F_\infty}^\sigma \longrightarrow \text{Aut}_{\mathbb{R}}(V)$$

and let $G_{F_\infty}^\sigma \rightarrow \text{Aut}_{\mathbb{R}}(\text{Res}_{M_\infty/F_\infty} V)$ be the representation obtained by restriction of scalars. We have the following proposition:

PROPOSITION 8.1. *Suppose that $\Phi \in C_c^\infty(U(\mathbb{A}_F))$ and $f \in C_c^\infty(G(\mathbb{A}_F))$ are factorable and that Φ_v matches f_v for all finite places v of F . Consider the following assumptions:*

(1) *One has*

$$f_\infty = f_1 \times f_2 \in C_c^\infty(G^\sigma(F_\infty) \times G^\sigma(F_\infty)) = C_c^\infty(G(F_\infty))$$

with $f_1^{-\tau} * f_2 = f_{L, \text{Res}_{M_\infty/F_\infty} V, \tau}$.

(2) *One has*

$$\Phi_\infty = \Phi_1 \times \Phi_2 \in C_c^\infty(U^\sigma(F_\infty) \times U^\sigma(F_\infty)) = C_c^\infty(U(F_\infty))$$

with $\Phi_1 * \Phi_2^{-1} = c_\infty f_{EP, V}$ for some $c_\infty \in \mathbb{R}_{>0}$.

If the assumptions hold, then for an appropriate choice of $c_\infty \in \mathbb{R}_{>0}$ one has

$$\text{RT}_e(\Phi) = 2 \text{TRT}_e(f).$$

In the proposition, $f_{L, \text{Res}_{M_\infty/F_\infty} V, \tau}$ is the Lefschetz function attached to the representation $\text{Res}_{M_\infty/F_\infty} V$ of $G_{F_\infty}^\sigma$ and the involution τ of $G_{F_\infty}^\sigma$ (see [BLS, Proposition 8.4]). Moreover $f_{EP, V}$ is the Euler-Poincaré function attached to V ; this is simply the Lefschetz function in the case that the associated automorphism is trivial.

Proof. The functions $f_{L, \text{Res}_{M_\infty/F_\infty} V, \tau}$ and $f_{EP, V}$ are stable in the sense of [La1, Définition 3.8.2] (see [CL, Théorème A.1.1] and [La2, Théorème 7.1]). Thus, in view of Proposition 4.8 and assumptions (2) and (1), any γ_0 (resp. δ_0) contributing a nonzero summand to $\text{RT}_e(\Phi)$ (resp. $\text{TRT}_e(f)$) is relatively regular elliptic (resp. relatively τ -regular elliptic) at F_∞ . Applying [La1, Proposition 1.9.6 and Lemme 1.9.7] we conclude that for these places the set ∞ is (H, H_{γ_0}) (resp. (G, G_δ))-essential for any γ_0 (resp. δ_0) contributing a nonzero summand. Using the fact that $f_{L, \text{Res}_{M_\infty/F_\infty} V, \tau}$ and $f_{EP, V}$ are stable and Proposition 4.8 again, we conclude that the κ -orbital integrals for $\kappa \neq 1$ in (8.3.4) all vanish, and hence

$$\text{RT}_e(\Phi) = \sum_{\gamma_0} \frac{\tau(U^\sigma \times U^\sigma)}{d(U_{\gamma_0}, U^\sigma \times U^\sigma)} \text{SRO}_{\gamma_0}(\Phi)$$

$$\mathrm{TRT}_e(f) = \sum_{\delta_0} \frac{\tau(G^\sigma \times G^\theta)}{d(G_{\delta_0}, G^\sigma \times G^\theta)} \mathrm{STRO}_{\delta_0}(f).$$

By [La1, Corollaire A.1.2] for $v|^\infty$ the functions f_v and $c_v \Phi_v$ match for an appropriate constant $c_v \in \mathbb{R}_{>0}$ (see also [La2, Théorème 7.1]). We henceforth assume that f_v and Φ_v match for all v .

Since every relatively regular semisimple γ_0 is a norm of a δ_0 by Lemma 3.11 and every relatively τ -regular elliptic semisimple δ_0 that has local norms at infinity has a global norm by Proposition 3.10, to complete the proof it suffices to show that

$$(8.4.2) \quad \frac{\tau(U^\sigma \times U^\sigma)}{d(U_{\gamma_0}, U^\sigma \times U^\sigma)} \mathrm{SRO}_{\gamma_0}(\Phi) = 2 \frac{\tau(G^\sigma \times G^\theta)}{d(G_{\delta_0}, G^\sigma \times G^\theta)} \mathrm{STRO}_{\delta_0}(f)$$

if γ_0 is a norm of δ_0 . By the definition of matching we have $\mathrm{SRO}_{\gamma_0}(\Phi) = \mathrm{STRO}_{\delta_0}(f)$. Moreover

$$(8.4.3) \quad \begin{aligned} d(U_{\gamma_0}, U^\sigma \times U^\sigma) &= \# \mathrm{coker} [H_{\mathrm{ab}}^1(\mathbb{A}_F/F, U^\sigma \times U^\sigma) \rightarrow H_{\mathrm{ab}}^1(\mathbb{A}_F/F, U^\sigma \times U^\sigma)] = 1 \\ d(G_{\delta_0}, G^\sigma \times G^\theta) &= \# \mathrm{coker} [H_{\mathrm{ab}}^1(\mathbb{A}_F/F, U^\sigma \times G^\theta) \rightarrow H_{\mathrm{ab}}^1(\mathbb{A}_F/F, G^\sigma \times G^\theta)] = 1 \end{aligned}$$

by the fact that γ_0 and δ_0 are relatively elliptic and relatively τ -elliptic, respectively, and [La1, Corollaire 1.9.3]. By [La1, Corollaire 1.7.4], we have

$$(8.4.4) \quad \begin{aligned} \tau(U^\sigma \times U^\sigma) &= \frac{\# H_{\mathrm{ab}}^1(\mathbb{A}_F/F, U^\sigma \times U^\sigma)}{\# \ker_{\mathrm{ab}}^1(F, U^\sigma \times U^\sigma)} = \frac{4}{\# \ker_{\mathrm{ab}}^1(F, U^\sigma \times U^\sigma)} \\ \tau(G^\sigma \times G^\theta) &= \frac{\# H_{\mathrm{ab}}^1(\mathbb{A}_F/F, G^\sigma \times G^\theta)}{\# \ker_{\mathrm{ab}}^1(F, G^\sigma \times G^\theta)} = \frac{2}{\# \ker_{\mathrm{ab}}^1(F, G^\sigma \times G^\theta)} \end{aligned}$$

Here we are using the fact that $H_{\mathrm{ab}}^1(\mathbb{A}_F/F, \mathrm{GL}_n) = 1$ and $H_{\mathrm{ab}}^1(\mathbb{A}_F/F, H) = 2$ if H is a (nonsplit) unitary group (see [HL, Lemma 1.2.1(i)] for the latter statement).

Since G^σ is an inner form of a general linear group, the Hasse principle is valid for it. On the other hand, G^θ and U^σ are unitary groups, so the Hasse principle is valid for them as well [HL, Lemma 1.2.1(i)], so $\ker_{\mathrm{ab}}^1(F, U^\sigma \times U^\sigma) = \ker_{\mathrm{ab}}^1(F, G^\sigma \times G^\theta) = 1$. In view of (8.4.3) and (8.4.4), this completes the proof of the proposition. $\square$

9. Relative trace formulae.

9.1. A simple relative trace formula. For $f \in C_c^\infty(G(\mathbb{A}_F))$ and a cuspidal unitary automorphic representation Π of $G(\mathbb{A}_F)$, Arthur has shown [Ar1, Lemmas 4.5 and 4.8] that there is a (unique) function $K_{\Pi(f)}(x, y) \in L_0^2(G(F) \backslash {}^1G(\mathbb{A}_F) \times$

$G(F) \backslash {}^1G(\mathbb{A}_F)$) that is smooth in x and y separately with L^2 -expansion

$$(9.1.1) \quad K_{\Pi(f^1)}(x, y) = \sum_{\phi \in \mathcal{B}(\Pi)} \Pi(f^1)\phi(x)\overline{\phi(y)}.$$

Here $\mathcal{B}(\Pi)$ is an orthonormal basis of the Π -isotypic subspace $V_{\Pi} \leq L_0^2(G(F) \backslash {}^1G(\mathbb{A}_F))$ with respect to the pairing

$$(9.1.2) \quad V_{\Pi} \times V_{\Pi} \longrightarrow \mathbb{C} \\ (\phi_1, \phi_2) \longmapsto \int_{G(F) \backslash {}^1G(\mathbb{A}_F)} \phi_1(y)\overline{\phi_2(y)}dy,$$

with dy induced by the Tamagawa measure. We emphasize that this expansion (9.1.1), in general, is only convergent in the L^2 sense. Following [H], we define

$$(9.1.3) \quad \text{TRT}(\Pi(f^1)) = \iint_{G^{\sigma}(F) \backslash {}^2G^{\sigma}(\mathbb{A}_F) \times G^{\theta}(F) \backslash {}^2G^{\theta}(\mathbb{A}_F)} K_{\Pi(f^1)}(x, y) dx dy.$$

The integral is absolutely convergent by [AGR, Section 2, Proposition 1]. The following simple relative trace formula is proved in [H] via a modification of the argument used to prove the usual simple trace formula:

THEOREM 9.1. *Let $f = f_{v_1} \otimes f_{v_2} \otimes f^{v_1 v_2} \in C_c^{\infty}(G(\mathbb{A}_F))$ be a factorable function such that*

- f_{v_1} is F -supercuspidal.
- f_{v_2} is supported on relatively τ -regular elliptic elements of $G(F_{v_2})$.

Then

$$(9.1.4) \quad \text{TRT}_e(f) := \sum_{\delta} \tau(G_{\delta}) \text{TRO}_{\delta}(f) = \sum_{\Pi} \text{TRT}(\Pi(f^1))$$

where the sum on the left is over a set of representatives for the relatively τ -regular elliptic classes in $G(F)$ and the sum on the right is over a set of representatives for the equivalence classes of cuspidal automorphic representations Π of ${}^1G(\mathbb{A}_F)$.

Here we say that f_v is F -supercuspidal if f_v has zero integral along the unipotent radical of any proper parabolic of G_{F_v} that is defined over F ; i.e., is the base change to F_v of a parabolic subgroup of G . As usual, we allow τ to be trivial in the theorem above. By convention, an automorphic representation of ${}^1G(\mathbb{A}_F)$ is the restriction to ${}^1G(\mathbb{A}_F)$ of an automorphic representation of $G(\mathbb{A}_F)$, and we consider two such to be equivalent if they are equivalent as representations of ${}^1G(\mathbb{A}_F)$.

Let $K_{\infty} \leq G(F_{\infty})$ be a maximal compact subgroup. We note that if Π is cuspidal and f is K_{∞} -finite or $\Pi(f^1)$ has finite rank then we have

$$(9.1.5) \quad \text{TRT}(\Pi(f^1)) = \sum_{\phi \in \mathcal{B}(\Pi)} \mathcal{P}_{G^{\sigma}}(\Pi(f^1)\phi)\overline{\mathcal{P}_{G^{\theta}}(\phi)}$$

where the sum is over an orthonormal basis $\mathcal{B}(\Pi)$ of the Π -isotypic subspace of $L_0^2(G(F) \backslash {}^1G(\mathbb{A}_F))$ consisting of smooth vectors (see (1.1.1) for the definition of $\mathcal{P}_{G^\sigma}$). For any $f \in C_c^\infty(G(\mathbb{A}_F))$, if $\text{TRT}(\Pi(f^1))$ is nonzero then Π is both G^σ and G^θ -distinguished.

9.2. Comparison. Upon combining Proposition 8.1 and Theorem 9.1, we obtain the following proposition:

PROPOSITION 9.2. *Suppose that $\Phi = \otimes'_v \Phi_v \in C_c^\infty(U(\mathbb{A}_F))$ and $f = \otimes'_v f_v \in C_c^\infty(G(\mathbb{A}_F))$ are factorable, and satisfy the following conditions:*

- Φ_v matches f_v for all v .
- Φ_∞ and f_∞ satisfy conditions (1-2) of Proposition 8.1.
- There is a finite place v_1 such that Φ_{v_1} is F -supercuspidal.
- There is a place v_2 of F such that f_{v_2} is F -supercuspidal.
- There is a place v_3 of F such that Φ_{v_3} is supported on relatively regular elliptic semisimple elements.
- There is a place v_4 of F such that f_{v_4} is supported on relatively τ -regular elliptic semisimple elements.

Under the above assumptions, we have

$$\sum_{\pi} \text{RT}(\pi(\Phi^1)) = 2 \sum_{\Pi} \text{TRT}(\Pi(f^1)),$$

where the sums are over equivalence classes of cuspidal automorphic representations π of $U(\mathbb{A}_F)$ and Π of ${}^1G(\mathbb{A}_F)$, respectively.

Note that we do not require that the places v_i be distinct. Here $\text{RT}(\pi(\Phi^1))$ is defined to be $\text{TRT}(\Pi(f^1))$ in the “ $\tau = 1$ ” case.

10. Application. For this entire section we will place ourselves in the following special case of the construction exposed in the previous sections. Let E/F be a quadratic extension of totally real fields and let M/F be a CM extension. Let U^σ be a unitary group over F as in (8.4.1) and $U := \text{Res}_{E/F} U^\sigma$. We let σ be the automorphism of U induced by the generator of $\text{Gal}(E/F)$, which we will also denote by σ . The groups $G^\sigma = \text{Res}_{M/F} U^\sigma$ and $G = \text{Res}_{M/F} U$ are isomorphic to inner forms of $\text{Res}_{M/F} \text{GL}_n$ and $\text{Res}_{ME/F} \text{GL}_n$, respectively, for some n .

When we refer to the base change map below, we will mean the (partially defined) functorial lifting with respect to the map of L -groups

$$b : {}^L U \longrightarrow {}^L G$$

induced by the natural inclusion $U \rightarrow G$ (see [HL]).

Definition 10.1. Let π be a cuspidal automorphic representation of $U^\sigma(\mathbb{A}_E) = U(\mathbb{A}_F)$. We say that a cuspidal automorphic representation Π of $G^\sigma(\mathbb{A}_E) = G(\mathbb{A}_F)$

is a *weak base change* of π if Π_w is the base change of π_w for all places w of E satisfying the following:

- w is infinite,
- ME/E is split at w , or
- π_w and $U_{E_w}^\sigma$ are unramified.

By strong multiplicity one for G [Ba], if a weak base change of π exists then it is unique. Suppose that π and π' are cuspidal and both admit weak base changes to $G(\mathbb{A}_F)$. If π and π' are moreover *nearly equivalent*, i.e., $\pi_v \cong \pi'_v$ for almost all places v , then their weak base changes are obviously equal.

10.1. Statement of theorem. To state the main theorem of this section, it is convenient to introduce a definition. Let $K_{U^\infty} \leq U(F_\infty)$ be a maximal compact subgroup and let V be a representation of $U(F_\infty)$. Let $\mathfrak{u} := \text{Lie}(U_{F_\infty}) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real Lie group $U(F_\infty)$.

Definition 10.2. An automorphic representation π of $U(\mathbb{A}_F)$ has *nonzero cohomology with coefficients in V* if $H^*(\mathfrak{u}, K_{U^\infty}; \pi_\infty \otimes V)$ is not identically zero.

We have the following theorem:

THEOREM 10.3. *Let π be a cuspidal automorphic representation of $U^\sigma(\mathbb{A}_E) = U(\mathbb{A}_F)$ that admits a weak base change Π to $G(\mathbb{A}_F)$. Suppose that π satisfies the following conditions:*

- *There is a finite-dimensional representation V of U_{F_∞} such that π has nonzero cohomology with coefficients in V .*
- *There is a finite place v_1 of F totally split in ME/F such that π_{v_1} is supercuspidal.*
- *There is a finite place $v_2 \neq v_1$ of F totally split in ME/F such that π_{v_2} is in the discrete series.*
- *For all places v of F such that ME/F is ramified and M/F , E/F are both nonsplit at v the weak base change Π of π to $G(F)$ has the property that Π_v is relatively τ -regular.*

If the automorphic representation Π is both G^σ and G^θ -distinguished then there is a cuspidal automorphic representation π' of $U^\sigma(\mathbb{A}_E) = U(\mathbb{A}_F)$ that is U^σ -distinguished and nearly equivalent to π . Moreover, we can take π' to have nonzero cohomology with coefficients in V .

Theorem 10.3 will be proven later in this section. Combining it with the work of Jacquet, Lapid and their collaborators, and Flicker and his collaborators, we obtain the following corollary:

COROLLARY 10.4. *Assume that U^σ is quasi-split or more generally that G^θ and G^σ are quasi-split. Let π be a cuspidal automorphic representation of $U(\mathbb{A}_F)$ satisfying the conditions (1)–(4) of Theorem 10.3. The representation π admits a*

weak base change Π to $\mathrm{GL}_n(\mathbb{A}_{ME})$. If the partial Asai L -function $L^S(s, \Pi; r)$ has a pole at $s = 1$ then some cuspidal automorphic representation π' of $U(\mathbb{A}_F)$ nearly equivalent to π is U^σ -distinguished. Moreover, we can take π' to have nonzero cohomology with coefficients in V .

We require the following lemma for the proof of Corollary 10.4 and also below in the proof of Theorem 10.3:

LEMMA 10.5. *Suppose that Π is a cuspidal automorphic representation of $G(\mathbb{A}_F)$. If Π is G^θ -distinguished, then $\Pi \cong \Pi^{\theta\vee}$. If Π is G^σ -distinguished, then $\Pi^\sigma \cong \Pi^\vee$.*

Proof. Both of these are well-known. For the first, see [J1, Section 3]. For the second, assume that Π is G^σ -distinguished. For places v of F split in E/F , it is trivial to check that $\Pi_v^\sigma \cong \Pi_v^\vee$ using an analogue of the proof of Proposition 10.6 given below. If v is inert (and unramified) in E/F , the fact that $\Pi_v^\sigma \cong \Pi_v^\vee$ is [P, Corollary 2] (see also [F13]). Thus $\Pi^\sigma \cong \Pi^\vee$ by strong multiplicity one. $\square$

We also require the following proposition in the proof of Theorem 10.3:

PROPOSITION 10.6. *Let Ξ be the set of places of F that split in E/F , let π be a cuspidal automorphic representation of $U(\mathbb{A}_F)$. If π is U^σ -distinguished, then $\pi_\Xi^\vee \cong (\pi_\Xi)^\sigma$.*

Proof. We may assume, without loss of generality, that the ϕ_0 in the space of π with nonzero period over $U^\sigma(\mathbb{A}_F)$ is factorable. Write $\phi_0 = \otimes_v \phi_{0,v}$, where the tensor product is over all places of F . For $v \in \Xi$, choose an isomorphism $U_{F_v} \cong U_{F_v}^\sigma \times U_{F_v}^\sigma$ intertwining σ with $(x, y) \mapsto (y, x)$. Using this isomorphism, we can and do decompose

$$(10.1.1) \quad \pi_v \cong \pi'_{1,v} \otimes \pi'_{2,v}$$

for some admissible representations $\pi'_{i,v}$ of $U^\sigma(F_v)$. Thus

$$\pi_v^\sigma \cong \pi'_{2,v} \otimes \pi'_{1,v}.$$

For a fixed $v \in \Xi$, let $V_{\pi'_{1,v}}$, $V_{\pi'_{2,v}}$, and V_π be the spaces of the representations $\pi'_{1,v}$, $\pi'_{2,v}$, and π , respectively. Consider the bilinear pairing

$$V_{\pi'_{1,v}} \otimes V_{\pi'_{2,v}} \hookrightarrow V_\pi \longrightarrow \mathbb{C}.$$

Here the first map is $(\phi_1, \phi_2) \mapsto (\otimes_{v' \neq v} \phi_{0,v'}) \otimes (\phi_1 \otimes \phi_2)$ and the second is $\phi \mapsto \mathcal{P}_{U^\sigma}(\phi)$. This bilinear pairing is $U^\sigma(F_v)$ -equivariant, and is nonzero by hypothesis. By the irreducibility of π_v , we conclude that

$$\pi_{2,v} \cong \pi_{1,v}^\vee$$

and thus

$$\pi_v^\sigma \cong \pi_v^\vee. \quad \square$$

We give the proof of Corollary 10.4 now:

10.2. Proof of Corollary 10.4. We assume the notation and hypotheses of Corollary 10.4. Let Π be the weak base change of π ; it exists by the proof of [HL, Theorem 3.1.4]. Since Π is a weak base change we have $\Pi_v \cong \Pi_v^\tau$ for almost all places v . By strong multiplicity one we conclude that $\Pi \cong \Pi^\tau$. View Π as an automorphic representation of $\text{Res}_{ME/M} \text{GL}_n(\mathbb{A}_M)$. Let $r : {}^L \text{Res}_{ME/M} \text{GL}_n \rightarrow \text{GL}_{n^2}(\mathbb{C})$ be the Asai representation (see [R, Section 6] or [Fl2] for the definition of this representation). By assumption the partial Asai L -function

$$L^S(s, \Pi; r)$$

has a pole at $s = 1$. Here S is a finite set of places of M containing the infinite places and all finite places where ME/M or Π_v is ramified. We note that we are using the fact that $L^S(s, \Pi; r)$ admits a meromorphic continuation to a closed right half-plane containing $s = 1$, a fact established by Flicker and Flicker-Zinoviev [Fl2] [FIZ, Theorem] using the Langlands-Shahidi method [Sh] and an adaptation of the Rankin-Selberg method of Jacquet, Piatetski-Shapiro, and Shalika [JPSS]. As a biproduct of this method, Flicker and Flicker-Zinoviev also establish in [FIZ, Theorem] that if $L^S(s, \Pi; r)$ has a pole at $s = 1$, then Π is distinguished by $\text{GL}_{n/M}$. This implies that $\Pi^\sigma \cong \Pi^\vee$ by Lemma 10.5.

Writing L for the subfield of ME fixed by $\theta := \sigma \circ \tau$, note that ME/L splits at all infinite places. Since $\Pi^\sigma \cong \Pi^\vee$ and $\Pi^\tau \cong \Pi$ we have $\Pi^{\theta^\vee} \cong \Pi$. Thus, applying results of Jacquet ([J1, Theorems 3 and 4] and [J2]) we conclude that Π is distinguished by “the” quasi-split unitary group in n -variables with respect to ME/L .

The corollary now follows from Theorem 10.3. $\square$

10.3. Preparations for the proof of Theorem 10.3. We isolate three steps in the proof of Theorem 10.3 in the following lemmas. If the reader so desires, (s)he can skip this subsection and refer back to it as needed during the following subsection.

For the purpose of stating a lemma we develop some notation. Assume that we are in the biquadratic case of Section 3.1 and let v be a place of F split in E/F . Suppose Π_v is an irreducible admissible representation of $G(F_v)$. Write

$$\mathcal{H}_{\Pi_v} := \text{Im}(C_c^\infty(G(F_v)) \longrightarrow \text{End}_{\mathbb{C}}(V_{\Pi_v})),$$

where the map sends f to $\Pi_v(f)$. For any $x, y \in G(F_v)$ we have linear left multiplication by x and right multiplication by y maps

$$L_x, R_y : \text{End}_{\mathbb{C}}(V_{\Pi_v}) \longrightarrow \text{End}_{\mathbb{C}}(V_{\Pi_v})$$

given by $L_x(T) := \Pi_v(x^{-1}) \circ T$ and $R_y(T) := T \circ \Pi_v(y^{-1})$.

LEMMA 10.7. *The space of linear forms*

$$\ell : \mathcal{H}_{\Pi_v} \longrightarrow \mathbb{C}$$

satisfying $L_x(\ell) = R_y(\ell) = \ell$ for $(x, y) \in G^{\sigma}(F_v) \times G^{\theta}(F_v)$ is at most one-dimensional.

A linear form as in the lemma is known as a $(G^{\sigma}(F_v), G^{\theta}(F_v))$ -invariant *linear form*.

Proof. The natural isomorphism

$$\text{End}_{\mathbb{C}}(V_{\Pi_v})^{\vee} \cong V_{\Pi_v} \boxtimes V_{\Pi_v}^{\vee}$$

is $G(F_v) \times G(F_v)$ -equivariant. A linear form ℓ as in the lemma is sent to

$$(10.3.1) \quad (V_{\Pi_v})^{G^{\sigma}(F_v)} \boxtimes (V_{\Pi_v}^{\vee})^{G^{\theta}(F_v)}$$

under this isomorphism. Since $\Pi_v \cong \Pi_{1v} \otimes \Pi_{2v}$ for some admissible representations Π_{iv} of $G^{\sigma}(F_v)$, it follows that (10.3.1) is at most one dimensional. This proves that the space of linear forms satisfying the hypotheses of the lemma is at most one dimensional. $\square$

We now record a lemma on supercuspid forms. Let v be a finite place of F that splits completely in ME/F . Thus we have isomorphisms

$$(10.3.2) \quad \begin{aligned} U_{F_v} &\cong (U^{\sigma})_{F_v}^2 \\ G_{F_v} &\cong U_{F_v}^2 \cong (U^{\sigma})_{F_v}^4 \end{aligned}$$

intertwining σ with $(x, y) \mapsto (y, x)$ (resp. $(x, y, z, w) \mapsto (z, w, x, y)$) and τ with $(x, y, z, w) \mapsto (y, x, w, z)$. Let π_v be a unitary admissible representation of $U(F_v)$ satisfying $\pi_v^{\vee} \cong \pi_v^{\sigma}$; thus we can and do decompose

$$\pi_v \cong \pi'_v \otimes \pi_v'^{\vee}$$

for some admissible representation π'_v of $U^{\sigma}(F_v)$ using (10.3.2). Let Π_v be the base change of π_v to $G(F_v)$. We can and do factor

$$\Pi_v = \pi_v \otimes \pi_v^{\vee} \cong \pi'_v \otimes \pi_v'^{\vee} \otimes \pi_v'^{\vee} \otimes \pi'_v$$

with respect to the second line of (10.3.2). We also write $\Pi'_v = \pi'_v \otimes \pi_v'^{\vee}$.

Let H be a reductive F -group. As above, we say that a function $f \in C_c^\infty(H(F_v))$ is F -supercuspidal if the integral of f along the F_v -points of the unipotent radical of any F -rational proper parabolic subgroup of H vanishes.

LEMMA 10.8. *If π'_v is supercuspidal, then there are matching functions*

$$f = f_1 \times f_2 \in C_c^\infty(G^\sigma \times G^\sigma(F_v)) \cong C_c^\infty(G(F_v))$$

and $\Phi \in C_c^\infty(U(F_v))$, both F -supercuspidal, such that

$$\mathrm{tr}(\Pi_{v'}(f_1^{-\tau} * f_2) \circ \tau) \neq 0.$$

Proof. Let Φ_1 be a truncated diagonal matrix coefficient of π'_v [HL, Section 1.9]. We simply let

$$\begin{aligned} f_1 &= \Phi_1^{-1} \times \mathrm{ch}_{K'_{U_v}} \\ f_2 &= \mathrm{meas}(K'_{U_v})^{-2} \mathrm{ch}_{K'_{U_v}} \times \mathrm{ch}_{K'_{U_v}} \\ \Phi &= \Phi_1^{-1} \times \mathrm{ch}_{K'_{U_v}} \end{aligned}$$

for a sufficiently small compact open subgroup $K'_{U_v} \leq U^\sigma(F_v)$ such that $\Phi_1 \in C_c^\infty(U^\sigma(F_v)/K'_{U_v})$. The functions $f_1 \times f_2$ and Φ match by Proposition 4.8. $\square$

We require an analogous lemma in the discrete series case:

LEMMA 10.9. *If π'_v is in the discrete series, then there are matching functions*

$$f = f_1 \times f_2 \in C_c^\infty(G^\sigma \times G^\sigma(F_v)) \cong C_c^\infty(G(F_v))$$

and $\Phi \in C_c^\infty(U(F_v))$, supported on the relatively τ -regular elliptic subset of $G(F_v)$ and the relatively regular elliptic subset of $U(F_v)$, respectively, such that

$$\mathrm{tr}(\Pi_{v'}(f_1^{-\tau} * f_2) \circ \tau) \neq 0.$$

Proof. Let $G^{re}(F_v) \subset G(F_v)$ (resp. $U^{re}(F_v) \subset U(F_v)$) denote the subset of relatively τ -regular elliptic elements (resp. regular elliptic elements). By definition, $G^{re}(F_v)$ is the preimage of the set

$$\{(x, y, y^{-1}, x^{-1}) \in (U^\sigma)^4(F_v) \cong G(F_v) : xy \text{ is elliptic regular}\} \subset S(F_v)$$

under the map $g \mapsto gg^{-\theta}$. Similarly, $U^{re}(F_v)$ is the preimage of the set

$$\{(x, x^{-1}) \in (U^\sigma)^2(F_v) \cong U(F_v) : x \text{ is elliptic regular}\} \subset Q(F_v)$$

under the map $g \mapsto gg^{-\sigma}$. It follows that both $G^{re}(F_v) \subset G(F_v)$ and $U^{re}(F_v) \subset U(F_v)$ are open and intersect arbitrarily small neighborhoods of the identity.

Let $\Theta_{\pi'_v}$ be the character of π'_v . Using a well-known result of Harish-Chandra, we view $\Theta_{\pi'_v}$ as a locally constant function on the elliptic regular set of $U^\sigma(F_v)$

and as a locally integrable function on all of $U^\sigma(F_v)$. Since π'_v is in the discrete series, there is a regular elliptic $\gamma_0 \in U^\sigma(F_v)$ such that $\Theta_{\pi'_v}(\gamma_0) \neq 0$ [Ro1, Proposition 5.5]. Note $(\gamma_0, 1, 1, 1) \in U^\sigma(F_v)^4 \cong G(F_v)$ is relatively τ -elliptic regular. By the openness statement above, we can and do choose a sufficiently small compact open subgroup $K'_{U_v} \leq U^\sigma(F_v)$ and a function $\Phi_1 \in C_c^\infty(U^\sigma(F_v))$ supported in the elliptic regular set of $U^\sigma(F_v)$ such that

$$f_1 \times f_2 = (\Phi_1^{-1} \times \text{ch}_{K'_{U_v}}) \times \text{ch}_{K'_{U_v}} \times \text{ch}_{K'_{U_v}}$$

is supported in $G^{re}(F_v)$ and $\text{tr}(\pi'_v(\Phi_1)) \neq 0$. Here we are using the fact that $\Theta_{\pi'_v}$ is a locally constant function on the elliptic regular set of $U^\sigma(F_v)$. By Proposition 4.8 the functions $f_1 \times f_2$ and $\Phi := \Phi_1^{-1} \times \text{ch}_{K'_{U_v}}$ match, and hence $f_1 \times f_2$ and $\Phi := \Phi_1^{-1} \times \text{ch}_{K'_{U_v}}$ satisfy the requirements of the lemma. $\square$

10.4. Proof of Theorem 10.3: separating Hecke characters. In this section we use our trace formula identity Proposition 9.2 together with an adaptation of the argument of [JL, Section 3] (see also [Hak1, Section 13]) to reduce the proof of Theorem 10.3 to a nonvanishing statement that will be proved in the following subsection. We assume the hypotheses of Theorem 10.3.

By assumption, π has nonzero cohomology with coefficients in V . In view of Proposition 10.6 this implies that there is a representation V_1 of $U_{F_\infty}^\sigma$ such that $V \cong V_1 \boxtimes V_1^\vee$. Let $K_\infty \leq G(F_\infty)$ (resp. $K_{U_\infty} \leq U(F_\infty)$) be a maximal compact subgroup. We claim that we can choose $f_\infty \in G(F_\infty)$ and $\Phi_\infty \in C_c^\infty(U(F_\infty))$ that match, are K_∞ and K_{U_∞} -finite, respectively, and satisfy the hypotheses (1)–(2) of Proposition 8.1 for the representations V_1 . To see this we recall that Lefschetz functions are finite under the left and right action of the relevant maximal compact subgroup [BLS, Proposition 8.4]. In particular for each irreducible admissible representation Π'_∞ of $G^\sigma(F_\infty)$ the operator $\Pi'(f_{L, \text{Res}_{M_\infty/F_\infty}} V_1, \tau)$ is a finite rank operator. Thus it follows from the Jacobson density theorem that we can choose $K_\infty^\sigma := K_\infty \cap G^\sigma(F_\infty)$ -finite $f_1, f_2 \in C_c^\infty(G^\sigma(F_\infty))$ such that $f_1^{-\tau} * f_2 = f_{L, \text{Res}_{M_\infty/F_\infty}} V_1, \tau$. A similar argument with $G(F_\infty)$ replaced by $U(F_\infty)$ together with Proposition 8.1 implies our claim. We henceforth assume Φ_∞ and f_∞ satisfying the conclusion of our claim.

Now let

$$\begin{aligned} \Phi &= \Phi_\infty \otimes \Phi^\infty \in C_c^\infty(U(\mathbb{A}_F)) \\ f &= f_\infty \otimes f^\infty \in C_c^\infty(G(\mathbb{A}_F)) \end{aligned}$$

be factorable functions satisfying the hypotheses of Proposition 9.2. Such functions exist by Lemma 4.5, Proposition 4.8, Theorem 5.1, Lemma 10.8 and Lemma 10.9. Thus by Proposition 9.2 we have

$$(10.4.1) \quad \sum_{\pi'} \text{RT}(\pi'(\Phi^1)) = 2 \sum_{\Pi'} \text{TRT}(\Pi'(f^1)).$$

Here the sum on the left is over equivalence classes of cuspidal automorphic representations of ${}^1U(\mathbb{A}_F) = U(\mathbb{A}_F)$ and the sum on the right is over equivalence classes of cuspidal automorphic representations of ${}^1G(\mathbb{A}_F)$. Let S be a finite set of places containing all infinite places and all places where π is ramified. Choose compact open subgroups $K_U^S \leq U(\mathbb{A}_F^S)$, $K^S \leq G(\mathbb{A}_F^S)$ such that $K_{U_v}^S$ and K_v^S are hyperspecial for all $v \notin S$ and π^S (resp. Π^S) contains the unit representation of K_U^S (resp. K^S). We assume that $f^S \in C_c^\infty(G(\mathbb{A}_F^S)/K^S)$ and $\Phi^S \in C_c^\infty(U(\mathbb{A}_F^S)/K_U^S)$; then (10.4.1) implies the following identity:

$$(10.4.2) \quad \sum_{\pi'} \mathrm{tr}(\pi'(\Phi^S)) \mathrm{RT}(\pi'(\Phi_S^1 \mathrm{ch}_{K_U^S})) = 2 \sum_{\Pi'} \mathrm{tr}(\Pi'(f^S)) \mathrm{TRT}(\Pi'(f_S^1 \mathrm{ch}_{K^S}))$$

(compare [JL, Section 3(2)]). The reason (10.4.1) simplifies to (10.4.2) is simply that if π'_v (resp. Π'_v) is unramified then it contains a unique fixed vector under the hyperspecial subgroup K_{U_v} (resp. K_v).

We have the following lemma:

LEMMA 10.10. *Any Π' contributing a nonzero summand to the right of (10.4.2) has nonzero cohomology with coefficients in $\mathrm{Res}_{M_\infty/F_\infty} V$. Any π' contributing a nonzero summand to the left of (10.4.2) has nonzero cohomology with coefficients in V .*

Proof. Choose an isomorphism $G(F_\infty) \cong G^\sigma(F_\infty) \times G^\sigma(F_\infty)$ equivariant with respect to τ and intertwining σ with $(x, y) \mapsto (y, x)$. Write $\Pi'_\infty = \Pi_1 \otimes \Pi_2$ for some representations Π_i of $G^\sigma(F_\infty)$ using this isomorphism. For a fixed $f_S^\infty \in C_c^\infty(G(F_S^\infty))$, consider the linear forms

$$\begin{aligned} \ell_i : \mathcal{H}_{\Pi_\infty} &\longrightarrow \mathbb{C} \\ (\Pi_1(f_1), \Pi_2(f_2)) &\longmapsto \mathrm{tr}(\Pi_{v'}(f_1^{-\tau} * f_2) \circ \tau) \\ (\Pi_1(f_1), \Pi_2(f_2)) &\longmapsto \sum_{\phi \in \mathcal{B}(\Pi')^{K_S}} \mathcal{P}_{G^\sigma}(\Pi_1 \times \Pi_2((f_1 \times f_2)^1) \Pi'(f_S^\infty) \phi) \overline{\mathcal{P}_{G^\sigma}(\phi)}. \end{aligned}$$

They are both $(G^\sigma(F_\infty), G^\theta(F_\infty))$ -invariant, and hence equal up to a constant multiple (possibly zero) by Lemma 10.7. Thus the first assertion of the lemma follows from the defining property of Lefschetz functions [BLS, Proposition 8.4] and Lemma 10.5. The proof of the second assertion is similar; one uses Proposition 10.6 instead of Lemma 10.5. $\square$

By the lemma, the collection of Π' on the right of (10.4.2) is finite in a sense independent of $f^S \in C_c^\infty(G(\mathbb{A}_F^S)/K^S)$ for fixed f_S by our assumption on f_∞ . Indeed, the sum can be thought of as being over automorphic representations contributing to the cohomology of a locally symmetric space depending only on f_S with coefficients in a fixed local system depending only on f_∞ by Lemma 10.10. Using the supply of matching f^S and Φ^S provided by Corollary 4.6 and Corollary

4.9 we separate strings of Hecke eigenvalues “outside S ” in (10.4.2) to arrive at the following refined identity:

$$(10.4.3) \quad \sum_{\substack{\pi' \\ \pi'^S \cong \pi^S}} \text{tr}(\pi'(\Phi^S)) \text{RT}(\pi'(\Phi_S^1 \text{ch}_{K_U^S})) = 2 \text{tr}(\Pi(f^S)) \text{TRT}(\Pi(f_S^1 \text{ch}_{K^S})).$$

Here we are using Lemma 10.5 and strong multiplicity one for $G(\mathbb{A}_F)$ to isolate the contribution of Π on the right hand side. We are also using the fact that when ME/F , U , and G are unramified at a finite place v then the base change map from irreducible admissible unramified representations of $U(F_v)$ to irreducible admissible unramified representations of $G(F_v)$ is injective [M, Corollary 4.2].

Let $S(ME)$ be the set of finite places of F such that ME/F is ramified and both E/F and M/F are nonsplit. We let $S_0 = \infty \cup \{v_1, v_2\}$ and $S_0(ME) = S(ME) \cup S_0$. Enlarging S if necessary, we assume that $S_0(ME) \subseteq S$. To complete the proof of the theorem, we show that the left side of (10.4.3) is nonzero for some $f \in C_c^\infty(G(\mathbb{A}_F))$ satisfying the various conditions we have placed earlier. In view of Theorem 5.1, it suffices to show that upon enlarging S if necessary we can choose f_S^∞ so that

$$\text{TRT}(\Pi(f_S^1 \text{ch}_{K^S})) \neq 0$$

where $f_{S(ME)}$ is supported on relatively τ -regular semisimple elements admitting norms, f_{v_1} is F -supercuspidal and matches an F -supercuspidal Φ_{v_1} and f_{v_2} is supported on relatively τ -regular elliptic semisimple elements and matches a function Φ_{v_2} supported on relatively regular elliptic semisimple elements. This is done in the following subsection.

10.5. Proof of Theorem 10.3: nonvanishing. We assume all of the notation and conventions of the previous section and the hypotheses of Theorem 10.3. In particular, Π is both G^σ and G^θ -distinguished. We prove the following proposition:

PROPOSITION 10.11. *With notation as in Section 10.4, upon possibly enlarging S we can choose a function $f_S^\infty \in C_c^\infty(G(F_S^\infty))$ so that*

$$\text{TRT}(\Pi(f_S^1 \text{ch}_{K^S})) \neq 0$$

where $K^S \leq G(\mathbb{A}_F^S)$ is a hyperspecial subgroup, $f_{S(ME)}$ is supported on relatively τ -regular semisimple elements admitting norms, f_{v_1} is F -supercuspidal and matches an F -supercuspidal Φ_{v_1} , and f_{v_2} is supported on relatively τ -regular elliptic semisimple elements and matches a function $\Phi_{v_2} \in C_c^\infty(U(F_{v_2}))$ supported on relatively regular elliptic semisimple elements.

This proposition completes the proof of Theorem 10.3 as noted at the end of the previous subsection.

Proof. We claim that we can choose a pure tensor $\phi_0 \in V_\Pi$ such that the restriction of the two forms

$$\mathcal{P}_{G^\sigma}(\cdot) : V_\Pi \longrightarrow \mathbb{C} \quad \text{and} \quad \mathcal{P}_{G^\theta}(\cdot) : V_\Pi \longrightarrow \mathbb{C}$$

to $\phi_0^S \otimes V_{\Pi,S}$ are nonzero. Indeed, by assumption, there are vectors $\phi_1, \phi_2 \in V_\Pi$ such that $\mathcal{P}_{G^\sigma}(\phi_1) \neq 0$ and $\mathcal{P}_{G^\theta}(\phi_2) \neq 0$. If either $\mathcal{P}_{G^\sigma}(\phi_2) \neq 0$ or $\mathcal{P}_{G^\theta}(\phi_1) \neq 0$ then we are done, otherwise $\mathcal{P}_{G^\sigma}(\phi_1 + \phi_2)\mathcal{P}_{G^\theta}(\phi_1 + \phi_2) \neq 0$. Thus there is a vector $\phi_0 \in V_\Pi$ such that $\mathcal{P}_{G^\sigma}(\phi_0)\mathcal{P}_{G^\theta}(\phi_0) \neq 0$. We may assume that ϕ_0 is a pure tensor, which implies the claim.

Enlarging S if necessary, we choose a function $f_S^{S_0(ME)} \in C_c^\infty(G(F_S^{S_0(ME)}))$ such that $\Pi(f_S^{S_0(ME)})$ is the orthogonal projection onto $\phi_S^{S_0(ME)}$; this is possible by the Jacobson density theorem and the matching statements Lemma 4.5 and Proposition 4.8.

With this choice, for an orthonormal basis $\mathcal{B}(\Pi)$ of the Π -isotypic subspace of $L_0^2(G(F) \backslash {}^1G(\mathbb{A}_F))$ we have

$$(10.5.1) \quad \begin{aligned} & \sum_{\phi \in \mathcal{B}(\Pi)^{K^S}} \mathcal{P}_{G^\sigma}(\Pi'(f^1)\phi) \overline{\mathcal{P}_{G^\theta}(\phi)} \\ &= \text{tr}(\Pi(f^S)) \sum_{a_i} \mathcal{P}_{G^\sigma}(\phi_0^{S_0(ME)} \otimes \Pi'(f_{S_0(ME)}^1)a_i) \overline{\mathcal{P}_{G^\theta}(\phi_0^{S_0(ME)} \otimes a_i)} \end{aligned}$$

where the sum is over an orthonormal basis $\mathcal{B}(\Pi)$ of $V_{\Pi_{S_0(ME)}}$ with respect to the Hermitian pairing

$$(\cdot, \cdot) : V_{\Pi_{S_0(ME)}} \times V_{\Pi_{S_0(ME)}} \longrightarrow \mathbb{C}$$

given by

$$\begin{aligned} & (\psi_1, \psi_2) \\ & \longmapsto \int_{G(F) \backslash {}^1G(\mathbb{A}_F)} \phi_0(g^{S_0(ME)}) \otimes \psi_1(g_{S_0(ME)}) \overline{\phi_0(g^{S_0(ME)}) \otimes \psi_2(g_{S_0(ME)})} dg d\bar{g}, \end{aligned}$$

where dg is the Tamagawa measure. Note that only finitely many of the a_i terms will have a nonzero contribution to (10.5.1).

For the moment let f_{S_0} be chosen so that $\Pi_{S_0}(f_{S_0})$ is the projection to the space spanned by ϕ_{0,S_0} . Consider the linear functional

$$(10.5.2) \quad \begin{aligned} & \Theta : C_c^\infty(G(F_{S(ME)})) \longrightarrow \mathbb{C} \\ & f_{S(ME)} \longmapsto \sum_{a_i} \mathcal{P}_{G^\sigma}(\phi_0^{S_0} \otimes \Pi(f_{S_0(ME)}^1)a_i) \overline{\mathcal{P}_{G^\theta}(\phi_0^{S_0} \otimes a_i)}. \end{aligned}$$

For each $v \in S(ME)$ this defines a spherical matrix coefficient of Π_v in the sense of Section 6. It is nonzero because we can choose $f_{S(ME)} \in C_c^\infty(G(F_{S(ME)}))$ such that $\Pi(f_{S(ME)})$ is the projection onto the space spanned by $\phi_{0S(ME)}$. By assumption, Π_v is relatively τ -regular for $v \in S(ME)$, so we can and do choose $f_{S(ME)} \in C_c^\infty(G(F_{S(ME)}))$ that is supported in the set of τ -regular semisimple elements in $G(F_{S(ME)})$ that admit norms and such that $\Theta(f_{S(ME)}) \neq 0$. This $f_{S(ME)}$ admits a matching function $\Phi_{S(ME)} \in C_c^\infty(H(F_{S(ME)}))$ by Theorem 5.1 above.

We are left with choosing f_{S_0} . Fix an isomorphism

$$(10.5.3) \quad G(F_{S_0}) \xrightarrow{\sim} G^\sigma(F_{S_0}) \times G^\sigma(F_{S_0})$$

intertwining σ with $(x, y) \mapsto (y, x)$ and τ with $(x, y) \mapsto (x^\tau, y^\tau)$. By Lemma 10.5 we can and do factor

$$\Pi_{0S_0} \cong \Pi_{1S_0} \otimes \Pi_{1S_0}^\vee$$

with respect to (10.5.3) for some irreducible admissible representation Π_{1S_0} of $G^\sigma(F_{S_0})$. The map

$$\begin{aligned} \ell_1 : \mathcal{H}_{\Pi_{0S_0}} &\longrightarrow \mathbb{C} \\ \Pi_{1v}(f_1) \otimes \Pi_{1v}^\vee(f_2) &\longmapsto \prod_{v \in S_0} \text{tr}(\Pi_{1v}(f_{1v} * f_{2v}^{-\tau})) \end{aligned}$$

is $(G^\sigma(F_{S_0}), G^\theta(F_{S_0}))$ -invariant in the sense of Section 10.3, and clearly not identically zero.

Notice the linear functional

$$\begin{aligned} \ell_2 : \mathcal{H}_{\Pi_{S_0}} &\longrightarrow \mathbb{C} \\ \Pi(f_{S_0}) &\longmapsto \sum_{a_i} \mathcal{P}_{G^\sigma}(\phi_0^{S_0} \otimes \Pi(f_{S_0(EF)}^1) a_i) \mathcal{P}_{G^\theta}(\overline{\phi_0^{S_0}} \otimes a_i) \end{aligned}$$

is also $(G^\sigma(F_{S_0}), G^\theta(F_{S_0}))$ -invariant. By Lemma 10.7 we have that

$$\ell_2 = c\ell_1$$

for some constant $c \in \mathbb{C}$. By our choice of $f_{S(ME)}$ above the linear functional ℓ_2 is not identically zero and hence we have that $c \neq 0$.

Choosing a factorable function

$$f_{S_0}^\infty = f_1 \times f_2 \in C_c^\infty(G^\sigma \times G^\sigma(F_{S_0}^\infty)),$$

we have

$$(10.5.4) \quad \ell_2(f_{S_0}) = c \left(\text{tr}(\Pi_{1\infty}(f_L, \text{Res}_{M_\infty/F_\infty} V_1, \tau)) \right) \left(\prod_{v \in S_0 \setminus \infty} \text{tr}(\Pi_{1v}(f_{1v} * f_{2v}^{-\tau})) \right)$$

with V_1 as in Section 10.4. We now show that we can choose a test function $f = f_1 \times f_2$ satisfying the conditions at ∞ , v_1 , and v_2 stipulated in the statement of the proposition such that (10.5.4) is nonzero; this will complete the proof of the proposition.

We work place by place. The factor corresponding to the infinite places is nonzero by [La2, Lemme 4.2]. Lemma 10.8 takes care of $v = v_1$, and Lemma 10.9 takes care of $v = v_2$. $\square$

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