ANALYTIC APPROXIMATIONS IN THE STUDY OF THE SOLAR MODULATION OF ELECTRONS

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Numerical solutions to the transport equation of galactic cosmic rays in the interplanetary medium have been used to investigate the applicability of commonly used approximate analytic solutions for electrons (10 MeV to 10 GeV). We find that for a given cosmic-ray diffusion coefficient: (a) the "force-field" approximation is in reasonable agreement with the numerical solution at energies $\gtrsim$ 200 MeV, but deviates significantly at lower energies, depending on the shape of the interstellar electron spectrum, (b) the "diffusion-convection" approximation agrees generally with the numerical solution within a factor of ~ 2 over the entire energy range.

1. Introduction. The transport of galactic cosmic-ray particles in the interplanetary medium has often been described in terms of the spherically symmetric transport equation:

$$\frac{V}{r^2}\frac{\partial}{\partial r}(r^2 U) - \frac{2V}{3r}\frac{\partial}{\partial T}(\alpha T U) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \kappa \frac{\partial U}{\partial r}\right) = 0 \qquad (1)$$

(see recent review, Jokipii, 1971), where V is the solar-wind velocity, assumed independent of heliocentric radius, r, U is the cosmic-ray particle density, T is particle kinetic energy, κ is the radial diffusion coefficient, and

$$\alpha(T) = \frac{T+2m}{T+m} \qquad (2)$$

where m is the particle rest energy. The long-term variations of the parameters of equation (1), particularly the diffusion coefficient, κ, are responsible for the observed solar modulation of galactic cosmic-ray intensities.

No general analytic solutions to equation (1) have been found. Several authors have made approximations to the equation which lead to analytic solutions. In the case of cosmic-ray electrons, these approximate solutions have frequently been used to derive information on the interstellar electron spectrum and on the interplanetary cosmic-ray diffusion coefficient.

In order to determine the validity of the analytic solutions for cosmic-ray electron spectra, we have investigated the most commonly used analytic approximations, i.e., the "force-field" and the "diffusion-convection" solutions, for electrons in the energy range 10 MeV - 10 GeV, by comparing them to numerical solutions of the complete transport equation.

2. Solutions of the Transport Equation.

a) The "Force-Field" (FF) Approximation. Gleeson and Axford (1967, 1968) have derived an approximate solution for the case of small modulation by making use of the radial differential current density (or streaming), S, defined as:

$$S = VU - \kappa\frac{\partial U}{\partial r} - \frac{V}{3}\frac{\partial}{\partial T}(\alpha TU) \tag{3}$$

The term $\kappa\frac{\partial U}{\partial r}$ represents the contribution from diffusion. The remaining two terms represent the effective radial current due to the transformation between a frame of reference at rest with respect to the solar wind and the observer's reference frame (the Compton-Getting effect).

Gleeson and Axford present arguments to show that S is negligible whenever $VL/\kappa \ll 1$, where L is a length characteristic of the radial variation of the diffusion coefficient. If one assumes $S = 0$ and that κ is a separable function of radius and energy, one obtains the so-called "force field" (FF) solution (Gleeson and Axford, 1968)

$$\frac{j(r,W)}{W^2 - m^2} = \frac{j(D,W+\Phi)}{(W+\Phi)^2 - m^2} \tag{4}$$

where $j = \beta cU/4\pi$ is the particle intensity, W is the total energy of the particle, m is its rest energy, βc is the particle velocity, and Φ is a spectral shift parameter which is determined from the diffusion coefficient.

b) The "Diffusion-Convection" (DC) Approximation. An even simpler approximation to equation (1) is obtained by neglecting the adiabatic deceleration term, i.e., $(2V/3r)\,\partial/\partial T(\alpha TU)$, which results in the "diffusion-convection" (DC) approximation:

$$\frac{\partial}{\partial r}\left(r^2 VU - r^2 \kappa\frac{\partial U}{\partial r}\right) = 0$$

In the absence of sources or sinks at the origin, this equation may be written:

$$VU = \kappa\frac{\partial U}{\partial r} \tag{5}$$

which is a statement of the balance between the outward current of particles due to convection and an inward current due to diffusion. If we assume a modulation boundary at heliocentric radius D beyond which V/κ is zero, the solution to the DC equation is:

$$U(r,T) = U(D,T)\ e^{-\psi(r,T)} \tag{6}$$

where the quantity ψ, defined by:

$$\psi(r,T) = \int_r^D \frac{V}{\kappa(r',T)}\, dr' \tag{7}$$

is called the "modulation parameter."

It is interesting to note that if the near-Earth and interstellar electron spectra are known, the modulation parameter at 1 AU, $\psi(1,T)$, is determined in the DC approximation from equation (6), i.e.,

$$\psi(1,T) = \ell n \left[\frac{U(D,T)}{U(1,T)} \right] \tag{8}$$

if the radial and energy dependences of the diffusion coefficient are separable, i.e. $\kappa(r,T) = \kappa_1(r)\ \kappa_2(T)$, the energy dependence of κ is determined from $\psi(1,T)$ (see equation (7)).

(c) The Numerical (FN) Solution. A numerical solution to the transport equation (1) has advantages over the analytic approximations. For example, numerical solutions can be readily obtained for any specified radial and energy dependence of κ, whereas the analytic approximations are often restricted to certain functional forms of the diffusion coefficient. In addition, the numerical solution can be obtained for all values of radius and energy of interest; these solutions can then be used to test the validity of the analytic approximations.

We have constructed a numerical solution to equation (1) based on the Crank-Nicholson technique, similar to the method of Fisk (1971).

3. Comparison of Solutions. In order to determine the region of applicability of the FF and DC approximations, we show in Figure 1, as an example, a comparison of these solutions with the numerical solution (FN) of the full transport equation. For all three models we used the same diffusion coefficient, assumed independent of radius, with a hypothetical boundary at 3 AU, and with the rigidity dependence:

$$\kappa(R)\ (\mathrm{cm}^2/\mathrm{sec}) = \begin{cases} 7.15 \times 10^{17}\ \beta R & R > R_c = 300\ \mathrm{MV} \\ 7.15 \times 10^{17}\ \beta R_c & R < R_c \end{cases}$$

where R is magnetic rigidity. This particular diffusion coefficient and the interstellar spectrum shown were used with the FF approximation by Meyer et al., (1971) in the analysis of their 1968 electron observations. It is evident that below ~ 200 MeV the FF result diverges significantly from the full numerical solution. It should be noted that adjustments of the diffusion coefficient and/or the assumed form of the interstellar electron spectrum, in order to achieve agreement between the FF solution and the near-Earth observations, does not resolve the basic conflict, i.e., the disagreement between the FF and FN solutions below ~ 200 MeV. It is difficult to predict under which circumstances the force-field solution is a reasonable approximation at low energies. We find that the numerical and force-field solutions are more consistent if the interstellar electron spectrum is flatter than $T^{-2.5}$ below a few hundred MeV. To illustrate this improvement we show in Figure 2 a similar comparison of solutions as in Figure 1 except that we have used the interstellar positron spectrum calculated by Ramaty and Lingenfelter (R&L) (1968). This galactic spectrum flattens out gradually below ~ 1 GeV and eventually turns over below ~ 50 MeV. Both the DC and the FF solutions are within a factor of ~ 2 of the full numerical solution over most of the energy range from 10 MeV to 10 GeV. The region of validity of the FF solution probably also depends on the rigidity dependence of the diffusion coefficient (Urch and Gleeson, 1972).

We find the DC solution to be a fairly good approximation for a wide range of interstellar spectra, boundary locations, and diffusion coefficients (see Figures 1 and 2). The DC solution differs from the FN solution by its omission of the adiabatic deceleration term. The effect of this omission is clearly discernable in Figures 1 and 2, and can be approximately described by an energy shift of ~ 25%. This energy loss is much smaller than in the case of low-energy nuclei because of the electrons' much higher velocities.

The DC solution, therefore, represents a useful tool for a zeroth order estimate of

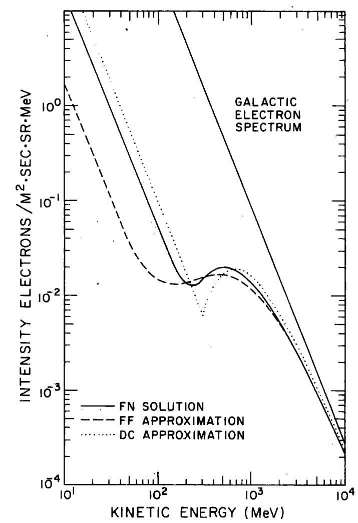

Fig. 1. Comparison of the 1 AU electron spectra derived from the FF, DC and FN solutions for a power-law type interstellar electron spectrum.

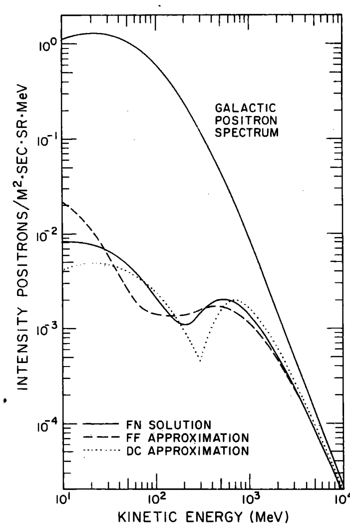

Fig. 2. Comparison of the 1 AU positron spectra derived from the FF, DC, and FN solutions for the R&L interstellar positron spectrum.

the rigidity dependence $\kappa(R)$ of the cosmic-ray diffusion coefficient. Assuming κ to be a separable function of radius r and energy T, we can use equations (7) and (8) together with data on the electron spectrum near 1 AU and in interstellar space (see Cummings et al., 1973) to derive $\kappa(T)$. This diffusion coefficient, with minor corrections for adiabatic-deceleration effects, may be used as an input parameter for iteration in the FN solution.

4. Summary. We have used numerical solutions of the full transport equation describing cosmic-ray propagation in the interplanetary medium to discuss the applicability of two analytic approximations to the equation for galactic electrons in the 10 MeV - 10 GeV energy range. We have found that:

a) The "force-field" approximation is in reasonable agreement with the numerical solution at energies $\gtrsim$ 200 MeV, but deviates significantly at lower energies, depending on the shape of the interstellar electron spectrum.

b) The "diffusion-convection" approximation yields a reasonable first-order solution of the transport equation for a wide range of interstellar spectra and of diffusion coefficients. Given the interstellar electron spectrum, e.g., derived from the galactic non-thermal radio background, and near-Earth electron spectra, the diffusion-convection approximation may be used to estimate the rigidity dependence of the interplanetary cosmic-ray diffusion coefficient.

5. References.

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