The gravitational wave strain in the characteristic formalism of numerical relativity

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The extraction of the gravitational wave signal, within the context of a characteristic numerical evolution is revisited. A formula for the gravitational wave strain is developed and tested, and is made publicly available as part of the PITT code within the Einstein Toolkit. Using the new strain formula, we show that artificial non-linear drifts inherent in time integrated waveforms can be reduced for the case of a binary black hole merger configuration. For the test case of a rapidly spinning stellar core collapse model, however, we find that the drift must have different roots.

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I. INTRODUCTION

The characteristic approach to numerical relativity has been developed over a number of years (see, for example, the review [1]), and the method is often used, as characteristic extraction [2–5], in the computation of waveforms from astrophysical events (see e.g., [6–9]). The origin of the approach is the work of Bondi *et al.* [10] in which coordinates particularly suitable for the description of gravitational radiation were introduced, and the corresponding Einstein field equations were derived. It was also shown that it is always possible to impose additional conditions on the coordinates, now called the Bondi gauge, so as to greatly simplify the asymptotic form of the metric. In this gauge, the description of gravitational radiation in terms of the metric, is very simple. In addition, Bondi gauge is the coordinate frame in which a gravitational wave detector would measure gravitational radiation.

One difficulty faced by characteristic numerical relativity is that, in general, it is not possible to impose the Bondi gauge conditions since they are asymptotic conditions on the coordinates. In practice, the gauge is fixed in the interior of the spacetime. For example, in characteristic extraction [1, 3, 5] the coordinates and therefore the gauge are induced by the gauge driver of the 3+1 Cauchy evolution system at the extraction worldtube. It is non-trivial to derive an expression from metric data in a general gauge that describes the gravitational radiation field as measured by an asymptotic inertial observer. This was done for the gravitational news $\mathcal{N}$ at the same time as the development of the first 3D characteristic code, now known as the PITT code [11], and it was only quite recently, in 2009 [12], that a formula for ψ_4 (which is commonly used to describe gravitational radiation in numerical relativity) was developed. It should also be mentioned that there is an alternative approach to calculating $\mathcal{N}$ based on an explicit construction of the coordinate transformation between the Bondi and general gauges [13], but the method has not been further developed or implemented.

While formulas have been developed for $\mathcal{N}$ and ψ_4 within characteristic numerical relativity, there is no specific formula in the literature for the gravitational wave strain $(h_+, h_\times)$, which is what would be measured by a gravitational wave detector. Instead, the strain is found by time integration of $\mathcal{N}$ or ψ_4, and it appears that this process introduces aliasing or other numerical effects so reducing the accuracy of the result [14–17].

This work re-investigates gravitational radiation within the characteristic formalism, and derives a formula for the wave strain. In addition, alternative formulas for some intermediate variables are found that avoid time integration. In this way it is possible to reduce, but not to eliminate, the number of time integrals used in the computation of gravitational radiation descriptors.

Sec. II summarizes previous relevant work, and then Sec. III introduces and evaluates the properties of the coordinate transformation between the Bondi and general gauges. Sec. IV derives a formula for the gravitational wave strain, and Sec. IV C finds an explicit formula (rather than an evolution equation) for the phase of $\mathcal{N}$ and ψ_4. Numerical tests, and other approaches to validation, are discussed in Sec. V. Sec. VI is the Conclusion. This work makes extensive use of computer algebra scripts; details are given in an Appendix, and the scripts are available in the online supplement.

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II. REVIEW OF PREVIOUS WORK

A. The characteristic formalism

The formalism for expressing Einstein's equations as an evolution system based on characteristic, or null-cone, coordinates is based on work originally due to Bondi *et al.* [10, 18] for axisymmetry, and extended to the general case by Sachs [19]. The formalism is covered in the review by Winicour [1]. We start with coordinates based upon a family of outgoing null hypersurfaces. Let u label these hypersurfaces, x^A $(A = 2, 3)$ label the null rays, and r be a surface area coordinate. In the resulting $x^\alpha = (u, r, x^A)$ coordinates, the metric takes the Bondi-Sachs form

$$ds^2 = -\left(e^{2\beta}(1 + W_c r) - r^2 h_{AB} U^A U^B\right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \tag{1}$$

where $h^{AB} h_{BC} = \delta^A_C$ and $\det(h_{AB}) = \det(q_{AB})$, with q_{AB} a metric representing a unit 2-sphere; for the computer algebra calculations presented later it is necessary to have specific angular coordinates, and for that purpose stereographic coordinates are used with $x^A = (q, p)$ and

$$q_{AB} dx^A dx^B = \frac{4}{(1 + q^2 + p^2)^2} \left(dq^2 + dp^2\right). \tag{2}$$

W_c is a normalized variable used in the code, related to the usual Bondi-Sachs variable V by $V = r + W_c r^2$. It should be noted here that different references use various notations for what is here denoted as W_c, and in particular ref. [11] uses W with $W = r^2 W_c$. It is convenient to describe angular quantities and derivatives by means of complex numbers using the spin-weighted formalism and the eth ($\eth$) calculus [11, 20, 21]. To this end, q_{AB} is represented by a complex dyad q_A with, for example, $q_A = (1, i)2/(1 + q^2 + p^2)$ in stereographic coordinates. Then h_{AB} can be represented by its dyad component $J = h_{AB} q^A q^B / 2$. We also introduce the fields $K = \sqrt{1 + J\bar{J}}$ and $U = U^A q_A$.

We will be concerned with asymptotic properties, and so make the coordinate transformation r to $\rho = 1/r$, and consider the behaviour of quantities near $\rho = 0$. We have

$$ds^2 = \rho^{-2} \left(-\left(e^{2\beta}(\rho^2 + \rho W_c) - h_{AB} U^A U^B\right) du^2 + 2e^{2\beta} du d\rho - 2h_{AB} U^B du dx^A + h_{AB} dx^A dx^B\right). \tag{3}$$

In contravariant form,

$$g^{11} = e^{-2\beta} \rho^3 (\rho + W_c), \;\; g^{1A} = \rho^2 e^{-2\beta} U^A, g^{10} = \rho^2 e^{-2\beta}, \;\; g^{AB} = \rho^2 h^{AB}, \;\; g^{0A} = g^{00} = 0. \tag{4}$$

B. The Bondi gauge

The spacetimes considered here are asymptotically flat; but even so, in general, the metric near $\rho = 0$ may be complicated. Matters are greatly simplified by using coordinates which naturally exhibit the property of asymptotic flatness. As shown in the original work of Bondi *et al.* [10], such coordinates always exist, and have become known as the Bondi gauge. We will use $\tilde{}$ to denote a quantity in the Bondi gauge, e.g. the Bondi gauge coordinates are denoted by $(\tilde{u}, \tilde{\rho}, \tilde{x}^A)$. The Bondi gauge can be defined by conditions on the metric quantities

$$\lim_{\tilde{\rho}=0} (\tilde{J}, \tilde{U}, \tilde{\beta}, \tilde{W}_c) = 0; \tag{5}$$

since flat Minkowski spacetime has $J, U, \beta, W_c = 0$ everywhere, it is clear that the Bondi gauge gives a natural expression of asymptotic flatness. Actually, we can go further and use the asymptotic Einstein vacuum equations, that is evaluate the Ricci tensor as a power series in $\tilde{\rho}$ and keep only the leading order terms. In this way the dependence of the metric coefficients to leading order on $\tilde{\rho}$ is found to be [22, 23]

$$\tilde{J} = 0 + \mathcal{O}(\tilde{\rho}), \; \tilde{U} = 0 + \mathcal{O}(\tilde{\rho}^2), \; \tilde{\beta} = 0 + \mathcal{O}(\tilde{\rho}^2), \; \tilde{W}_c = 0 + \mathcal{O}(\tilde{\rho}^3). \tag{6}$$

It will be convenient to describe gravitational radiation with respect to an orthonormal null tetrad, so that formulas are independent of specific coordinates. The normalization adopted is

$$\ell^a \ell_a = n^a n_a = m^a m_a = \ell^a m_a = n^a m_a = 0, \ell^a n_a = m^a \bar{m}_a = 1, \tag{7}$$

which is consistent with common practice in numerical relativity, although Refs. [1, 11, 12] normalize to 2 rather than to 1. In the Bondi gauge, the leading order terms of the natural null tetrad are

$$\tilde{\ell}^\alpha = \left(0, -\frac{\tilde{\rho}^2}{\sqrt{2}}, 0, 0\right), \; \tilde{n}^\alpha = \left(\sqrt{2}, \frac{\tilde{\rho}^2}{\sqrt{2}}, 0, 0\right), \; \tilde{m}^\alpha = \left(0, 0, \frac{\tilde{\rho}\tilde{q}^A}{\sqrt{2}}\right). \tag{8}$$

III. COORDINATE TRANSFORMATION BETWEEN THE BONDI AND GENERAL GAUGES

We construct quantities in the general gauge by means of a coordinate transformation to the Bondi gauge. The form of the transformation has been used previously [13], and is written as a series expansion in ρ with coefficients arbitrary functions of the other coordinates. Thus it is a general transformation, and the requirements that g^{ab} must be of Bondi-Sachs form, and that $\tilde{g}^{ab}$ must be in the Bondi gauge, imposes conditions on the transformation coefficients. The transformation is

$$u \to \tilde{u} = u + u_0 + \rho A^u, \rho \to \tilde{\rho} = \rho\omega + \rho^2 A^\rho, x^A \to \tilde{x}^A = x^A + x_0^A + \rho A^A, \tag{9}$$

where $u_0, A^u, \omega, A^\rho, x_0^A, A^A$ are all functions of u and x^A only. We obtain

$$\tilde{g}^{ab} = \frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial \tilde{x}^b}{\partial x^d} g^{cd} \text{ and } g_{ab} = \frac{\partial \tilde{x}^c}{\partial x^a}\frac{\partial \tilde{x}^d}{\partial x^b}\tilde{g}_{cd}. \tag{10}$$

We find the following formulas (many of which have been given previously [11, 13])

$$\tilde{\rho}^2 + \mathcal{O}(\tilde{\rho}^3) = \tilde{g}^{01}, \;\text{ thus } \tilde{\rho}^2 = \omega e^{-2\beta}(1 + u_{0,u} + U^A u_{0,A})\rho^2, \;\text{ so that } (\partial_u + U^B\partial_B)u_0 = \omega e^{2\beta} - 1, \tag{11}$$

$$\mathcal{O}(\tilde{\rho}^3) = \tilde{g}^{A1}, \;\text{ thus } 0 = \omega e^{-2\beta}(x^A_{0,u} + U^A + U^B x^A_{0,B}) \text{ so that } (\partial_u + U^B\partial_B)x_0^A = -U^A, \tag{12}$$

$$\mathcal{O}(\tilde{\rho}^4) = \tilde{g}^{11}, \;\text{ thus } 0 = \omega e^{-2\beta}(2\omega_{,u} + 2U^A\omega_{,A} + \omega W_c) \text{ so that } (\partial_u + U^B\partial_B)\omega = -\omega W_c/2, \tag{13}$$

$$0 = \tilde{g}^{00} \;\text{ so that to } \mathcal{O}(\tilde{\rho}^2), \;\; 2\omega A^u = \frac{J\bar{\eth}^2 u_0 + \bar{J}\eth^2 u_0}{2} - K\eth u_0 \bar{\eth} u_0. \tag{14}$$

In the next equations, $X_0 = q_A x_0^A, A = q_A A^A$; the introduction of these quantities is a convenience to reduce the number of terms in the formulas, since x_0^A, A^A do not transform as 2-vectors. As a result, the quantity $\zeta = q + ip$ also appears, and the formulas are specific to stereographic coordinates.

$$\begin{aligned} 0 &= \tilde{q}_A \tilde{g}^{0A} \quad \text{so that to } \mathcal{O}(\tilde{\rho}^2), \\ 0 &= 2A\omega + 2A_u X_0 U\bar{\zeta}e^{-2\beta} + K\eth u_0(2 + \bar{\eth}X_0 + 2X_0\bar{\zeta}) + K\bar{\eth}u_0\eth X_0 - J\bar{\eth}u_0(2 + \bar{\eth}X_0 + 2X_0\bar{\zeta}) - \bar{J}\eth u_0 \eth X_0\,. \end{aligned} \tag{15}$$

The following formulas were derived using the transformation of the covariant metric from Bondi to general gauge

$$\det(q_{AB}) = \det(g_{AB})\rho^4 \;\text{ so that } \omega = \frac{1+q^2+p^2}{1+\tilde{q}^2+\tilde{p}^2}\sqrt{1 + q_{0,q} + p_{0,p} + q_{0,q}p_{0,p} - q_{0,p}p_{0,q}}\,, \tag{16}$$

$$J = \frac{q^A q^B g_{AB}}{2}\rho^2 \text{ so that } J = \frac{(1+q^2+p^2)^2}{2(1+\tilde{q}^2+\tilde{p}^2)^2\omega^2}\eth X_0(2 + \eth\bar{X}_0 + 2\bar{X}\zeta). \tag{17}$$

IV. FORMULAS FOR GRAVITATIONAL RADIATION

Formulas for ψ_4 and $\mathcal{N}$ have been derived in earlier work [11, 12], and take the form

$$\mathcal{N} = \frac{e^{-2i\delta}F^a F^b N_{ab}}{2\omega^2}, \qquad \lim_{r\to\infty} r\psi_4 = \lim_{\rho\to 0}\frac{e^{2i\delta}\rho C_{abcd}n^a\bar{F}^b n^c \bar{F}^d}{\omega^3}, \tag{18}$$

where F^a is related to the null tetrad vector m^a by

$$m^a = e^{-i\delta(u,x^A)}\rho F^a \;\text{ with }\; F^a = \left(0, 0, \frac{q^A\sqrt{K+1}}{2} - \frac{\bar{q}^A J}{2\sqrt{(K+1)}}\right). \tag{19}$$

The Weyl tensor C_{abcd}, the null tetrad vector n^a and the quantity N_{ab} are not specified explicitly here because these details will not be needed. The quantity $\delta(u, x^A)$ is a phase factor which has been fixed in previous work [11, 12] by an evolution equation derived from the condition that the null tetrad vector m^a be parallel propagated along the generators of $\mathcal{I}^+$; an alternative expression for δ, which has the advantage of being explicit, will be derived below.

A gravitational wave detector does not measure $\mathcal{N}$ or ψ_4, but instead measures the gravitational wave strain h (i.e. h_+ and $h_\times$), which in the Bondi gauge is related to the metric variables used here by

$$h \equiv \lim_{\tilde{r}\to\infty} \tilde{r}\,(h_+ + ih_\times) = \tilde{J}_{,\tilde{\rho}}. \tag{20}$$

In the Bondi gauge, expressions for $\mathcal{N}$ and ψ_4 are particularly simple

$$\mathcal{N} = \frac{\tilde{J}_{,\tilde{\rho}\tilde{u}}}{2}, \;\; \lim_{\tilde{r}\to\infty} \tilde{r}\psi_4 = \bar{\tilde{J}}_{,\tilde{\rho}\tilde{u}\tilde{u}}\;, \tag{21}$$

from which it follows that h, $\mathcal{N}$ and ψ_4 are simply related

$$h_{,\tilde{u}\tilde{u}} = 2\mathcal{N}_{,\tilde{u}} = \left(\lim_{\tilde{r}\to\infty} \tilde{r}\bar{\psi}_4\right). \tag{22}$$

In previous work, h has been found by integrating $\mathcal{N}$ or ψ_4 with respect to Bondi gauge time $\tilde{u}$. In practice, time integration has been observed to introduce aliasing [14, 15], that is a non-linear drift of the mean of the wave oscillation away from zero. Thus, it is useful to derive a direct formula for h.

A. Formula for the gravitational wave strain h

The calculation starts from $\tilde{g}^{ab}$ as a series in $\tilde{\rho}$ and applies the coordinate transformation (9) to find g^{ab}, now as a series in ρ. Then

$$J = \frac{q_A q_B g^{AB}}{2\rho^2} \tag{23}$$

is also expressed as a series, and the second term of the series is $J_{,\rho}$. The expression found for $J_{,\rho}$ is linear in $\tilde{J}_{,\tilde{\rho}}$, and takes the form

$$C_1 J_{,\rho} = C_2 \tilde{J}_{,\tilde{\rho}} + C_3 \bar{\tilde{J}}_{,\tilde{\rho}} + C_4, \tag{24}$$

which may be inverted to give

$$\begin{aligned} h = \tilde{J}_{,\tilde{\rho}} &= \frac{C_1\bar{C}_2 J_{,\rho} - C_3\bar{C}_1\bar{J}_{,\rho} + C_3\bar{C}_4 - \bar{C}_2 C_4}{\bar{C}_2 C_2 - \bar{C}_3 C_3} \quad \text{where the coefficients are} \\ C_1 &= \frac{4\omega^2(1+\tilde{q}^2+\tilde{p}^2)^2}{(1+q^2+p^2)^2},\; C_2 = \omega(2+\eth\bar{X}_0+2\bar{X}_0\zeta)^2,\; C_3 = \omega(\eth X_0)^2, \\ C_4 &= \eth A(4+2\eth\bar{X}_0+4\bar{X}_0\zeta) + \eth X_0(2\eth\bar{A}+4\bar{A}\zeta) + 4\eth\omega\eth u_0. \end{aligned} \tag{25}$$

The above formula for the wave strain involves intermediate variables, and the procedure for calculating them is to solve equations for the variable indicated in the following order: Eq. (12) for x_0^A and thus X_0, Eq. (16) for ω, Eq. (11) for u_0, Eq. (14) for A^u, and Eq. (15) for A.

The linearized approximation may be used for the Einstein field equations and for the description of gravitational radiation when $|J|, |U|, |\beta|, |W_c| \ll 1$. This case is important because it often applies in astrophysical problems in which characteristic extraction is used [3], and formulas for $\mathcal{N}$ and ψ_4 have been given previously [12, 24]. In this approximation, the wave strain Eq. (25) simplifies to

$$h = J_{,\rho} - \eth A, \tag{26}$$

and linearization reduces Eq. (15) to $A + \eth u_0 = 0$ so that

$$h = J_{,\rho} + \eth^2 u_0\,. \tag{27}$$

B. Gauge freedom of the wave strain h

From Eq. (11), it is clear that the quantity u_0 defined in the coordinate transformation Eq. (9) is subject to the gauge freedom $u_0 \to u_0' = u_0 + u_G$, provided that $(\partial_u + U^A\partial_A)u_G = 0$; in the linearized case this is simply $\partial_u u_G = 0$ so that $u_G = u_G(x^A)$. Thus the wave strain h is subject to the gauge freedom $h \to h' = h + h_G$, where, in the linearized case,

$$h_G = \eth^2 u_G(x^A). \tag{28}$$

In the general case, the relation between h_G and u_G is via Eqs. (14), (15) and (25) and is much more complicated, but these details will not be needed. Decomposing u_G into spherical harmonics, it follows that, in the linearized case, the gauge freedom is represented by the addition of a constant to each spherical harmonic component of h. In the common case of a numerical simulation in which the final state is the Kerr geometry, these constants would normally be set so that $h = 0$ at the final time.

The same gauge freedom would occur if the wave strain h is obtained by time integration of the news $\mathcal{N}$, in this case appearing as an arbitrary "constant" of integration $f(\tilde{x}^A)$.

C. Explicit formula for the phase factor δ

We now revisit the computation of the phase factor δ introduced in Eq. (18). We find an expression for δ which does not require solution of an ODE, and thus reduces the number of time integrals required for computing ψ_4 or $\mathcal{N}$. The null tetrad vector m^a in Eq. (19) and $\tilde{m}^a$ in Eq. (8) must be related by the coordinate transformation Eq. (9). Thus

$$\frac{\tilde{q}^A}{\sqrt{2}} = \frac{\partial x_0^A}{\partial x^B} e^{-i\delta} \frac{F^B}{\omega} \tag{29}$$

Then δ is fixed by contracting Eq. (29) with $\bar{\tilde{q}}_A/\sqrt{2}$, since the LHS evaluates to 1 and the only unknown is δ,

$$e^{i\delta} = \frac{F^B \bar{\tilde{q}}_A}{\omega\sqrt{2}} \frac{\partial x_0^A}{\partial x^B} = \frac{1+q^2+p^2}{4\omega(1+\tilde{q}^2+\tilde{p}^2)} \sqrt{\frac{2}{K+1}} \left((K+1)(2+\eth\bar{X}_0+2\bar{X}_0\zeta) - J\bar{\eth}\bar{X}_0\right). \tag{30}$$

V. NUMERICAL TESTS

A. Analytic verification

At the analytic level there are some simple consistency tests that can be used to check aspects of the formulas derived here. First, in the linearized case, differentiation of the LHS of Eq. (27) with respect to $\tilde{u}$ is $2\mathcal{N}$; now in the linearized case, $\partial_{\tilde{u}} = \partial_u$ so $2\mathcal{N} = J_{,u\rho} + \bar{\eth}^2 u_{0,u}$, and then applying Eq. (11) (linearized) gives

$$2\mathcal{N} = J_{,u\rho} + \bar{\eth}^2\omega + 2\bar{\eth}^2\beta, \tag{31}$$

which is consistent with previous results [12, 24].

The validity of Eq. (29) can be checked by contraction with $\tilde{q}_A$, since the LHS reduces to 0. Computer algebra was used to replace J, ω on the RHS with terms of the form $x^A_{0,B}$ using Eqs. (17) and (16); the result was identically 0. The RHS of Eq. (30) must be a complex number of modulus 1. In this case the computer algebra was unable sufficiently to simplify the expression, so instead random values for the variables were generated and the resulting formula evaluated numerically. The result for the absolute value was found to be 1 (to within computational precision).

B. Tests of the strain formula

We test the new strain formula (27) in real-world applications of Cauchy-characteristic extraction (CCE) [1–3]. In CCE, the strong field region of a spacetime is evolved via standard 3+1 Cauchy evolutions. Here, we use the 3+1 code described in [25, 26]. During 3+1 Cauchy evolution, at some finite coordinate radius defining the worldtube Γ, metric data is collected and used as inner boundary data for a subsequent characteristic evolution to integrate the Einstein equations out to $\mathcal{I}^+$ where the gravitational wave signal can be unambiguously extracted.

We use the strain formula (27) to extract gravitational radiation at $\mathcal{I}^+$ via CCE from simulations of three different binary black hole (BBH) merger configurations and a rapidly rotating stellar core collapse model. For the BBH merger test, we consider an equal-mass non-spinning black hole system with an initial separation of $D = 11M$ that completes ~ 8 orbits before merger. The simulation, the initial parameters, and the grid setups are described in detail in [25, 27]. We also consider a mass-ratio $M_1/M_2 = 4$ non-spinning configuration with an initial separation of $D = 12M$ that completes ~ 15 orbits before merger. The simulation is described in [28]. Finally, we consider an equal-mass spinning configuration with individual black hole spins that are (anti-)aligned with the orbital angular momentum. The z-component of the black hole spins is given by $(a_1, a_2) = (0.8, -0.4)$. The initial separation is $D = 12.5M$, and the binary completes ~ 12 orbits before merger. The simulation is described in [9]. In all cases, characteristic metric boundary data is collected at worldtubes Γ of radii $R_\Gamma = 100M$ and $R_\Gamma = 250M$ for subsequent characteristic wave extraction at $\mathcal{I}^+$.

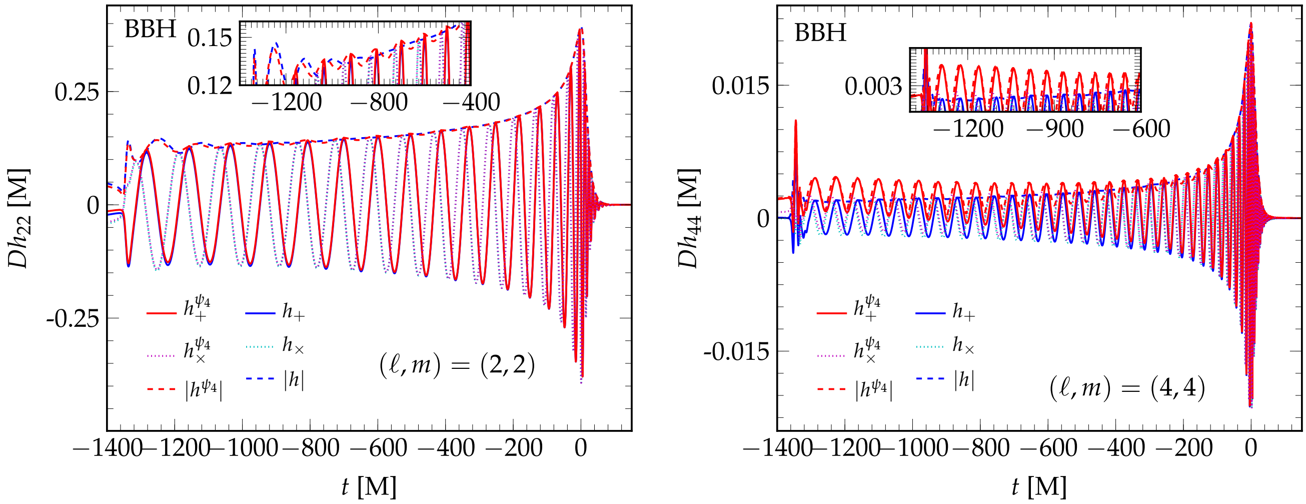

FIG. 1: *Left*: The "+" and "×" polarizations, as well as the amplitude $|h|$ of the $(\ell, m) = (2, 2)$ mode of the GW strain emitted from a non-spinning equal-mass binary black hole merger (i) computed using formula (27), and (ii) time integrated from ψ_4. The inset shows a close-up of the amplitude evolution. The amplitude of the time-integrated h^{ψ_4} is subject to larger oscillations compared to the amplitude of h computed using formula (27), which however decay during the late inspiral. *Right*: The same as in the left panel, but for the $(\ell, m) = (4, 4)$ mode. The time-integrated h^{ψ_4} shows significant non-linearities which result in strong amplitude oscillations. The strain h obtained via (27), on the other hand, is essentially free of this effect.

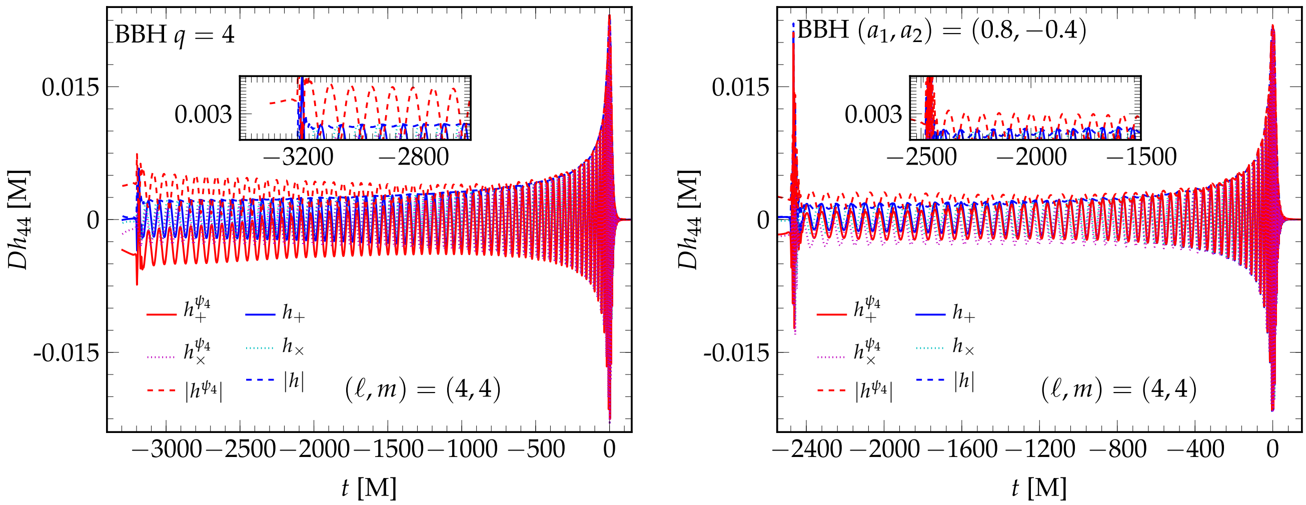

FIG. 2: The same as the right panel of Fig. 1, but for a BBH system with mass-ratio $M_1/M_2 = 4$ (left), and an equal-mass spinning configuration $(a_1, a_2) = (0.8, -0.4)$. The time-integrated h^{ψ_4} shows significant non-linearities which result in strong amplitude oscillations. For the strain h obtained via (27), on the other hand, this effect is strongly reduced.

The rotating stellar core collapse model is taken from [29], and is labeled by "A3B3G3". This particular model consists of a rapidly differentially spinning iron core at the onset of collapse, and produces a pronounced core bounce signal followed by strong fundamental mode excitation in the nascent protoneutron star. The initial stellar model is discussed in detail in [26, 29]. The simulation and grid setup are detailed in [26], and a comparison between different wave extraction methods for this model has been performed in [15]. Characteristic metric boundary data is collected on worldtubes of radii $R_\Gamma = 1000M$, $R_\Gamma = 1500M$, $R_\Gamma = 2500M$, and $R_\Gamma = 3500M$.

We compare the strain as extracted via the new formula (denoted by h) with the strain as time-integrated from the Weyl scalar ψ_4 [12, 30] in the time domain (denoted by h^{ψ_4}) and in the frequency domain using fixed-frequency integration (FFI) [14] (denoted by "h^{ψ_4} (FFI)"). In Fig. 1, we show the "+" and "×" polarizations, as well as the amplitude of h and h^{ψ_4} for the equal-mass binary black hole merger configuration using worldtube data from $R_\Gamma = 100M$ (similar results also hold for

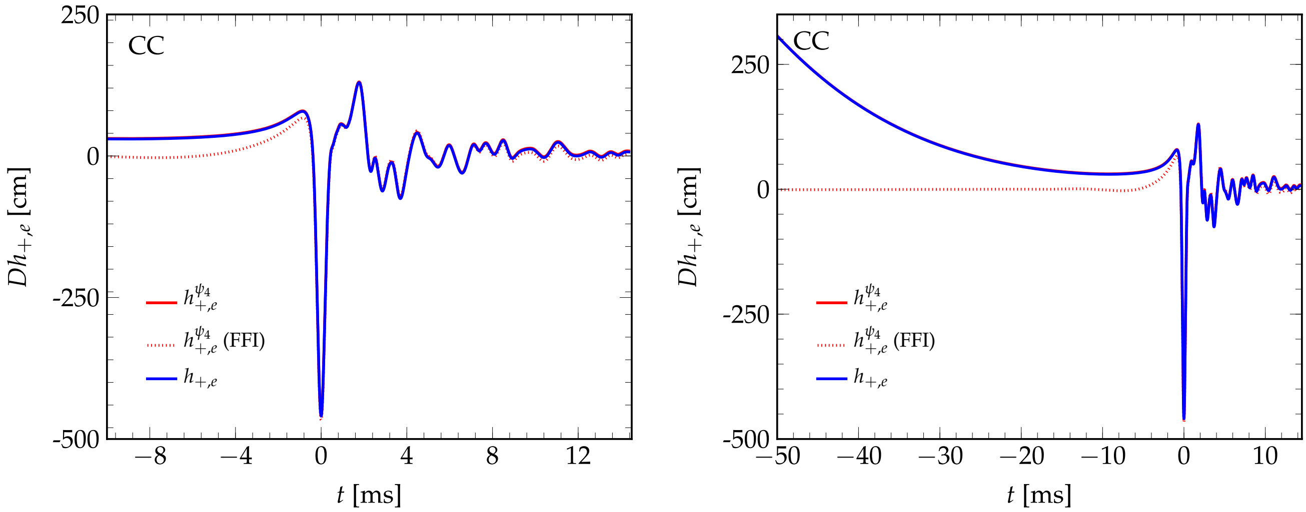

FIG. 3: *Left*: the distance rescaled "+" polarization of the gravitational wave strain h as emitted in the equatorial plane from stellar core collapse model "A3B3G3". The strain is (i) computed via (27), (ii) integrated in the time domain from ψ_4, and (iii) time-integrated in the Fourier domain from ψ_4 using FFI. *Right*: The same as in the left panel, but including the entire early collapse phase. Both, $h_{+,e}$ and $h^{\psi_4}_{+,e}$, should exhibit a strictly positive monotonic growth during th early collapse phase. Instead, both waveforms show a clear non-linear drift that leads to a significant offset.

$R_\Gamma = 250M$). We see that h^{ψ_4} as time-integrated from ψ_4 is subject to non-linear drifts. This is particularly visible in the higher-order $(\ell, m) = (4, 4)$ mode (right panel of Fig. 1; Fig. 2), where the time-integrated $h^{\psi_4}_+$ does not oscillate about zero. Note that we have picked the two integration constants such that $h^{\psi_4} = 0$ at late times after ring-down. This artificial non-linear drift is also noticeable in the amplitude $|h^{\psi_4}|$ where it leads to long-lasting transient oscillations (see inset of left and right panel of Fig. 1 and inset of left and right panel of Fig. 2). The strain h as directly obtained via formula (27), on the other hand, shows significantly better behavior for the $(\ell, m) = (4, 4)$ mode, where essentially no artificial non-linear drifts are visible, and the amplitude $|h|$ remains monotonic (right panel of Fig. 1, and left and right panel of Fig. 2). In the case of the $(\ell, m) = (2, 2)$ mode for the equal-mass non-spinning configuration, there are some amplitude oscillations of $|h|$ in the initial $-1400M \lesssim t \lesssim 1000M$ due to non-linear drift. At around $t \simeq -1000M$, it becomes quickly monotonic, however, whereas the time-integrated $|h^{\psi_4}|$ is still subject to long lasting amplitude oscillations (inset of left panel of Fig. 1). Similar behavior also holds for the other two BBH configurations (not shown). We note that the overall behavior, especially in the higher-order modes, is independent of the particular BBH merger configuration considered (compare right panel of Fig. 1 with Fig. 2).

For the equal-mass non-spinning case, we quantify the differences in h as obtained via the various methods more precisely in the following. When the two waveforms are aligned at merger, we measure a relative amplitude difference between h and h^{ψ_4} of $\Delta A/A \simeq 0.004\%$ for the $(\ell, m) = (2, 2)$ mode, and $\Delta A/A \simeq 0.5\%$ for the $(\ell, m) = (4, 4)$ mode at the peak of the signal ($t = 0$). To avoid the artifacts associated with time-integrated waveforms, we also directly compare ψ_4 to get a different handle on the differences between formula (27) and ψ_4 extraction. Therefore, we take the second time derivative of h obtained via formula (27). We arrive at relative peak amplitude differences between $\ddot{h}$ and ψ_4 of $\Delta A/A \simeq 0.003\%$ for the $(\ell, m) = (2, 2)$ mode, and $\Delta A/A \simeq 0.2\%$ for the $(\ell, m) = (4, 4)$ mode. To measure the phase difference, we align the signals during inspiral in the interval $t \in [-1100, 800]$ using the method presented in [31], and measure the dephasing accumulated at the merger peak. Between h and h^{ψ_4}, we measure a relative dephasing of $\Delta\phi/\phi \simeq 0.2\%$ for the $(\ell, m) = (2, 2)$ mode. For the $(\ell, m) = (4, 4)$ mode, no meaningful dephasing can be obtained due to the large non-linear drift. Between $\ddot{h}$ and ψ_4, we measure a relative dephasing of $\Delta\phi/\phi \simeq 0.0002\%$ for the $(\ell, m) = (2, 2)$ mode, and $\Delta\phi/\phi \simeq 0.001\%$ for the $(\ell, m) = (4, 4)$ mode. The results based on the differences between $\ddot{h}$ and ψ_4 indicate that the new formula is accurate within the numerical and systematic errors inherent in CCE [32], at least for the particular case considered here. Similar results also hold for the unequal-mass configuration and for the spinning configuration.

We now turn to the test case of rotating stellar core collapse. In Fig. 3, we show the "+" polarization of the strains h, h^{ψ_4}, and "h^{ψ_4} (FFI)" as emitted in the equatorial plane for the rotating stellar core collapse model "A3B3G3". Instead of the expected monotonically rising signal, both, h_+ and $h^{\psi_4}_+$, exhibit a noticeable non-linear drift in the first ~ 40 ms during collapse, which leads to a significant offset. Otherwise, h_+ and $h^{\psi_4}_+$ are practically identical. Note that we have shifted the waveforms such that they oscillate about zero at late times. The strain "h^{ψ_4} (FFI)" does not show non-linear drifts due to the filtered low-frequency components. As detailed in [15], this filter approach also removes frequency components that are physical. Unfortunately, it is not possible to disentangle the artificial low frequency components from the physical ones. Given that directly extracting h by using formula (27) does not reduce the unphysical non-linear drifts in the particular core collapse model that we consider, we conclude that this drift must have different roots, perhaps stemming from artifacts in the Cauchy initial data. This is further supported by the fact that the drift is practically independent of numerical Cauchy and characteristic resolutions. In addition, by changing the characteristic initial data from conformally flat $J = 0$ to data which vanishes at $\mathcal{I}^+$ and smoothly blends to Cauchy data via some polynomial function, we practically observe no difference in the behavior of the drift. A detailed analysis of this must be left to future work.

VI. CONCLUSION

This work re-investigated formulas describing gravitational radiation in the characteristic formulation of numerical relativity. A new formula for the gravitational wave strain, $(h_+, h_\times)$, was derived. Further, alternative procedures, that reduce the use of time integrals, were found for calculating intermediate variables needed for any description of the gravitational radiation. As in previous work, it is still necessary to solve the evolution equations Eq. (12) for x_0^A and Eq. (11) for u_0, but the introduction of explicit formulas for ω (Eq. (16)) and δ (Eq. (30)) reduces the number of time integrals from 4 to 2 in the computation of $\mathcal{N}$ and ψ_4, and from 5 to 2 in the computation of the wave strain.

The numerical tests show that formula (27) for directly extracting the strain h at $\mathcal{I}^+$ yields results which are comparable to the "traditional" strain computed via a double time-integration from the Weyl scalar ψ_4. In the case of three representative binary black hole merger simulations including an equal-mass non-spinning case, a mass-ratio $M_1/M_2 = 4$ configuration, and an equal-mass spinning case with spins $(a_1, a_2) = (0.8, -0.4)$, the new formula leads to reduced artificial non-linear drifts, especially in higher-order modes, which are typically observed in numerical simulations [14, 16, 17]. In the case of a rapidly rotating stellar core collapse model, however, artificial drifts persist, and are thus unrelated to time integration.

The new formula has been implemented in the PITT Nullcode and is available as part of the Einstein Toolkit [33, 34].

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Appendix A: Computer algebra scripts and output

The Maple scripts used to derive some of the results in this paper are included in the online supplement. The script files are `J_rho.map` with output `J_rho.out` for Eqs. (25) and (26), and `delta.map` with output `delta.out` for Eq. (30); both these script files start by running the script file `common.map`. The file `READ.ME` contains a description of the variable names used in the scripts in terms of the notation used in this paper.

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