COMPACT ARITHMETIC QUOTIENTS OF THE COMPLEX 2-BALL AND A CONJECTURE OF LANG

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Introduction

Let $\mathcal{H}_{\mathbb{C}}^2$ be the 2-dimensional complex hyperbolic space, represented by the unit ball in $\mathbb{C}^2$ equipped with the Bergmann metric of constant holomorphic sectional curvature -1, on which the Lie group $\mathrm{U}(2,1)$ acts in a natural way. Let Γ be a cocompact arithmetic discrete subgroup of $\mathrm{U}(2,1)$. By the Baily-Borel theorem the quotient $X_\Gamma := \Gamma\backslash\mathcal{H}_{\mathbb{C}}^2$ has a structure of a normal, projective surface, in fact over a number field.

A projective surface X defined over a number field k is said to be *Mordellic* if, and only if, for any finite extension k' of k, the set $X(k')$ of k'-rational points of X is finite. Lang conjectured in [L, Conjecture VIII.1.2] (see also [T, p.xviii]) that any *hyperbolic*, smooth X is Mordellic; this is consistent with the general philosophy of Vojta. The following result makes use of certain key theorems of Faltings [F1, F2] (see also [V]) and Rogawski [R1, R2], as well as the hyperbolicity of X_Γ.

Theorem. *Let M be a CM quadratic extension of a totally real number field F and let G be a unitary group over F defined by an anisotropic hermitian form on M^3 of signature $(2,1)$ at one infinite place and $(3,0)$ at the others. Let Γ be (the image in $\mathrm{U}(2,1)$ of) an arithmetic subgroup of $G(F)$ such that all its torsion elements are scalar. Then X_Γ is Mordellic.*

The paper is organized as follows. In §1 we give a criterion for X_Γ to be a smooth hyperbolic variety, that can always be achieved by taking a finite cover. Such X_Γ have an ample canonical bundle and it seems to be a folklore result that they are of general type, which we verify for completeness in Proposition 1.4 using the Enriques-Kodaira classification. For the convenience of the reader we provide a proof that X_Γ is defined over a number field, based on Yau's algebro-geometric characterisation of compact Kähler surfaces covered by $\mathcal{H}_{\mathbb{C}}^2$ (see Proposition 1.3). In §2 we give a precise definition of the arithmetic groups Γ under consideration and prove that the irregularity of $X_{\Gamma'}$ becomes arbitrarily large over explicite finite covers of X_Γ (see Proposition 2.3), based on Rogawski's classification of cohomological automorphic forms on G contributing to $\mathrm{H}^1(X_\Gamma)$ for congruence subgroups Γ. The importance of this is evident given Mazur's path breaking work on the modular curves in the seventies. The *Bombieri-Lang conjecture* predicts that $X_\Gamma(k)$ is not Zariski dense, implying that the k-rational points are contained in a finite union of curves. This we establish in Proposition 3.1 using a deep Theorem due to Faltings, when X_Γ is smooth and does not admit a dominant map to its Albanese variety, in particular when its irregularity is > 2. Finally in §3 we deduce our main Theorem from Propositions 2.3 and 3.1, and from Faltings' proof of the Mordell's conjecture for curves.

Note that even though our theorem only concerns arithmetic subgroups, because F and M can vary, it can be applied to infinitely many pairwise non-commensurable (cocompact) discrete subgroups in $\mathrm{U}(2,1)$.

Our results do *not* apply, however, to the analogous case of (a cocompact discrete subgroup of) a unitary group G' defined by a division algebra of dimension 9 over an imaginary quadratic field M with an involution of the second kind. In that case it is known that the Albanese of X_Γ is zero for any congruence subgroup $\Gamma \subset G'(\mathbb{Z})$; this was proved by Rapoport and Zink [RZ] under a ramification hypothesis, and later by Rogawski [R1] using a different method, without the hypothesis.

In a related paper [DR], we analyze the more complicated case when G is defined by an *isotropic* hermitian form in 3 variables on M^3. Then, for the arithmetic quotient $Y_\Gamma := \Gamma\backslash\mathcal{H}^2_{\mathbb{C}}$ to be well defined, it is necessary that $F = \mathbb{Q}$, and Y_Γ is always *non-compact*. Its toroidal compactification X_Γ is not hyperbolic, being a union of Y_Γ with a finite number of elliptic curves E_κ indexed by the cusps κ. If we proceed as above, one cannot rule out having curves of genus ≤ 1 on X_Γ (in fact there are rational curves at low levels), except to know that by the hyperbolicity of the open surface Y_Γ, they must meet some E_κ on the boundary; the nature of the meeting points is also crucial. Nevertheless we exhibit in [DR] an infinite class $\mathcal{C}$ of X_Γ of congruence type which are not covers of each other, such that $X_\Gamma(M)$ is finite; of course X_Γ cannot be Mordellic in this situation as over some finite extension the E_κ will have positive Mordell-Weil rank. For the finiteness statement over M for the members of $\mathcal{C}$, we do not appeal to Faltings, and instead employ the full force of the theory of automorphic forms on $\mathrm{U}(3)$, exhibiting suitable abelian variety quotients A of $\mathrm{Alb}(X_\Gamma)$ with finite Mordell-Weil group (as in Mazur's work on modular curves), and establish a formal immersion over $\mathbb{C}$ at the torsion points of E_κ where any curve C in the inverse image of $A(M)$ may meet E_κ. We do the last part by analyzing the Fourier-Jacobi coefficients of certain associated *residual* Picard modular forms.

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1. General type and hyperbolicity

Definition 1.1. Given a discrete subgroup $\Gamma \subset \mathrm{U}(2,1)$ we let $\bar{\Gamma} = \Gamma/\Gamma \cap \mathrm{U}(1)$ denote its image in the adjoint group $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ is centrally embedded in $\mathrm{U}(2,1)$. We put $X_{\bar{\Gamma}} := X_\Gamma = \Gamma \backslash \mathcal{H}^2_{\mathbb{C}}$

Conversely any discrete subgroup $\bar{\Gamma} \subset \mathrm{PU}(2,1) = \mathrm{PSU}(2,1)$ is the image of a discrete subgroup of $\mathrm{U}(2,1)$, namely $\mathrm{U}(1)\bar{\Gamma} \cap \mathrm{SU}(2,1)$.

Lemma 1.2. *Let Γ be a cocompact discrete subgroup of* $\mathrm{U}(2,1)$.

(i) *The analytic surface X_Γ is an orbifold and one has the following implications:*

Γ *is neat* $\Rightarrow$ Γ *is torsion free* $\Rightarrow$ $\bar{\Gamma}$ *is torsion free* $\Rightarrow$ X_Γ *is a hyperbolic manifold.*

(ii) *Assume that $\bar{\Gamma}$ is torsion free. Then the natural projection $\mathcal{H}^2_{\mathbb{C}} \to X_\Gamma$ is an étale covering of group $\bar{\Gamma}$. Moreover for every finite index normal subgroup Γ' of Γ the natural morphism $X_{\Gamma'} \to X_\Gamma$ is an étale covering of group $\bar{\Gamma}/\bar{\Gamma}'$.*

Proof. The stabiliser in $\mathrm{U}(2,1)$ of any point of $\mathcal{H}^2_{\mathbb{C}}$ is a compact group, hence its intersection with the discrete subgroup Γ is finite, showing that X_Γ is an orbifold. If Γ is neat, then no element of it has a non trivial root of unity as an eigenvalue, in particular Γ is torsion free. Since $\Gamma \cap \mathrm{U}(1)$ is finite, this implies that $\bar{\Gamma}$ is torsion free too. Under the latter assumption, $\Gamma \cap \mathrm{U}(1)$ acts trivially on $\mathcal{H}^2_{\mathbb{C}}$, and $\bar{\Gamma}$ acts freely and properly discontinuously on it, hence X_Γ is a manifold. Since $\mathcal{H}^2_{\mathbb{C}}$ is simply connected, it is a universal covering space of X_Γ with group $\bar{\Gamma}$. In particular, X_Γ is hyperbolic. The last claim follows from the exact sequence: $1 \to \bar{\Gamma}' \to \bar{\Gamma} \to \bar{\Gamma}/\bar{\Gamma}' \to 1$. □

It is a well known fact that any compact orbifold admits a finite cover which is a manifold and Lemma 2.2 provides such a cover explicitly for arithmetic quotients.

Proposition 1.3. *The projective variety X_Γ can be defined over a number field.*

Proof. Without loss of generality one may assume that $\bar{\Gamma}$ is torsion free. Calabi and Vesentini [CV] have proved that X_Γ is locally rigid, hence by Shimura [Sh1] it can be defined over a number field.

We will now provide a second, more direct proof when $\bar{\Gamma}$ is arithmetic. Since $X_{\bar{\Gamma}}$ is uniformized by $\mathcal{H}^2_{\mathbb{C}}$ it has ample canonical bundle and the Chern numbers c_1, c_2 of its complex tangent bundle satisfy the relation $c_1^2 = 3c_2$. Since everything can be defined algebraically, for any automorphism σ of $\mathbb{C}$, the variety $X^\sigma_{\bar{\Gamma}}$ also has ample canonical bundle and $c_1^{\sigma 2} = 3c_2^\sigma$. By a famous result of Yau [Y, Theorem 4], this is equivalent to the fact that $X^\sigma_{\bar{\Gamma}}$ may be realized as $\bar{\Gamma}^\sigma \backslash \mathcal{H}^2_{\mathbb{C}}$ for some cocompact discrete irreducible torsion free subgroup $\bar{\Gamma}^\sigma$.

Since $\bar{\Gamma}$ is arithmetic, it has infinite index in its commensurator in $\mathrm{PU}(2,1)$, denoted $\mathrm{Comm}(\bar{\Gamma})$. For every element $g \in \mathrm{Comm}(\bar{\Gamma})$ there is a Hecke correspondence

$$X_{\bar{\Gamma}} \leftarrow X_{\bar{\Gamma} \cap g^{-1}\bar{\Gamma} g} \xrightarrow[g\cdot]{\sim} X_{g\bar{\Gamma} g^{-1} \cap \bar{\Gamma}} \to X_{\bar{\Gamma}} \tag{1}$$

and the correspondences for g and g' differ by an isomorphism $X_{g\bar{\Gamma} g^{-1}\cap\bar{\Gamma}} \xrightarrow{\sim} X_{g'\bar{\Gamma} g'^{-1}\cap\bar{\Gamma}}$ over $X_{\bar{\Gamma}}$ if, and only if, $g' \in \bar{\Gamma} g$. By Chow (1) is defined algebraically, hence yields a correspondence on $X^\sigma_{\bar{\Gamma}} = X_{\bar{\Gamma}^\sigma}$:

$$X_{\bar{\Gamma}^\sigma} \leftarrow X_{\bar{\Gamma}_1} \xrightarrow{\sim} X_{\bar{\Gamma}_2} \to X_{\bar{\Gamma}^\sigma},$$

for some finite index subgroups $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ of $\bar{\Gamma}^\sigma$. By the universal property of the covering space $\mathcal{H}^2_{\mathbb{C}}$, the middle isomorphism is given by an element of $g_\sigma \in \mathrm{PU}(2,1) \simeq \mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}})$. Since $\mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}}/X_{\bar{\Gamma}_i}) = \bar{\Gamma}_i$ $(i = 1, 2)$, it easily follows that $\bar{\Gamma}_2 = g_\sigma \bar{\Gamma}_1 g_\sigma^{-1}$, and by applying σ^{-1} one sees that $\bar{\Gamma}_1 = \bar{\Gamma}^\sigma \cap g_\sigma^{-1} \bar{\Gamma}^\sigma g_\sigma$. It follows that $g_\sigma \in \mathrm{Comm}_G(\Gamma^\sigma)$ and one can check that $g'_\sigma \in \bar{\Gamma}^\sigma g_\sigma$ if, and only if, $g' \in \bar{\Gamma} g$. Therefore $\mathrm{Comm}(\bar{\Gamma}^\sigma)/\bar{\Gamma}^\sigma \simeq \mathrm{Comm}(\bar{\Gamma})/\bar{\Gamma}$ is infinite too, which by a major theorem of Margulis implies that Γ^σ is arithmetic, providing an alternative proof of a result of Kazhdan.

Consider now the action of $\mathrm{Aut}(\mathbb{C})$ on the set of equivalence classes of cocompact arithmetic subgroups Γ (modulo their center and up to conjugation by an element of

$\mathrm{U}(2,1)$). The group $\mathrm{U}(2,1)$ has only countably many $\mathbb{Q}$-forms, classified by central simple algebras of dimension 9 over a CM field, endowed with an involution of a second kind and verifying some conditions at infinity (see [PR, pp. 87-88]). Finally, there are only countably many arithmetic subgroups for a given $\mathbb{Q}$-form, since those are all finitely generated and contained in their common commensurator, which is countable. It follows that Γ is fixed by an open subgroup of $\mathrm{Aut}(\mathbb{C})$, allowing one to conclude that X_Γ is defined over a number field. □

Proposition 1.4. *Assume that $\bar{\Gamma}$ is torsion free. Then X_Γ is of general type.*

Proof. Our assumption on Γ implies that $X = X_\Gamma$ admits $\mathcal{H}^2_{\mathbb{C}}$ is a universal covering space with group $\bar{\Gamma}$. Hence X is hyperbolic, implying in particular that it is a minimal surface.

We will use the Enriques-Kodaira classification of smooth projective surfaces. The *Kodaira dimension* $\kappa(X)$ of X has four possible values, namely in $\{-\infty, 0, 1, 2\}$, and we have to rule out the occurrence of the first three situations, as X is of general type if, and only if, $\kappa(X) = 2$. Denote by $\mathcal{K}_X$ the canonical divisor of X.

First suppose $\kappa(X) = -\infty$. Then all the plurigenera $P_m := \dim \mathrm{H}^0(X, \mathcal{K}_X^m)$ vanish. If $q = 0$, then by the Castelnuovo criterion, having $P_2 = q = 0$ implies that X is a rational surface. If $q > 0$, then the fibers of the Albanese map $X \to \mathrm{Alb}(X)$ are rational curves, i.e., X is ruled. In either case, we would get a non-constant holomorphic map $\mathbb{C} \to X$, which is impossible. Kodaira's class VII surfaces do not occur here as X is algebraic.

Next look at the case when $\kappa(X) = 0$. The only algebraic surfaces in characteristic zero are among the following types: (i) $K3$, (ii) Enriques, (iii) abelian, and (iv) hyperelliptic. Suppose we are in case (i). Then one knows (see [MM]) that there is always a rational curve in any complex $K3$ surface, which is impossible in our situation. In case (ii), X is a necessarily a quotient of a $K3$ surface X' by an involution, which would again give a non-constant holomorphic map from $\mathbb{C}$ to X, furnishing a contradiction. In case (iii), X is a quotient of $\mathbb{C}^2$ by a lattice, which is again impossible. Finally, in case (iv), such a surface is, over $\mathbb{C}$, a quotient of a product of two elliptic curves by a finite group of automorphisms. One again this would result in a non-constant holomorphic map $\mathbb{C} \to X$, and we are done eliminating $\kappa(X) \leq 0$.

Finally, let $\kappa(X) = 1$. Here X is an elliptic surface, admitting a fibration over a base curve with all but finitely many fibers being elliptic curves. Again this is impossible by the hyperbolicity of X. $\square$

Remark 1.5. When Lang originally made his conjecture on Mordellicity, his definition of a variety X over $k \subset \mathbb{C}$ being hyperbolic required the Kobayashi semi-distance on $X(\mathbb{C})$ to be in fact a metric. Later it was established by R. Brody [B] that in the compact case this was equivalent to requiring that there is no non-constant holomorphic map from $\mathbb{C}$ to $X(\mathbb{C})$. It is expected that every smooth projective irreducible surface X of general type over $\mathbb{C}$ containing no curve of genus ≤ 1 is hyperbolic, and this is known when X does not admit a dominant map to its Albanese variety.

2. Positive irregularity for arithmetic covers

Let F be totally real number field of degree d and ring of integers $\mathcal{O}_F$, and let M be a CM quadratic extension of F. Consider a 3-dimensional M-vector space V endowed with an anisotropic hermitian form $\langle\ ,\ \rangle$ of signature $(2,1)$ at one infinite place and $(3,0)$ at the others. The reductive group G over F is defined as

$$G = \{g \in \mathrm{GL}(V) \mid \langle g(v), g(v')\rangle = \langle v, v'\rangle \text{ , for all } v, v' \in V\},$$

For example, if M is any quadratic CM extension of a real quadratic field F of discriminant D, then one might consider a hermitian form given by $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \sqrt{D} \end{pmatrix}$.

Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^d$ and let $\iota : F \hookrightarrow \mathbb{R}$ be the embedding corresponding to the first factor in the identification $G(F_\infty) = \mathrm{U}(2,1) \times \mathrm{U}(3)^{d-1}$. For simplicity we keep the same notation for a fixed extension $\iota : M \hookrightarrow \mathbb{C}$. We will complete this extension ι to a CM type Φ of M. By choosing a basis of $V \otimes_{F,\iota} \mathbb{R} \simeq \mathbb{C}^3$ where the hermitian form is represented by the matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$, we identify $\mathcal{H}^2_{\mathbb{C}}$ with the hermitian symmetric space of negative lines in $V \otimes_{F,\iota} \mathbb{R}$.

Since G is defined by a hermitian form on M^3, we have an embedding $G(F) \hookrightarrow \mathrm{GL}(3, M)$, through which we may view elements of $G(F)$ as 3×3 matrices. When we compose this embedding with (the extension to M of) ι, the image of $\iota(G(F))$ lands in $\mathrm{U}(2,1) \subset \mathrm{GL}(3,\mathbb{C})$. The standard left action of $\mathrm{GL}(3,\mathbb{C})$ on $\mathbb{P}^2(\mathbb{C})$ yields a left transitive action of the first component $G(F_\iota) = \mathrm{U}(2,1)$ of $G(F_\infty)$ on $\mathcal{H}^2_{\mathbb{C}}$. The standard maximal compact subgroup $K_\infty \subset G(F_\infty)$ is defined as the stabilizer of a distinguished point in $\mathcal{H}^2_{\mathbb{C}}$ and we have $\mathcal{H}^2_{\mathbb{C}} \simeq G(F_\infty)/K_\infty$. Since K_∞ also stabilizes the orthogonal of the corresponding negative line, one has $K_\infty \simeq \mathrm{U}(2) \times \mathrm{U}(1) \times$

$\mathrm{U}(3)^{d-1}$. Any discrete subgroup Γ of $G(F)$ acts properly discontinuously on $\mathcal{H}^2_{\mathbb{C}}$ (via ι). We will use the same notation for an arithmetic subgroup $\Gamma \subset G(F)$ and for its image $\iota(\Gamma) \subset G(F_\iota) = \mathrm{U}(2,1)$.

Definition 2.1. A subgroup Γ of $G(F)$ is called a *congruence* subgroup if there exists an integer N such that Γ contains the principal congruence subgroup of level N, defined as:

$$\Gamma(N) = \ker\left(G(\mathcal{O}_F) \to G(\mathcal{O}_F/N\mathcal{O}_F)\right).$$

A subgroup Γ of $G(F)$ is *arithmetic* if it is commensurable with $G(\mathcal{O}_F)$.

Lemma 2.2. *The principal congruence subgroup* $\Gamma(3)$ *is torsion free.*

Proof. Suppose that $\gamma \in \Gamma(3)$ has order a prime number ℓ. Since at least one of the eigenvalues ζ of γ is a root of unity of order ℓ, the congruence condition on γ implies that $\ell = 3$. Denote by a the highest power of 3 dividing all the coefficients of $\gamma - 1$. Then the highest power of 3 dividing all the coefficients of $\gamma^3 - 1$ is $a+1$, leading to a contradiction. $\square$

Denote by $q = q(X_\Gamma)$ the *irregularity* of X_Γ, given by the dimension of $\mathrm{H}^0(X_\Gamma, \Omega^1_{X_\Gamma})$. By the Hodge decomposition of $\mathrm{H}^1(X_\Gamma)$, the first Betti number of X_Γ is $2q(X_\Gamma)$. The positivity of $q(X_\Gamma)$ is an essential ingredient in the proof of our Diophantine results and is the object of following proposition.

Proposition 2.3. *For any arithmetic subgroup* Γ *of* $G(F)$ *and for any* $r > 0$*, there is a finite index torsion free subgroup* Γ' *of* Γ *such that* $q(X_{\Gamma'}) > r$.

The starting point for the arithmetic application of this paper was our knowledge that Rogawski's theory [R1, R2] allowed one to prove such an unboundedness result for q over coverings. Many examples were known earlier, via the theta correspondence, by the works of Kazhdan [K] and Shimura [Sh2, Theorem 8.1]. Each step of the proof below can be carried out explicitly thus giving a precise level, depending on M, at which the irregularity is at least r. After the completion of the work on this paper, we learned of Marshall's interesting work [Ma] giving sharp asymptotic bounds for $q(X_\Gamma)$ when Γ shrinks, also by using Rogawski's theory.

Proof. Note that it suffices to prove that $\Gamma(3)$ contains a subgroup Γ' of finite index such that $q(X_{\Gamma'}) > r$, since then, for any arithmetic Γ, the natural morphism $X_{\Gamma\cap\Gamma'} \to X_{\Gamma'}$ is finite and surjective, hence $q(X_{\Gamma\cap\Gamma'}) \geq q(X_{\Gamma'}) > r$.

By [BW, Chap.XIII] there is a decomposition:

$$\mathrm{H}^1(X_\Gamma, \mathbb{C}) \simeq \bigoplus_\pi \mathrm{H}^1(\mathrm{Lie}(G(F_\infty)), K_\infty; \pi)^{\oplus m(\pi,\Gamma)}, \tag{2}$$

where π runs over irreducible unitary representations of $G(F_\infty)$ occurring with multiplicity $m(\pi, \Gamma)$ in $L^2(\Gamma\backslash G(F_\infty))$, which is discrete by virtue of Γ being cocompact.

At the distinguished archimedean place ι where $G(F_\iota) = \mathrm{U}(2,1)$ there are exactly two irreducible non-tempered unitary representations of $G(F_\iota)$, denoted π^+ and π^-, with non-zero relative Lie algebra cohomology in degree 1; π^+ occurs in $\mathrm{H}^{1,0}$ and π^- in $\mathrm{H}^{0,1}$. So we need to show that $m(\pi^+, \Gamma')$ is non-zero for sufficiently many $\Gamma' \subset \Gamma$. In fact it will suffice to show that there are infinitely many $\Gamma' \subset \Gamma$ of relatively prime auxiliary levels (that is to say levels outside the level of Γ) such that $m(\pi^+, \Gamma') \neq 0$.

The restriction to $\mathbb{C}^*$ of the Langlands parameter of π^+ is given by (see [La, p.64]):

$$z \mapsto \begin{pmatrix} z^{-1} & 0 & 0 \\ 0 & z/\overline{z} & 0 \\ 0 & 0 & \overline{z} \end{pmatrix} \in {}^L G^0 = \mathrm{GL}_3(\mathbb{C}). \tag{3}$$

We will work adelically and construct infinitely many automorphic representations $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{A}_F)$ of relatively prime auxiliary levels such that $\pi_\iota = \pi^+$. As usual, $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$ denotes the adele ring of F, and K is a open compact subgroup of $G(\mathbb{A}_{F,f})$. The adelic quotient

$$X_K := G(F)\backslash G(\mathbb{A}_F)/K_\infty K \tag{4}$$

is a finite disjoint union of surfaces, each of the form X_Γ for some congruence subgroup $\Gamma \subset G(F)$. The decomposition (2) can be rewritten as:

$$\mathrm{H}^1(X_K, \mathbb{C}) \simeq \oplus_\pi \mathrm{H}^1(\mathrm{Lie}(G(F_\infty)), K_\infty; \pi_\infty)^{\oplus m(\pi_f, K)},$$

and by the Multiplicity One Theorem of Rogawski one deduces that:

$$q(X_K) = \sum_{\substack{\pi \text{ automorphic} \\ \pi_\iota \simeq \pi^+}} \dim(\pi_f^K) = \sum_{\substack{\pi \text{ automorphic} \\ \pi_\iota \simeq \pi^-}} \dim(\pi_f^K). \tag{5}$$

Again by Rogawski [R2], we have an explicit description of the automorphic representations π of $G(\mathbb{A})$ such that $\pi_\iota \simeq \pi^+$, that we will now present. Let W_F (resp. W_M) be the global Weil group of F (resp. M). (Due to the simple nature of our situation, we may write the parameter in terms of W_F instead of the conjectural global Langlands group $\mathcal{L}_F$.) Let ρ denote the non-trivial automorphism of M/F, written as $x \to \overline{x}$. The corresponding global Arthur parameter

$$\psi : W_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G = \mathrm{GL}(3, \mathbb{C}) \rtimes \{1, \rho\}$$

is such that its restriction ψ_M to W_M is given by the 3-dimensional representation $(\lambda \otimes \mathrm{St}) \oplus (\chi_M \otimes 1)$, where St (resp. 1) is the standard 2-dimensional (resp. trivial) representation of $\mathrm{SL}(2, \mathbb{C})$, λ is a character of W_M whose transfer to W_F is the quadratic character $\delta_{M/F}$ associated to M/F, and $\chi_M = \chi/\chi \circ \rho$ is the base change of an idele class character χ of $\mathrm{U}(1)/M$. We have $\mathrm{Hom}_{\mathbb{Q}}(M, \mathbb{C}) = \Phi \cup \Phi^\rho$, and at the distinguished place ι in Φ, $\psi_{M,\iota}$ agrees with (3) on $\mathbb{C}^*$. For reasons which will become apparent when we do the transfer from the quasi-split group, we take ψ_M to have the same type at all $v \in \Phi$. Then λ is an algebraic (unitary) Hecke character of M of weight 1, whereas χ is an algebraic character of $\mathrm{U}(1)$ of weight -1.

Let $\tilde{G}$ denote the quasi-split unitary group associated to M/F, so that G is an inner form of $\tilde{G}$. (In [R1], [R2], the author uses (G, G') for our $(\tilde{G}, G)$.) Let $\tilde{\Pi}$ be the global Arthur packet associated to ψ, with $\tilde{\Pi}_v$ denoting the local Arthur packet at each place v. It arises by endoscopy from $\mathrm{U}(2) \times \mathrm{U}(1)$, and the embedding depends on the choice of a Hecke character μ of M whose restriction to F is the quadratic character $\delta_{M/F}$. When v is finite and remains prime in M, $\tilde{\Pi}_v = \{\pi_{s,v}, \pi_{n,v}\}$, where $\pi_{n,v}$ is a certain non-tempered representation and $\pi_{s,v}$ is in the discrete series. At a split place, $\tilde{\Pi}_v = \{\pi_{n,v}\}$, and for almost all v, $\pi_{n,v}$ is unramified. An element π of $\tilde{\Pi}$ is a tensor product $\otimes_v \pi_v$ with each π_v in $\tilde{\Pi}_v$ such that $\pi_v \simeq \pi_{n,v}$ for almost all v.

Since our G is an inner form defined by a hermitian form on M^3, we know from Rogawski (see [R1, §14.4] and [R2, p.397]) that there will be a corresponding Arthur packet Π of representations of $G(\mathbb{A}_F)$ *if, and only if,* certain condition is satisfied at each of the $d-1$ archimedean places v where G_v is the compact real group $\mathrm{U}(3)$. When such a correspondence is possible, we will say that Π exists, in which case $\Pi_v = \tilde{\Pi}_v$ at all the places v outside the $d-1$ archimedean places, and in each of the latter, Π_v is a singleton consisting of a finite-dimensional representation σ_v of $\mathrm{U}(3)$. As in [R2, p.397], we will write (at each $v \in \Phi$): $\mu_v(z) = (z/|z|)^{2t+1}$, and the one-dimensional representation of $\mathrm{U}(2) \times \mathrm{U}(1)$ associated to ψ at v by $(h, u) \to \det(h)^{r-t} u^s$. Then in our case λ, resp. χ, being of weight 1, resp. -1, implies that $r = -1$ and $s = 1$. Moreover, an explicit calculation shows in our case that μ has weight -1, implying that $t = 0$. Since $s > r$, by the recipe on the same page of [R2], the associated triple (a, b, c) is given by $(1, 0, -1)$. Since it is regular, i.e., with three distinct coordinates, the local condition is satisfied at every $v \neq \iota$ in Φ, and there is a corresponding σ_v of $\mathrm{U}(3)$. The regularity of (a,b,c) implies that there is a square integrable representation $\pi_{s,v}$ of $\mathrm{U}(2,1)$ in $\tilde{\Pi}_v$, which corresponds to σ_v. In fact, since its highest weight of σ_v is given by $\mathrm{diag}(x,y,z) \to x^{a-1}y^b z^{c+1}$, it is the trivial representation. In any case, the global Arthur packet Π exists for our ψ. Now let $\pi \in \Pi$ be such that $\pi_\iota = \pi^+ \in \Pi_\iota$. By [R2, Theorem 1.2], for such a $\pi \in \Pi$, we have

$$\pi \text{ automorphic} \iff W(\lambda\chi_M^{-1}) = (-1)^{d-1+s(\pi)}, \tag{6}$$

where $W(\lambda\chi_M^{-1}) \in \{\pm 1\}$ is the root number of the weight 3 algebraic Hecke character $\lambda\chi_M^{-1}$ of M, and $s(\pi)$ is the number of finite inert places v where $\pi_v \simeq \pi_{s,v}$.

Claim. *For any CM extension M/F and any CM type Φ, there exists an algebraic Hecke character λ of weight 1 and CM type Φ.*

Proof. Consider the character on M_∞^* given by $\lambda_\infty(z) = \prod_{v\in\Phi} \frac{\bar{z}_v}{|z_v|}$. Since M/F is a CM extension, the index m of $(\mathcal{O}_F^*)^2$ in $\mathcal{O}_M^*$ is finite, and λ_∞ is trivial on $(\mathcal{O}_M^*)^m$. By [C, Théorème 1] there exists an open compact subgroup U of $\mathbb{A}_{M,f}$ such that $U \cap \mathcal{O}_M^* \subset (\mathcal{O}_M^*)^m$, hence λ_∞ can be extended (trivially) to $M^*UM_\infty^*$. Finally, since $\mathbb{A}_M^*/M^*UM_\infty^*$ is a finite abelian (class) group, there exists a character λ of $\mathbb{A}_M^*/M^*$ extending λ_∞. ☐

The algebraic Hecke character $(\lambda\circ\rho)/\lambda$ has trivial restriction to F, hence is the base change of a weight -1 algebraic character of $\mathrm{U}(1)$, namely the restriction $\lambda_{|\mathrm{U}(1)}^{-1}$.

Let $\Pi(\lambda)$ be the global Arthur packet associated to a pair of characters $(\lambda, \lambda_{|\mathrm{U}(1)}^{-1})$ as in the above Claim. Fix a place v_0 of F, inert in M which is relatively prime to the conductor of λ and consider the open compact subgroup $K(\lambda) = \prod_v K(\lambda)_v$ of $G(\mathbb{A}_{F,f})$ such that $\pi_{n,v}^{K(\lambda)_v} \neq 0$ for all finite places $v \neq v_0$, with $K(\lambda)_v$ being the standard hyperspecial maximal compact for all $v \neq v_0$ relatively prime to the conductor of λ and $\pi_{s,v_0}^{K(\lambda)_{v_0}} \neq 0$ (one can take $K(\lambda)_{v_0}$ to be an Iwahori subgroup).

Denote by $\mathcal{P}$ the set of rational primes p which are totally split in F and such that every primes $\mathfrak{p}$ of F dividing p remains inert in M. Note that $\mathcal{P}$ is infinite, since M is a CM extension of F. Let $p \in \mathcal{P}$ be relatively prime to $2v_0$ and to the conductor of λ and denote by ε_p the unique quadratic Dirichlet character ramified only at p.

The quadratic Hecke character $\varepsilon_p \circ \mathrm{N}_{M/\mathbb{Q}}$ is trivial on F hence equal to the base change $\nu_{p,M}$ of a quadratic character ν_p of $\mathrm{U}(1)$, ramified only at places above p.

Consider the global Arthur packet Π^p attached to $(\lambda, \lambda_{|\mathrm{U}(1)}^{-1}\nu_p)$.

Claim. *For every prime $\mathfrak{p}$ of F dividing p, the representation $\pi_{n,\mathfrak{p}} \in \Pi_{\mathfrak{p}}^{p}$ has non-zero invariants under the open compact subgroup:*

$$K_0^1(\mathfrak{p}) = \left\{ k \in G(\mathcal{O}_{F,\mathfrak{p}})\,,\; k \equiv \begin{pmatrix} * & * & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix} \pmod{\mathfrak{p}} \right\}.$$

Proof. By [R1, §12.2] the representation of $G(F_{\mathfrak{p}})$ obtained by parabolic induction of the character

$$(\alpha, \beta) \mapsto \lambda_{\mathfrak{p}}(\alpha\beta^{-1})|\alpha|_{M_{\mathfrak{p}}}^{3/2}\nu_{p,\mathfrak{p}}(\beta)$$

of the standard torus $M_{\mathfrak{p}}^{*} \times \mathrm{U}(1)(F_{\mathfrak{p}})$ has $\pi_{n,\mathfrak{p}}$ as a unique irreducible quotient (the irreducible sub-representation being $\pi_{s,\mathfrak{p}}$). Since $\lambda_{\mathfrak{p}}$ is unramified and $\nu_{p,\mathfrak{p}}$ is tamely ramified, the claim follows by explicit computation [DR]. □

Consider an element $\pi^p = \otimes_v \pi_v^p \in \Pi^p$ such that $\pi_\iota^p = \pi^+$, $\pi_v^p = 1$ for every infinite $v \neq \iota$, $\pi_v^p = \pi_{n,v}$ for every finite $v \neq v_0$, and finally:

$$\pi_{v_0}^p = \begin{cases} \pi_{n,v_0} & \text{, if } W(\lambda^3\nu_{p,M}^{-1}) = (-1)^{d-1}, \textit{and} \\ \pi_{s,v_0} & \text{, if } W(\lambda^3\nu_{p,M}^{-1}) = (-1)^{d}. \end{cases} \tag{7}$$

By (6) π^p is automorphic, and by the claim $(\pi_f^p)^{K(\lambda,p)} \neq 0$, where $K(\lambda,p)$ is the subgroup of $K(\lambda)$ with the maximal compact $K(\lambda)_{\mathfrak{p}}$ being replaced by $K_0^1(\mathfrak{p})$ at all places $\mathfrak{p}$ of F dividing p.

Consider a strictly increasing sequence $(p_i)_{i\geq 1}$ of elements $p \in \mathcal{P}$ as above and for each $r \geq 1$ denote by X_r the adelic quotient (4) corresponding to the open compact subgroup $\bigcap_{i=1}^{r} K(\lambda, p_i)$. By (5) the automorphic representations $(\pi^{p_i})_{1\leq i\leq r}$ all contribute to $q(X_r)$ and since they are pairwise distinct we have that $q(X_r) \geq r$.

Although X_r may have several connected components, it is important to observe that the connected components remain constant as r grows, since by the above claim $\det(K(\lambda,p)) = \det(K(\lambda))$ for all $p \in \mathcal{P}$. The proposition then follows immediately. □

Remark 2.4. (a) The computation of an exact level K at which the automorphic representation π attached to a pair of characters (λ, χ) contributes to the H^1 is analyzed in detail in [DR] when M is an imaginary quadratic field. In particular, if λ is ramified only at the at the primes p dividing the discriminant of M and with minimal ramification there, we check that the level subgroup at any such p is precisely the one conjectured by B. Gross, namely the index 2 subgroup of the maximal parahoric subgroup with reductive quotient PGL(2).

(b) One consequence of Rogawski's theory is that the Albanese variety is of CM type for any congruence subgroup (see [MR]). If M is imaginary quadratic of discriminant a prime p, we will show in [DR] that the factor of the Albanese corresponding to the minimally ramified λ turns out (at an appropriate prime to p level) to be isogenous to the CM abelian variety $B(p)$ defined by B. Gross in [G].

(c) When Γ is not a congruence subgroup, there are examples of C. Schoen where the Albanese is not of CM type (see [Sc]).

3. Mordellicity of X

We will deduce our main Theorem from a more general Proposition which is a consequence of the following powerful *result of Faltings* on the rational points of subvarieties of abelian varieties.

Theorem F (Faltings [F2], [V])**.** *Suppose A is an abelian variety over a number field k, $Z \subset A$ a closed subvariety. Then there are finitely many translates Z_i of k-rational abelian subvarieties of A, such that $Z_i \subset Z$, and such that each k-rational point of Z lies on one of the Z_i.*

Proposition 3.1. *Let X be a smooth projective surface over a number field k which is geometrically irreducible and does not admit a dominant map to its Albanese variety. Then $X(k)$ is not Zariski dense in X.*

Moreover, if X does not contain curves of geometric genus at most one, then $X(k)$ is finite.

Proof of Proposition 3.1 . If $X(k)$ is empty, there is nothing to prove. Otherwise, use a k-rational point of X to define the Albanese map over k:

$$j : X \to \mathrm{Alb}(X).$$

Then $Z = j(X)$ is a closed, irreducible subvariety of $\mathrm{Alb}(X)$. Applying Faltings' Theorem F with $A = \mathrm{Alb}(X)$, we get a finite number, say $m \geq 1$, of k-rational translates Z_i of abelian subvarieties of $\mathrm{Alb}(X)$ such that

$$Z(k) \subset \bigcup_{i=1}^{m} Z_i(k) \quad \text{and} \quad Z_i \subset Z.$$

Since the Albanese map is defined over k, all the k-rational points of X are contained in those $j^{-1}(Z_i)$. Each $j^{-1}(Z_i)$ is either a point or a curve (possibly singular and reducible), since otherwise, the irreducibility of X would imply that $Z = Z_i = \mathrm{Alb}(X)$, contradicting the assumption that X does not admit a dominant map to its Albanese variety.

To show that $X(k)$ is finite it suffices to show the finiteness of k-rational points on each curve C occurring as irreducible component of $j^{-1}(Z_i)$ for some i. If $C(k)$ is infinite, then it would be a Zariski dense subset of C, implying that C is geometrically irreducible. Now by Faltings's celebrated proof of Mordell's conjecture [F1], the genus of the normalization $\tilde{C}$ of C is ≤ 1, contradicting the assumption made on X. □

Let X_Γ be as in the main Theorem. By Proposition 2.3 there exists a finite index subgroup Γ' of Γ such that $q(X_{\Gamma'}) > 2$, which can be assumed to be normal. It follows that $X_{\Gamma'}$ cannot admit a dominant map to its Albanese variety. Moreover $X_{\Gamma'}$ is a geometrically irreducible projective surface and, by Lemma 1.2(i), $X_{\Gamma'}(\mathbb{C})$ is a smooth hyperbolic manifold. Finally, if $X_{\Gamma'}$ contains contains a curve C, whose normalization $\tilde{C}$ is of genus $g \leq 1$, we will get a non-constant holomorphic map from $\mathbb{C}$ to $X_{\Gamma'}$ which is impossible by hyperbolicity. Therefore by Proposition 3.1 $X_{\Gamma'}(\mathbb{C})$ is Mordellic.

By Lemma 1.2(ii) the natural morphism $f : X_{\Gamma'} \to X_\Gamma$ is finite, etale and defined over a number field k. Denote S the finite set of places of k where f ramifies. Then, for any given number field $k' \supset k$,

$$f^{-1}(X_\Gamma(k')) \subset \bigcup_{k''} X_{\Gamma'}(k'')$$

where k'' runs over all field extensions of k' of degree at most the degree of f which are unramified outside S. Since $X_{\Gamma'}$ is Mordellic and there are only finitely many such extensions k'' (Hermite-Minkowski), it follows that $X_\Gamma(k')$ is finite, hence X_Γ is Mordellic. □

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