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COMPUTATION OF SURFACE WAVE DISPERSION FOR MULTILAYERED ANISOTROPIC MEDIA

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ABSTRACT

With the program described in this paper it is now possible to compute surface wave dispersion in a solid heterogeneous halfspace containing up to 200 anisotropic layers.

Certain discrepancies in surface wave observations, such as disagreement between Love and Rayleigh wave data and other independent evidence, suggest that anisotropy may be important in some seismological problems. In order to study the effect of anisotropy on surface wave dispersion a program was written for an IBM 7090 computer which will compute dispersion curves and displacements for Rayleigh waves in a layered halfspace in which each layer is transversely isotropic. A simple redefinition of parameters makes it possible to use existing programs to compute Love wave dispersion.

Introduction

The techniques described in previous papers (Dorman *et al.*, 1960; Press *et al.*, 1961) for computing dispersion of surface waves in multilayered isotropic media have been extended so that each layer may be transversely isotropic with an axis of symmetry perpendicular to the layering. The motivation for this extension is the suggestion (Anderson, 1961) that anisotropy may be responsible for certain discrepancies in surface wave data. These discrepancies include disagreement between Love wave and Rayleigh wave studies of both velocity and particle motion. The basic theory is developed in such a way that direct comparison with Haskell's (1953) original work on isotropic layers is possible at each stage. Final mathematical results are in the form of matrix elements which replace those of Haskell and are arranged in a manner convenient for computation.

Rayleigh Waves

As in the isotropic case (Haskell, 1953; Dorman *et al.*, 1960) the theoretical basis for the computation of Rayleigh type surface wave dispersion in a layered anisotropic halfspace is the vector equation relating the motion-stress vectors at the top and the bottom of each layer

$$\mathbf{U}_m = D_m E^{-1} \mathbf{U}^*_{m-1} \tag{1}$$

where $\mathbf{U}_m = (\dot{u}_m/c,\ \dot{w}_m/c,\ \sigma_m,\ \tau_m)$. The quantities $\dot{u}_m$ and $\dot{w}_m$ are the horizontal and vertical particle velocities associated with Rayleigh type motion. σ_m and τ_m are the normal and tangential stresses. The asterisk on the $\mathbf{U}^*_{m-1}$ vector indicates the side of the m^{th} layer next to the $(m-1)^{\text{th}}$ layer. D_m and E_m are given in Anderson (1961).

Since the boundary conditions and the form of the vector equation (1) are the same as for Haskell's (1953) isotropic layer formulation, we can make use of the matrix algebra performed by Haskell in obtaining the period equation. Thus all that needs to be done is to evaluate E_m^{-1}, which is the inverse of E_m, and the matrix product $D_m E_m^{-1}$, and then to give the period equation in terms of the matrix elements which are to replace Haskell's.

Taking the matrix relation (44) from Anderson (1961) and arranging columns as in Haskell (1953) we have

$$E_m = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & i\gamma_{1_m} & 0 & i\gamma_{2_m} \\ iX_{\alpha_m} & 0 & iX_{\beta_m} & 0 \\ 0 & Y_{\alpha_m} & 0 & Y_{\beta_m} \end{bmatrix} \tag{2}$$

where

$$X_{\alpha_m} = C_m\gamma_{1_m}r_{\alpha_m} + iF_m$$

$$X_{\beta_m} = C_m\gamma_{2_m}r_{\beta_m} + iF_m$$

$$Y_{\alpha_m} = L_m(r_{\alpha_m} - i\gamma_{1_m})$$

$$Y_{\beta_m} = L_m(r_{\beta_m} - i\gamma_{2_m})$$

The above E_m differs from Anderson's (1961) by the common factor ik which has been factored out and by two new quantities r_{α_m} and r_{β_m} defined by the relations

$$r_{\alpha_m} = \frac{\nu_{1_m}}{ik} \quad \text{and} \quad r_{\beta_m} = \frac{\nu_{2_m}}{ik} \tag{3}$$

which have the same sign criteria as Haskell's (1953) r_{α_m} and r_{β_m} or

$$r_{\alpha_m,\,\beta_m} = |\, r_{\alpha_m,\,\beta_m} \,| \qquad \text{for values of } c \text{ such that } r^2_{\alpha_m,\,\beta_m} > 0$$

$$r_{\alpha_m,\,\beta_m} = -i\,|\, r_{\alpha_m,\,\beta_m} \,| \qquad \text{for values of } c \text{ such that } r^2_{\alpha_m,\,\beta_m} < 0$$

For the isotropic case $r_{\alpha_m,\,\beta_m}$ reduce to Haskell's.

Rearranging Anderson's (1961) equation (46) to be consistent with E_m we have

$$D_m = \begin{bmatrix} \cos P_m & i \sin P_m & \cos Q_m & i \sin Q_m \\ -\gamma_{1_m} \sin P_m & i\gamma_{1_m} \cos P_m & -\gamma_{2_m} \sin Q_m & i\gamma_{2_m} \cos Q_m \\ iX_{\alpha_m} \cos P_m & -X_{\alpha_m} \sin P_m & iX_{\beta_m} \cos Q_m & -X_{\beta_m} \sin Q_m \\ iY_{\alpha_m} \sin P_m & Y_{\alpha_m} \cos P_m & iY_{\beta_m} \sin Q_m & Y_{\beta_m} \cos Q_m \end{bmatrix} \tag{4}$$

where

$$P_m = kr_{\alpha_m}\, d_m \quad \text{and} \quad Q_m = kr_{\beta_m}\, d_m$$

The inverse of E_m is then given by

$$E_m^{-1} = \begin{bmatrix} -a_m X_{\beta_m} & 0 & -ia_m & 0 \\ 0 & -ib_m Y_{\beta_m} & 0 & -b_m \gamma_{2_m} \\ a_m X_{\alpha_m} & 0 & ia_m & 0 \\ 0 & ib_m Y_{\alpha_m} & 0 & b_m \gamma_{1_m} \end{bmatrix} \tag{5}$$

where

$$a_m = \frac{1}{X_{\alpha_m} - X_{\beta_m}} \quad \text{and} \quad b_m = \frac{1}{\gamma_{1_m} Y_{\beta_m} - \gamma_{2_m} Y_{\alpha_m}}$$

and therefore by (2)

$$a_m = \frac{1}{C_m(\gamma_{1_m} r_{\alpha_m} - \gamma_{2_m} r_{\beta_m})} \tag{6}$$

and

$$b_m = \frac{1}{L_m(\gamma_{1_m} r_{\beta_m} - \gamma_{2_m} r_{\alpha_m})}$$

and by Anderson's (1961) equation (13) we have

$$\begin{aligned} \gamma_{1_m} &= i \frac{r_{\alpha_m} G_m}{(\rho_m c^2 - L_m - C_m r_{\alpha_m}^2)} = i g_{1m} r_{\alpha m} \\ \gamma_{2_m} &= i \frac{r_{\beta_m} G_m}{(\rho_m c^2 - L_m - C_m r_{\beta_m}^2)} = i g_{2_m} r_{\beta_m} \end{aligned} \tag{7}$$

which are real or imaginary depending on whether r_{α_m, β_m} are imaginary or real, since $g_{1_m, 2_m}$ are real for $r^2_{\alpha_m, \beta_m}$ real. From the above we see that X_{α_m} and X_{β_m} are always pure imaginary and therefore a_m is also imaginary. Thus we will define the real quantities $X^*_{\alpha_m}$, $X^*_{\beta_m}$ and $a_m{}^*$ by

$$X_{\alpha_m} = iX^*_{\alpha_m}, \qquad X_{\beta_m} = iX^*_{\beta_m} \quad \text{and} \quad a_m = ia_m{}^* \tag{8}$$

or

$$a_m{}^* = \frac{1}{X^*_{\beta_m} - X^*_{\alpha_m}}$$

Also from the above

$$\begin{aligned} Y_{\alpha_m} &= \rho_m \beta_m{}^2 r^2_{\alpha_m}(1 + g_{1_m}) \\ Y_{\beta_m} &= \rho_m \beta_m{}^2 r^2_{\beta_m}(1 + g_{2_m}) \end{aligned} \tag{9}$$

where $\beta_m^2 = L_m/\rho_m$; β_m being the velocity of horizontally traveling SV motion in the m^{th} layer.
therefore

$$b_m = -i \frac{1}{\rho_m \beta_m^2 r_{\alpha_m} r_{\beta_m} g_{12_m}} \tag{10}$$

where

$$g_{12_m} = g_{1_m} - g_{2_m}$$

Carrying out the matrix multiplication for $\boldsymbol{a}_m = D_m E_m^{-1}$ we have as the elements of this product

$$(\boldsymbol{a}_m)_{11} = a_m^*[X^*_{\beta_m} \cos P_m - X^*_{\alpha_m} \cos Q_m]$$

$$(\boldsymbol{a}_m)_{12} = i\left[-\frac{(1+g_{2_m})}{g_{12_m}}\left(\frac{\sin P_m}{r_{\alpha_m}}\right) + \frac{(1+g_{1_m})}{g_{12_m}}\left(\frac{\sin Q_m}{r_{\beta_m}}\right)\right]$$

$$(\boldsymbol{a}_m)_{13} = a_m^*[\cos P_m - \cos Q_m]$$

$$(\boldsymbol{a}_m)_{14} = i\left[-\frac{g_{2_m}}{\rho_m \beta_m^2 g_{12_m}}\left(\frac{\sin P_m}{r_{\alpha_m}}\right) + \frac{g_{1_m}}{\rho_m \beta_m^2 g_{12_m}}\left(\frac{\sin Q_m}{r_{\beta_m}}\right)\right]$$

$$(\boldsymbol{a}_m)_{21} = ia_m^*[-X^*_{\beta_m} g_{1_m}(r_{\alpha_m} \sin P_m) + X^*_{\alpha_m} g_{2_m}(r_{\beta_m} \sin Q_m)]$$

$$(\boldsymbol{a}_m)_{22} = \frac{g_{1_m}(1+g_{2_m})}{g_{12_m}} \cos P_m - \frac{g_{2_m}(1+g_{1_m})}{g_{12_m}} \cos Q_m$$

$$(\boldsymbol{a}_m)_{23} = ia_m^*[-g_{1_m}(r_{\alpha_m} \sin P_m) + g_{2_m}(r_{\beta_m} \sin Q_m)]$$

$$(\boldsymbol{a}_m)_{24} = \frac{g_{1_m} g_{2_m}}{\rho_m \beta_m^2 g_{12_m}} [\cos P_m - \cos Q_m]$$

$$(\boldsymbol{a}_m)_{31} = -a_m^* X^*_{\alpha_m} X^*_{\beta_m}[\cos P_m - \cos Q_m]$$

$$(\boldsymbol{a}_m)_{32} = i\left[X^*_{\alpha_m} \frac{(1+g_{2_m})}{g_{12_m}}\left(\frac{\sin P_m}{r_{\alpha_m}}\right) - X^*_{\beta_m} \frac{(1+g_{1_m})}{g_{12_m}}\left(\frac{\sin Q_m}{r_{\beta_m}}\right)\right]$$

$$(\boldsymbol{a}_m)_{33} = a_m^*[-X^*_{\alpha_m} \cos P_m + X^*_{\beta_m} \cos Q_m]$$

$$(\boldsymbol{a}_m)_{34} = i\left[X^*_{\alpha_m} \frac{g_{2_m}}{\rho_m \beta_m^2 g_{12_m}}\left(\frac{\sin P_m}{r_{\alpha_m}}\right) - X^*_{\beta_m} \frac{g_{1_m}}{\rho_m \beta_m^2 g_{12_m}}\left(\frac{\sin Q_m}{r_{\beta_m}}\right)\right]$$

$$(\boldsymbol{a}_m)_{41} = ia_m^*[X^*_{\beta_m} \rho_m \beta_m^2 (1+g_{1_m})(r_{\alpha_m} \sin P_m) - X^*_{\alpha_m} \rho_m \beta_m^2 (1+g_{2_m})(r_{\beta_m} \sin Q_m)]$$

$$(\boldsymbol{a}_m)_{42} = -\rho_m \beta_m{}^2 \frac{(1 + g_{1_m})(1 + g_{2_m})}{g_{12_m}} [\cos P_m - \cos Q_m]$$

$$(\boldsymbol{a}_m)_{43} = i a_m{}^* \rho_m \beta_m{}^2 [(1 + g_{1_m})(r_{\alpha_m} \sin P_m) - (1 + g_{2_m})(r_{\beta_m} \sin Q_m)]$$

$$(\boldsymbol{a}_m)_{44} = -\frac{(1 + g_{1_m})}{g_{12_m}} g_{2_m} \cos P_m + \frac{(1 + g_{2_m})}{g_{12_m}} g_{1_m} \cos Q_m$$

For the isotropic case we have

$$Y_{\alpha_m} = \rho_m c^2 r_{\alpha_m} \gamma_m \qquad Y_{\beta_m} = -\frac{\rho_m}{r_{\beta_m}} c^2(\gamma_m - 1)$$

$$X^*_{\alpha_m} = -\rho_m c^2(\gamma_m - 1) \qquad X^*_{\beta_m} = -\rho_m c^2 \gamma_m$$

$$a^*_m = -\frac{1}{\rho_m c^2} \qquad g_{1_m} = 1$$

$$g_{2_m} = -\frac{1}{r^2_{\beta_m}} \qquad g_{12_m} = \frac{c^2}{\beta_m{}^2 r^2_{\beta_m}}$$

and substituting these values in $\boldsymbol{a}_m$ we obtain the $\boldsymbol{a}_m$ given by Haskell (1953).

Also the same matrix elements are real or imaginary in this horizontally isotropic case for the $\boldsymbol{a}_m$ matrix as in Haskell's pure isotropic case. Now, carrying through the boundary conditions, i.e. continuity of motion-stress vectors across welded solid to solid interfaces, no sources at infinity in the halfspace, and zero normal and tangential stresses at the free surface, we finally obtain as in Haskell

$$[KN - LM] + [NG - LH][T] = 0 \tag{12}$$

where

$$T = \frac{\sigma_l}{\dot{w}_l / c}$$

and σ_l and $\dot{w}_l$ are the normal stress and the vertical displacement respectively at the solid-liquid interface if a surface liquid layer is present. In that case

$$T = iT^* = i \frac{\rho_l}{r_{\alpha_l}} c^2 \tan P_l \tag{13}$$

If no surface liquid layer is present then $T = 0$ at the free surface and we obtain the same expression as Haskell. In this particular anisotropic case the following quantities are imaginary and are redefined as shown, where $A = \boldsymbol{a}_{n-1} \cdots \boldsymbol{a}_1$

$$A_{jk} = iA^*_{jk} \qquad \text{where } j + k \text{ is an odd integer}$$

$$L = iL^*$$

$$M = iM^*$$

$$G = iG^*$$

and relation (12) for the existence of surface waves becomes

$$F \equiv [KN + L^*M^*] - [G^*N - L^*H][T^*] = 0 \tag{14}$$

where

$$K = a_n{}^* X^*_{\beta_n} | r_{\alpha_n} | A^*_{12} - \frac{(1 + g_{2_n})}{g_{12_n}} A_{22} + a_n{}^* | r_{\alpha_n} | A^*_{32} - \frac{g_{2_n}}{\rho_n \beta_n{}^2 g_{12_n}} A_{42}$$

$$L^* = -a_n{}^* X^*_{\beta_n} | r_{\alpha_n} | A_{11} - \frac{(1 + g_{2_n})}{g_{12_n}} A^*_{21} - a_n{}^* | r_{\alpha_n} | A_{21} - \frac{g_{2_n}}{\rho_n \beta_n{}^2 g_{12_n}} A^*_{41}$$

$$M^* = -a_n{}^* X^*_{\alpha_n} A^*_{12} - a_n{}^* A^*_{32} + \frac{(1 + g_{1_n})}{g_{12_n} | r_{\beta_n} |} A_{22} + \frac{g_{1_n}}{\rho_n \beta_n{}^2 g_{12_n} | r_{\beta_n} |} A_{42}$$

$$G^* = -a_n{}^* X^*_{\beta_n} | r_{\alpha_n} | A_{13} - \frac{(1 + g_{2_n})}{g_{12_n}} A^*_{23} - a_n{}^* | r_{\alpha_n} | A_{32} - \frac{g_{2_n}}{\rho_n \beta_n{}^2 g_{12_n}} A^*_{42}$$

$$H = -a_n{}^* X^*_{\alpha_n} A_{13} - a_n{}^* A_{33} - \frac{(1 + g_{1_n})}{g_{12_n} | r_{\beta_n} |} A^*_{23} - \frac{g_{1_n}}{\rho_n \beta_n{}^2 g_{12_n} | r_{\beta_n} |} A^*_{43}$$

$$N = -a_n{}^* X^*_{\alpha_n} A_{31} - \frac{(1 + g_{1_n})}{g_{12_n} | r_{\beta_n} |} A^*_{21} - \frac{g_{1_n}}{\rho_n \beta_n{}^2 g_{12_n} | r_{\beta_n} |} A^*_{41}$$

and where the subscript n represents the halfspace.

Again, for the case of isotropic media, these relations reduce to the ones defined in Haskell (1953).

Love Waves

It is shown in Anderson (1962) that upon substitution of the following pseudo-parameters

$$d_m' = \left(\frac{N_m}{L_m}\right)^{1/2} d_m$$

and

$$\mu_m' = (L_m \cdot N_m)^{1/2} \tag{16}$$

into the motion-stress vector equation matrix $\boldsymbol{a}_m{}^L$ the matrix is equivalent to the isotropic matrix for Love waves (Haskell, 1953) where d_m, the m^{th} layer thickness, is replaced by d_m' and μ_{m_1} the m^{th} layer isotropic rigidity, is replaced by μ_m'. The quantities N_m and L_m are the m^{th} layer directional rigidities as defined in Anderson (1961) for horizontally isotropic layers. The isotropic layer is a special case of the above where $N_m = L_m = \mu_m$ and we have $d_m' = d_m$ and $\mu_m' = \mu_m$. Expressions in Haskell's Love formulation containing the isotropic shear velocity are the same in this anisotropic case with shear velocity being replaced by the shear velocity of horizontally traveling SH waves.

With the substitution of these pseudo-parameters in the halfspace relations, the period equation for Love type surface waves in $(n - 1)$ horizontally isotropic layers overlying a horizontally isotropic halfspace is identical with the period equation obtained in Haskell and used as the basis of calculation of Love wave dispersion in the computer program described in Press *et al.* (1961).

Therefore for the calculation of Love type dispersion in multilayered horizontally isotropic media, one simply substitutes the pseudo-parameters defined above for the individual layer rigidities and thicknesses into a program for calculating isotropic Love wave dispersion based on the Haskell matrix formulation such as the one described in Press *et al.* (1961).

Method of Computation

Unfortunately, for the general case of Rayleigh type dispersion, it is not possible to define pseudo-parameters and use a previous program for isotropic layers as in Love dispersion. Therefore a new program was written for horizontally isotropic layers. This program is written in FORTRAN for the IBM 7090 and can be used to find the dispersion (phase velocity versus period) and displacement at each interface for Rayleigh type motion in problems involving up to 200 solid horizontally isotropic layers over a solid horizontally isotropic halfspace. In addition, a liquid layer can be inserted at the free surface above this array. Since the case of a three-dimensionally isotropic layer is a special case of the horizontally isotropic layer where the vertical and horizontal elastic constants are equal, it is possible to use this program in problems involving a mixture of isotropic and horizontally isotropic layers.

The general computational procedure is to find the zeroes of the F function (14), which is a function of phase velocity, wave number, and the four elastic constants of the layers. The program flow is similar to the program described in Press *et al.* (1961), in that the zeroes of the F function are determined by initially specifying the phase velocity and a trial value of the wave number k. The elements of the $\boldsymbol{a}_m$ matrix are formed for each layer and then multiplied by the matrix for the layer below it, starting with the solid layer nearest the free surface. After the matrix product for all the layers have been calculated, the quantities for the halfspace are calculated and combined with the layer product matrix elements to obtain the quantities defined in (15). If a liquid surface layer is present, T^* is calculated next. These numerical values are then combined to obtain a numerical value of the F function (14). The F function is formed so that its value is positive for k's less than the root in the gravest mode. New trial values of k (of increasing or decreasing size

depending on the sign of the initial F value) specified by Δk, which is an input parameter, are used to calculate new values of the F function until a change in sign of F is detected. Linear interpolation and extrapolation are then repeatedly used to find smaller F values until k's of different F sign are within the precision interval desired. The resulting interpolated value of k is the root to the required precision.

The root k is then used to recalculate the elements used to evaluate F. These new elements are combined to form the horizontal to vertical particle velocity amplitude ratio at the free surface or, if a liquid surface layer is present, to form the same ratio in the solid at the liquid solid interface. The group velocity is computed by numerical differentiation of the phase velocity values. This is done by perturbing the phase velocity by a small amount and then finding a new k root to be used in the difference relation. If displacements at depth are desired, the program will now calculate the particle velocity ratios at each interface normalized to the surface vertical velocity by iterative use of (1) using the previously calculated horizontal to vertical surface ratio as the initial value.

The phase velocity is decremented by an input specified amount, and then a new k using the previous root as a trial k and group velocity are calculated by repeating the above procedure. The order of normal mode calculated is controlled by an input which determines the sign of F for k's less than the root and the initial trial k and Δk being in a region such that the root finding process gives the desired mode root.

Some Programming Problems

As in Press *et al.* (1961), numbers of the order of $\exp\left(k\sum_{i=1}^{n-1} d_i\right)$ where d_i is the i^{th} layer thickness, are involved in the calculation of F. Therefore as the phase velocity decreases and the values of k increase, the larger root values of k will lead to machine overflow, even in floating point arithmetic, if the total number of layers remains constant. Thus when the elements in the F factor become large enough to cause a machine overflow, the program will automatically reduce the number of layers used in the calculation, starting with the halfspace and proceeding toward the free surface. This process will continue to eliminate layers with shear velocity greater than the phase velocity until the elements in the F function no longer cause an overflow. Then before proceeding to calculate the new root, the program will recalculate the root for the preceding phase velocity in order to verify that no loss in precision of k was caused by layer reduction.

For isotropic layers, the quantities r_{α_m} and r_{β_m} are either real or imaginary for all real values of phase velocity. However, as will be discussed in a later paper, for even very slight departures from isotropy, there exist regions of real phase velocity for which r_{α_m, β_m} will be complex for layers in the array. Unfortunately these regions may occur for phase velocities which would be part of the real dispersion curve if all the layers were completely isotropic. Computing complex dispersion for more than one layer is extremely difficult and therefore was not attempted in this program. Therefore, when a phase velocity is such that r_{α_m, β_m} becomes complex for one of the layers, the machine automatically converts this layer and all other layers in the array for which r_{α_m, β_m} is complex to equivalent isotropic layers. The

new isotropic layers use the horizontal compressional and the horizontal SV velocities for their compressional and shear velocities respectively. The program then recalculates the root for the preceding phase velocity in order to estimate the degree of error in making the anisotropic layers isotropic. If the error is large the case can be re-run with a greater number of layers approximating a given velocity gradient.

Unlike the program described in Press *et al.* (1961), there is no double precision

```
ANISOTROPIC RAYLEIGH WAVE DISPERSION
 SWX-3-A

N=    21 DELT=  0.0000010 FK=    0.00100 FKD=    0.00500
NW=    -0 DKS=    1.0 CPERT=  1.00100

          D          ALPHA          BETA           RHO           PHI          NADA
       5.00         1.2040        0.2600        2.0000        1.0000        1.0000
       5.00         1.4880        0.4490        2.0000        1.0000        1.0000
       5.00         1.3930        0.7320        2.0000        1.0000        1.0000
       5.00         1.6060        1.0000        2.2500        1.0000        1.0000
       5.00         1.7000        1.0860        2.2500        0.9060        0.9500
       5.00         1.7000        1.0630        2.2500        0.9060        0.9500
       5.00         1.7950        1.0630        2.2500        0.9060        0.8400
       5.00         1.7950        0.9450        2.2500        0.9060        0.8400
       5.00         1.8660        1.0000        2.2500        0.6280        0.8400
       5.00         1.8660        1.0630        2.2500        0.6280        0.8400
       5.00         2.0000        1.4170        2.2500        0.6280        0.7520
       5.00         2.2440        1.4170        2.2500        0.6280        0.7520
       5.00         3.8970        1.6060        2.5000        0.9500        0.9300
       5.00         5.4080        1.6530        2.5000        0.9500        0.9300
       5.00         5.4080        1.6300        2.5000        0.8500        0.9300
       5.00         5.4080        1.6300        2.5000        0.8500        0.9300
       5.00         5.6920        2.0310        2.5000        0.8500        0.9000
       5.00         5.5970        1.7710        2.5000        0.8500        0.9000
       5.00         5.5030        1.8420        2.5000        0.9000        0.9300
       5.00         6.1170        1.7240        2.5000        0.9000        0.9300
      -0.           6.2350        2.0000        2.6000        0.9000        0.9300

     KDI          T             C             U             W*          N      SUM DI
  0.03894    448.220       1.8000        1.6376    -0.11380E 01     21      100.00
  0.05942    311.020       1.7000        1.3687    -0.15825E 01     21      100.00
  0.07602    258.299       1.6000        1.1176    -0.19655E 01     21      100.00
  0.09160    228.635       1.5000        0.9161    -0.22823E 01     21      100.00
  0.10787    208.024       1.4000        0.7670    -0.25464E 01     21      100.00
AOB2 IS NEGATIVE, I =      15
AOB2 IS NEGATIVE,LAYER I=      15 NOW ON ISO
AOB2 IS NEGATIVE,LAYER I=      16 NOW ON ISO
  0.10808    207.614       1.4000        0.7700    -0.25682E 01     21      100.00
  0.12650    191.040       1.3000        0.6637    -0.28195E 01     21      100.00
  0.14851    176.289       1.2000        0.5930    -0.31205E 01     21      100.00
  0.17656    161.755       1.1000        0.5534    -0.36153E 01     21      100.00
  0.21709    144.714       1.0000        0.8958    -0.45742E 01     21      100.00
  0.27699    126.019       0.9000        0.5565    -0.10734E 02     21      100.00
AOB2 IS NEGATIVE, I =       9
AOB2 IS NEGATIVE,LAYER I=       9 NOW ON ISO
AOB2 IS NEGATIVE,LAYER I=      10 NOW ON ISO
  0.28951    120.573       0.9000        0.5418    -0.27501E 02     21      100.00
  0.40105     97.919       0.8000        0.5279    -0.56306E 01     21      100.00

LAST OUTPUT FOR THIS DATA
```

FIG. 1. Output format for anisotropic Rayleigh wave dispersion calculation.

feature built into the program to handle a loss of significance in root differences for group velocity calculations. This error may occur in regions where k is a slowly varying function of phase velocity and can be corrected by increasing the phase velocity perturbation factor used in the group velocity calculation. Therefore all group velocities in regions of slowly varying k or period should be graphically verified.

RESULTS

For 20 horizontally isotropic layers overlying a halfspace, the program will calculate 10 values of phase and group velocity in roughly 2 minutes.

Figure 1 shows a sample calculation for 20 layers. At two phase velocities r_{α_m, β_m}

became complex and for the rest of the calculations the layers with complex r_{α_m}, β_m were assumed isotropic. These were layers 15 and 16 at a phase velocity of 1.4, and layers 9 and 10 at a phase velocity of 0.9. Figure 2 shows the results for a displacement at depth calculation.

```
ANISØTRØPIC RAYLEIGH WAVE DISPERSIØN
 SWX-3F

N=        4 DELT=   0.0000010 FK=     0.01000 FKD=     0.00500
NW=      -0 DKS=    1.0 CPERT=   1.00100
             D          ALPHA          BETA          RHØ          PHI          NADA
          20.00        1.7000        0.6100       2.0000       0.8300       0.9100
          40.00        2.6000        1.1400       2.2500       0.7500       0.8300
          40.00        6.5000        1.7400       2.5000       0.8300       0.9100
          0.           7.2000        2.0000       2.6000       0.8300       0.9100

    KD1        I           C         U                 W*          N         SUM DI
  0.11134   610.057     1.8500    1.7887         -0.87271E 00      4         100.00

                         DISPLACEMENT CALCULATIØN

                      M       UM/WO             WM/WØ
                      0 -0.87270635E 00  0.09999999E 01
                      1 -0.73212972E 00  0.10603922E 01
                      2 -0.49727625E-00  0.10966630E 01
                      3 -0.30205593E-00  0.11677102E 01

  0.20005   348.980     1.8000    1.6810         -0.11125E 01      4         100.00

                         DISPLACEMENT CALCULATIØN

                      M       UM/WO             WM/WØ
                      0 -0.11125333E 01  0.09999999E 01
                      1 -0.80927506E 00  0.11219122E 01
                      2 -0.39433942E-00  0.11328118E 01
                      3 -0.11585692E-00  0.11921527E 01

  0.27629   259.896     1.7500    1.5499         -0.12607E 01      4         100.00

                         DISPLACEMENT CALCULATIØN

                      M       UM/WO             WM/WØ
                      0 -0.12607308E 01  0.09999999E 01
                      1 -0.79325305E 00  0.11706448E 01
                      2 -0.26131526E-00  0.11057381F 01
                      3  0.21039486E-01  0.11042780E 01

  0.33900   218.055     1.7000    1.4061         -0.13211E 01      4         100.00

                         DISPLACEMENT CALCULATIØN

                      M       UM/WO             WM/WØ
                      0 -0.13211004E 01  0.09999999E 01
                      1 -0.72569170E 00  0.11978140E 01
                      2 -0.14351595E-00  0.10372628E 01
                      3  0.10065123E-00  0.96520985E 00

  0.39281   193.884     1.6500    1.2642         -0.13218E 01      4         100.00

                         DISPLACEMENT CALCULATIØN

                      M       UM/WO             WM/WØ
                      0 -0.13217677E 01  0.09999999E 01
                      1 -0.64051595E 00  0.12071598E 01
                      2 -0.53237729E-01  0.95253973E 00
```

FIG. 2. Output format of displacement at depth calculation.

The notation is as follows:

D	Thickness
ALPHA	Horizontal traveling compressional wave velocity
BETA	Horizontal traveling *SV* wave velocity
RHO	Density
PHI	Squared ratio of vertical to horizontal traveling compressional wave velocity
NADA	Squared ratio of 45° to horizontal traveling compressional wave velocity
KD	Dimensionless wave number with respect to top layer

T	Period in same time units as associated with layer velocities
C, U	Phase and group velocity in same units as layer velocities
W*	Ratio of horizontal to vertical displacement at top of solid nearest free surface
N	Number of layers including halfspace used in computation. Change in N indicates automatic layer reduction is operative.
M	Interface at which displacement is calculated starting at free surface. Repeated number indicates liquid-solid interface.
UM/WO, WM/WO	Horizontal and vertical displacement at M^{th} interface with respect to vertical displacement at free surface (FORTRAN floating point notation)

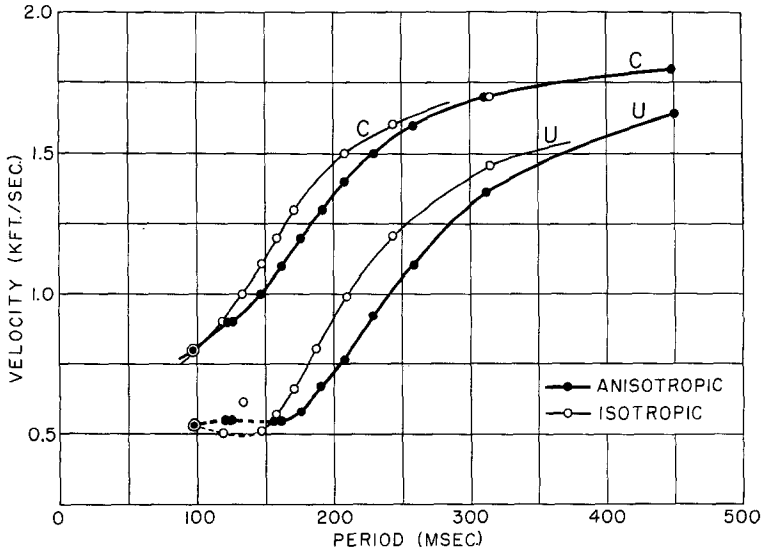

FIG. 3. Sample effect of horizontal isotropy on Rayleigh wave dispersion curves.

The dispersion curves in figure 3 illustrate the effect of horizontal isotropy for the case computed in figure 1. The anisotropic curve is drawn from the values in figure 1, and the isotropic curve is for the same case with the exception that the anisotropy factors PHI and NADA are equal to unity. Although the program assumption of making layers 9 and 10 isotropic appears to give a large error of approximately 6 sec at a phase velocity of 0.9 in figure 1, the actual effect on the phase velocity curve is slight as seen in figure 3. Note the scattering of group velocity points near the group velocity minimum due to a lack of significance in the root k differences used in the group velocity calculation.

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