PHYSICAL REVIEW E **89**, 042901 (2014)

Eliminating $1/f$ noise in oscillators

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(Received 1 November 2013; revised manuscript received 20 March 2014; published 2 April 2014)

We study $1/f$ and narrow-bandwidth noise in precision oscillators based on high-quality factor resonators and feedback. The dynamics of such an oscillator are well described by two variables, an amplitude and a phase. In this description we show that low-frequency feedback noise is represented by a single noise vector in phase space. The implication of this is that $1/f$ and narrow-bandwidth noise can be eliminated by tuning controllable parameters, such as the feedback phase. We present parameter values for which the noise is eliminated and provide specific examples of noise sources for further illustration.

DOI: 10.1103/PhysRevE.89.042901 PACS number(s): 05.45.−a, 84.30.−r

I. INTRODUCTION

Self-sustained oscillators have major technological significance. Such devices, generating a periodic signal at an inherent frequency, are often employed as highly accurate time or frequency references. The basic components of precision oscillators are a resonating element and a feedback loop, usually composed of an amplifier and a phase-shifting element. The amplifier provides the energy necessary for sustaining the oscillations, while the phase shifter ensures the energy injection at a phase that sustains the oscillations, while also providing a convenient control parameter for setting the operating point. Since oscillators are driven through this feedback mechanism, and not by an external clock, they posses a phase invariance property which makes their phase sensitive to stochastic perturbations. The stochastic phase dynamics broadens the peaks in the power spectrum of the oscillator's output and degrades its performance [1].

There are several standard noise sources to consider in feedback oscillators. One is thermal noise associated with the resonator's damping. This noise is spread over a wide frequency range (often "white") [2] and causes uncorrelated fluctuations in both the amplitude and the phase quadratures. The additive noise in the phase quadrature is inversely proportional to the oscillator power and can be suppressed by operating at high amplitudes [3]. The amplitude fluctuations do not contribute to the oscillator phase noise at operating points in which the oscillation frequency and amplitude are independent, and there is no amplitude-phase noise conversion [4,5]. Another source of phase noise might be fluctuations in the natural frequency of the resonator [6]. These fluctuations are expressed as multiplicative noise in the phase quadrature and cannot be removed by tuning parameters or driving the resonator to high amplitudes. Other noise sources originate from the electric components of the feedback loop. These often display a dependency on the frequency, which increases as the frequency decreases, often as a power law close to f^{-1} ("$1/f$ noise"). This type of noise is widely observed in nature and has been the subject of research in a large variety of physical systems [7–9].

In this paper we focus on feedback noise and thus consider $1/f$ noise. Our results also apply to other low-frequency noise sources with a bandwidth that is small compared with the oscillator frequency. Generally, amplifier noise is expressed as fluctuations in both the magnitude and the phase of the feedback. One way to eliminate it was shown by Greywall *et al.* [10]. Their scheme is based on quenching magnitude noise by using a saturated amplifier or a limiter, so that the feedback level is independent of the oscillation amplitude, while simultaneously tuning a parameter such as the feedback phase so that the resonator nonlinearity cancels noise in the feedback phase. For $1/f$ noise we show that it is possible to eliminate feedback noise, even for an unsaturated amplifier. This is important, because a saturated amplifier providing a nonfluctuating feedback level is only an ideal limit, and in practice, driving the amplifier to saturation may degrade the performance. The reason the effect of $1/f$ noise can be eliminated is that in the amplitude-phase description (sometimes described as the rotating-wave approximation), it leads to fluctuations along a particular direction in the complex plane, rather than to a disk of fluctuations. For high-frequency noise (i.e., with a correlation time that is short compared with the oscillator period), the amplitude and phase quadratures of the noise are uncorrelated, since they result from the uncorrelated noise kicks acting on different portions of the limit cycle, leading to the noise disk in the amplitude-phase description. If, on the other hand, the correlation time of the noise is long compared with the period, the noise forces acting on different portions of a period of the limit cycle remain correlated, leading to fluctuations along a line. This intuitive picture is confirmed by our detailed calculations.

Our analysis is made by using a new method for calculating the phase noise of oscillators composed of a high-Q, weakly nonlinear resonator sustained by a feedback loop. This is a common architecture since oscillators are commonly operated near the linear regime of the resonator and the large quality factor Q improves the oscillator precision [3]. We project the noise onto a slow time scale, given by the relaxation time scale of the resonator, and average out the complicated fast-scale dynamics of the resonator-feedback system. This provides great simplification in modeling the noisy feedback system coupled to the resonator. Although the high-Q resonator is characterized by a weakly nonlinear response over a narrow frequency range, the amplifier in the feedback loop will typically have a strongly nonlinear response that is uniform over a wide frequency range. Therefore, the resonator motion is described by a slow envelope function, whereas the amplifier

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is considered to have an instantaneous response to the central periodic signal, over which slow modulations can be neglected. Combining these two approaches allows us to analyze the cyclostationary noise generated by the harmonic production of the nonlinear amplifier and resolve its effect on the ultimately stationary output of the whole oscillator system.

II. EFFECTIVE NOISE OF OSCILLATORS WITH HIGH-Q RESONATORS

The equation of motion for the resonator displacement $x(t)$ with feedback $d(t)$ is

$$\ddot{x}(t) + Q^{-1}\dot{x}(t) + x(t) + \cdots = Q^{-1}d(t), \tag{1}$$

where the ellipsis includes nonlinear terms. The factor Q^{-1} multiplies the feedback since this compensates for the dissipation term $Q^{-1}\dot{x}$ when oscillations occur. We consider noise in the feedback by taking $d \to d + \xi$, with $\xi(t)$ a stochastic variable. We use the method of multiple scales to project the equation of motion onto an equation for the slow modulation of the resonator displacement about a pure oscillation at the linear resonance frequency, an approximation justified by the high-quality factor of the resonator (see, for example, the review by Lifshitz and Cross [11] for a discussion of this approach in the context of driven nanomechanical resonators). Since the derivation of the resonator terms is standard [11], here we focus on the feedback drive and stochastic terms. The signal and the feedback are expressed as

$$x(t) = \frac{1}{2}A(T)e^{it} + \text{c.c.} + \epsilon x_1(t), \tag{2}$$

$$d(t) = \frac{i}{2}g(a(T))e^{i(\Phi(T)+\Delta)}e^{it} + \text{c.c.} + d_1(t), \tag{3}$$

$$\xi(t) = \frac{i}{2}\Xi(T)e^{i(\Phi(T)+\Delta+\Phi_N)}e^{it} + \text{c.c.} + \xi_1(t), \tag{4}$$

with $T = \epsilon t$ the slow time scale with $\epsilon = Q^{-1} \ll 1$, $A = ae^{i\Phi}$, Δ a controlled phase shift, and Φ_N an additional constant phase defined to simplify the noise correlations. In these equations c.c. denotes the complex conjugate. The terms x_1, d_1, and ξ_1, in addition to the slow modulations of the basic oscillation, involve harmonics $e^{\text{int}}, n \neq \pm 1$: these are small relative to the main term in Eq. (2) due to the weak nonlinearity of the resonator but need not be small in Eqs. (3) and (4), although they will have a small effect on the resonator motion since they are nonresonant. The factor of i is introduced in Eqs. (3) and (4) so that the feedback compensates for the energy loss proportional to $\dot{q}$ for a phase shift Δ close to 0.

Substituting Eqs. (2)–(4) into Eq. (1), along with the appropriate scaling of the other terms in the equation, gives, at leading order in ϵ,

$$\ddot{x_1} + x_1 = \left[-\frac{dA}{dT} - \frac{1}{2}A + \cdots + \frac{1}{2}(g + \Xi e^{i\Phi_N})e^{i(\Phi+\Delta)}\right]e^{it} + \text{nonsecular terms} + \text{c.c.} \tag{5}$$

Requiring that the terms in the braces sum to 0 gives an equation for the slow dynamics of the oscillator system

$$\frac{dA}{dT} = f(A) + \frac{1}{2}[g(a) + \Xi e^{i\Phi_N}]e^{i(\Phi+\Delta)}, \tag{6}$$

with $f(A) = -\frac{1}{2}A[1 - \frac{1}{4}(3i - \eta)|A|^2]$, describing a resonator which features a nonlinear elastic force and nonlinear damping [11,12]. We have used similar amplitude equations to study oscillator phase noise due to noise acting directly on the resonator and feedback noise treated phenomenologically [4,13,14]. Here we give a systematic derivation for $1/f$ feedback noise and demonstrate new methods for noise elimination in this case. The term $\Xi = \Xi_R + i\Xi_I$ gives the complex noise in the feedback. In the oscillatory state, the frequency is slightly shifted from the natural frequency of the resonator so $\Phi(T) = \Omega_0 T + \phi(T)$, where ϕ is a stochastic correction induced by the noise. For weak noise, this phase can be approximated as constant over a period $\phi \simeq \phi_0$, and so the noise Ξ defined in Eq. (4) is extracted by the integral

$$\Xi(T) = \frac{e^{-i\Omega_0 T}}{\pi}\int_{\epsilon^{-1}T-\pi}^{\epsilon^{-1}T+\pi} \xi(t)e^{-i\psi_N}e^{-it}dt, \tag{7}$$

with $\psi_N = \Delta + \Phi_N + \pi/2 + \phi_0$ a constant phase.

We consider the oscillator to be affected by a single dominant stationary noise source $\xi_s(t)$ with spectrum $Q_{s_0}(\omega) = \mathcal{F}[\langle\xi_s(t)\xi_s(0)\rangle]$ ($\mathcal{F}$ denotes the Fourier transform), occurring somewhere in the feedback system, which then passes through various of the feedback components and transforms to the feedback noise $\xi(t)$. In cases in which multiple noise sources are active in the oscillator system, our scheme may be used to eliminate the most significant one and reduce the total phase noise. Since both ξ_s and ξ are assumed to be small perturbations, they are related through the linear response function $h(t,t')$ of the time-varying system consisting of the amplifier components between the noise source and the output of the feedback system driven by the periodic output signal,

$$\xi(t) = \int_{-\infty}^{\infty} h(t,t')\xi_s(t')dt', \tag{8}$$

with h a periodic function, unchanged by adding the oscillation period to t and t' [15]. The response function can then be written as the summation

$$h(t,t') = \sum_n e^{\text{in}\phi_0}h_n(t-t')e^{\text{in}\omega_0 t}, \tag{9}$$

with $\omega_0 = 1 + O(\epsilon)$ the oscillation frequency and ϕ_0 the phase of the input signal to the amplifier. The $n \neq 0$ terms in Eq. (9) give the mixing of the noise with the periodic signal. The harmonic transfer functions, defined relative to a zero input phase, are given by the Fourier transform $H_n(\omega) = \mathcal{F}[h_n(t)]$. The noise defined by Eq. (8) is *cyclostationary* with the correlation function

$$\langle\xi(t)\xi(t')\rangle = \sum_n R_n(t-t')e^{\text{in}\omega_0 t}, \tag{10}$$

giving the harmonic power spectral densities (HPSDs) $Q_n(\omega) = \mathcal{F}[R_n(t)]$. The HPSDs are then related to the stationary noise spectrum through [15]

$$Q_l(\omega) = e^{il\phi_0}\sum_n H_n(-\omega - n\omega_0) \times Q_{s_0}(\omega + n\omega_0)H_{l-n}(\omega + n\omega_0). \tag{11}$$

The slow noise, (7), we need to calculate the oscillator phase noise via the amplitude equation, (6), is characterized by the spectra $\langle\tilde{\Xi}_x(\Omega)\tilde{\Xi}_y(\Omega')\rangle = 2\pi\epsilon\delta(\Omega+\Omega')S_{xy}(\Omega)$, $\tilde{\Xi}_x(\Omega) = \mathcal{F}[\Xi_x(T)]$, $x,y = R,I$. As elaborated in Appendix A, these spectra are expressed in terms of the HPSDs Q_l as

$$\begin{aligned} S_{RR}(\Omega) &= Q_0(\epsilon\Omega-\omega_0) + Q_0(\epsilon\Omega+\omega_0) \\ &\quad + 2\mathrm{Re}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}], \\ S_{II}(\Omega) &= Q_0(\epsilon\Omega-\omega_0) + Q_0(\epsilon\Omega+\omega_0) \\ &\quad - 2\mathrm{Re}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}], \\ S^s_{RI}(\Omega) &= 4\mathrm{Im}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}], \end{aligned} \tag{12}$$

with $S^s_{RI}(\Omega) = S_{RI}(\Omega) + S_{IR}(\Omega)$. Note that the slow noise is *stationary*, as expected for a self-oscillating system. Its spectra are composed only of the HPSDs Q_0 and Q_2, indicating that the high-Q resonator acts similarly to a filter near its resonant frequency. Equations (6), (11), and (12) provide a complete description of the slow stochastic dynamics of the oscillator, giving the near-carrier phase noise spectrum, in terms of the spectral properties of the noise source Q_{s_0}.[1] In the complex amplitude phase space the noise components Ξ_R and Ξ_I represent perpendicular force vectors perturbing the limit-cycle orbit. The stochastic properties of these vectors are in general nontrivial, and eliminating the effect of the random perturbations of both vectors by tuning the feedback phase Δ seems generally impossible [16].

III. $1/f$ NOISE

In the present paper, we are concerned with a noise source whose intensity is significant only at low frequencies over a bandwidth much narrower than the oscillation frequency. In evaluating the spectra, Eqs. (12), the sum in Eq. (11) can be approximated in this case retaining only the terms with $\omega + n\omega_0 \simeq 0$. Physically, this corresponds to considering only the large noise at low frequencies up-converted to the oscillation frequency by the nonlinear amplifier. This leads to the result that the slow noise spectra, Eqs. (12), depend only on the first harmonic transfer functions $H_{\pm1}$, and these can be evaluated at zero frequency, $H_1(0) = H^*_{-1}(0) = |H_1(0)|e^{i\psi}$, since the feedback system is not expected to have strong dependence on frequencies that are low compared with ω_0. We write the phase of the first harmonic transfer functions as $\psi = \Phi_H + \Delta + \pi/2$, giving it the phase shift Φ_H relative to the feedback. This phase difference is determined by properties of the amplifier. Finally, setting $\Phi_N = \Phi_H$ in Eq. (6) gives

$$S_{RR}(\Omega) = 4Q_{s_0}(\epsilon\Omega)|H_1(0)|^2, \tag{13}$$

and $\Xi_I = 0$. Looking back at Eq. (6) we see that the low-frequency noise is represented by a single noise vector making an angle Φ_H relative to the feedback. It also inherits the $1/f$ spectrum of the noise source.

[1]Note that we retain the $O(\epsilon)$ corrections in the arguments of the Q_l since these functions may have a strong dependence on frequency shifts away from the oscillation frequency and its harmonics due to the $1/f$ spectrum of the noise source but may have set to 0 other $O(\epsilon)$ corrections. For broad-band noise sources these $\epsilon\Omega$ terms can also be set to 0.

Equation (13) is our key result since it implies that the noise can be eliminated by tuning a single parameter. This follows because the effect of the noise sources on the oscillator phase noise is given by projecting the noise onto a particular direction, the phase sensitivity vector $\mathbf{v}_\perp$, in the phase space. This vector is perpendicular to the isochrone at the limit cycle—the surfaces of perturbations to the limit cycle that decay to the same phase point on the cycle [17–20]—and can be calculated as the eigenvector with zero eigenvalue of the adjoint of the Jacobian of the evolution equations linearized about the no-noise limit cycle [16,21]. To implement this we write the complex amplitude equation as a pair of real equations for (a,Φ). The direction of the noise vector in the (a,Φ) space is

$$\mathbf{v}_R = \frac{1}{2}\left(\cos(\Delta+\Phi_H), \frac{\sin(\Delta+\Phi_H)}{a}\right). \tag{14}$$

The sensitivity of the phase to noise is then given by the projection $P(\Delta,\Phi_H) = \mathbf{v}_\perp\cdot\mathbf{v}_R$, and the oscillator spectrum at nonzero frequency offset Ω, given by solving Eq. (6) for the stochastic phase dynamics, is proportional to $S(\Omega) \sim P^2 S_{RR}(\Omega)\Omega^{-2}$ [4]. Note that in our amplitude-phase description both $\mathbf{v}_\perp$ and $\mathbf{v}_R$ are *constant* vectors. Therefore, the condition for eliminating the oscillator phase noise is $P(\Delta,\Phi_H) = 0$.

A. Constant feedback

Having derived a condition for phase noise elimination, we now explore ways to satisfy it as a function of the phase Φ_H characterizing the amplifier noise. We first consider a simplified case in which the feedback function is constant $g(a) = g_s$, and the resonator is only linearly damped ($\eta = 0$). In this setting the slow dynamic equations are

$$\begin{aligned} \frac{da}{dT} &= -\frac{a}{2} + \frac{g_s}{2}\cos\Delta = f_a(a), \\ \frac{d\Phi}{dT} &= \frac{3}{8}a^2 + \frac{\sin\Delta}{2}\frac{g_s}{a} = f_\Phi(a). \end{aligned} \tag{15}$$

These give the operating point $(da/dT = 0, d\Phi/dT = \Omega_0)$ at oscillation amplitude $a_0 = g_s\cos\Delta$ and frequency $\Omega_0 = f_\Phi(a_0)$. The vector $\mathbf{v}_\perp$ is then given by the zero-eigenvalue eigenvector of the adjoint of the Jacobean matrix linearizing the right-hand side of Eqs. (15) about the operating point,

$$\mathbf{v}_\perp = \left(-\frac{\left(\frac{\partial f_\Phi}{\partial a}\right)}{\left(\frac{\partial f_a}{\partial a}\right)}\bigg|_{a=a_0}, 1\right) = \left(\frac{3}{2}g_s\cos\Delta - \frac{\sin\Delta}{g_s\cos^2\Delta}, 1\right), \tag{16}$$

and the scalar product of this vector with (14) is

$$P = \frac{\sin\Phi_H}{2g_s\cos^2\Delta} + \frac{3}{4}g_s\cos\Delta\cos(\Delta+\Phi_H). \tag{17}$$

This equation presents the phase sensitivity of the standard Duffing oscillator to an arbitrary single noise vector. By reformulating it with the variable $x = \cos 2\Delta$ the condition for having zero phase sensitivity can be written as a quartic equation, and its real solutions are shown in Fig. 1. As shown in the figure, the equation $P = 0$ does not give a real solution for Δ for every value of g_s and Φ_H. However, the range

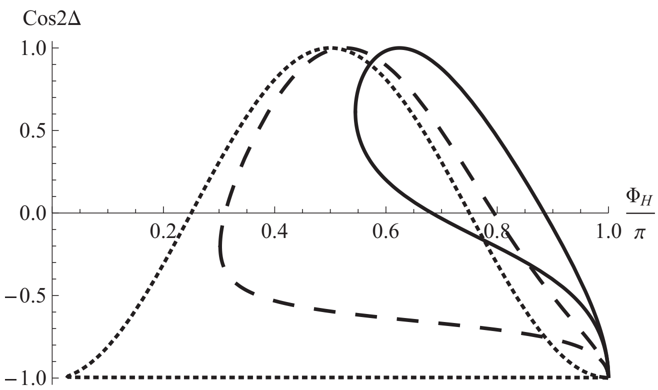

FIG. 1. Values of the feedback phase shift Δ giving zero phase noise ($P=0$) for constant feedback and $\eta=0$. Solid curve, $g_s = 4/\pi$; dashed curve, $g_s = 9/\pi$; dotted curve, $g_s = 300/\pi$. These values correspond to $g_s \simeq g_c \cdot (0.88,2,66)$, respectively.

of phases Φ_H for which the noise can be eliminated grows as the drive level g_s increases. For $g_s \to \infty$ there are two real solutions that solve the equation for the phase values $\Phi_H \in (0,\pi)$, and these approach $x_1 \to -1, x_2 \to -\cos 2\Phi_H$. Setting $\Phi_H = \pi/2$ in Eq. (17) corresponds to the situation in which the noise is purely in the phase of the feedback. For this case, it is necessary to drive the resonator to the nonlinear regime [$g_s > g_c = (4/3)^{5/4}$] in order to eliminate amplifier noise, as originally shown by Greywall *et al.* [10,22]. However, note that this physical requirement is not necessary in general, as exemplified by the solid line in Fig. 1, which is for $g_s < g_c$.

B. Dynamic feedback

To extend these findings to the more general case of dynamic feedback in which the feedback level depends on the oscillation amplitude, we consider a phenomenological model for the amplifier featuring an instantaneous amplification function,

$$x_{\text{out}} = x_s \mathcal{A}(G x_{\text{in}}/x_s), \tag{18}$$

with

$$\mathcal{A}(y) = r\frac{1-e^{-2y}}{r+e^{-2y}}. \tag{19}$$

We suppose this amplifier is followed by a phase shifter, giving the phase shift Δ but not adding additional noise to the system. The feedback function in Eq. (6) is then given by

$$g(a) = \frac{x_s}{\pi}\int_{-\pi}^{\pi} \mathcal{A}[Ga\cos(z)/x_s]\cos(z)dz, \tag{20}$$

and we use this expression to calculate the phase sensitivity vector $\mathbf{v}_\perp$. Although previous schemes for phase noise elimination in feedback oscillators relied on the saturation of the amplifier [4,10,13,22], Figure 2 demonstrates phase noise elimination in an oscillator using an unsaturated amplifier. The area in the (Φ_H, G) parameter space in which there is at least one phase shift value Δ satisfying $P=0$ is shown in white.

IV. FEEDBACK NOISE SOURCES

We now present specific examples of noise sources. The first example we consider is a noise source at the amplifier input ($q \to q + \xi_s$), with the amplifier given by Eqs. (18) and (19). Since noise at the input stage is amplified more than noise introduced at later stages, it is reasonable to assume that this noise may dominate in many cases. For this noise source the harmonic transfer functions are $H_n = F_n e^{\mathrm{i}n(\Delta+\pi/2)}$, with F_n the real expression

$$F_n = \frac{G}{2\pi}\int_{-\pi}^{\pi} \mathcal{A}'[Ga\cos(z)/x_s]\cos(nz)dz. \tag{21}$$

In this case $\Phi_H = 0$ and the equation $P(\Delta,0) = 0$ is satisfied for $\Delta = \Delta_R = -\arctan(3/\eta)$. Surprisingly, the phase-shift value Δ_R is independent of the amplifier parameters.

The second example we present is one for which there is a phase shift between the noise and the feedback ($\Phi_H \neq 0$). To examine such a case, we imagine that the dominant noise source is in a simple series RC filter placed after the amplifier in the feedback loop, with the voltage on the resistor as the feedback. We consider the effect of noise in the capacitor by adding to the capacitance stationary noise $C \to C(1+\xi_s)$.

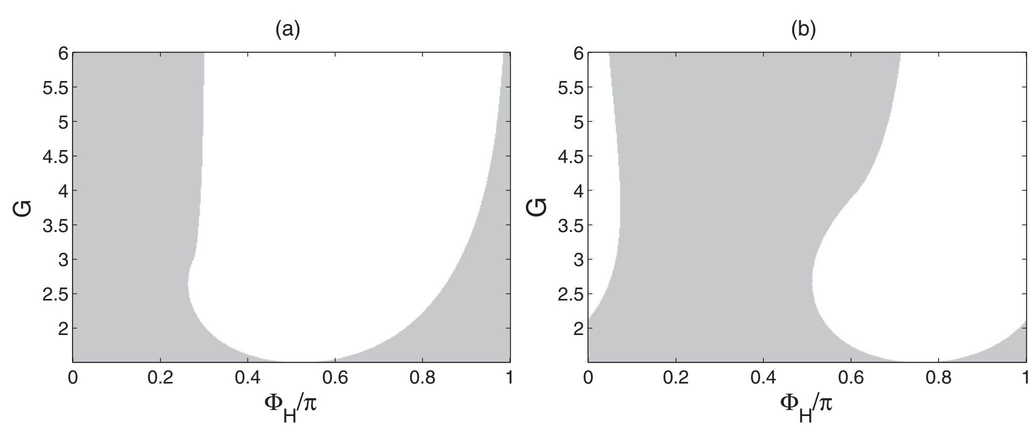

FIG. 2. Phase noise elimination in an oscillator using an unsaturated amplifier. The white area in the figures is the parameter range for which there are zero noise points ($P=0$) for $x_s = 3$ and $r = 0.5$ (corresponding to $g_s = 9/\pi$ in the large-gain limit). (a) $\eta = 0$; (b) $\eta = 3$.

The equation for the charge q on the noisy capacitor is

$$\frac{dq}{dt} = -\frac{q - q_{\rm in}}{\tau} + \frac{\xi_s(t) q}{\tau}, \qquad (22)$$

where $q_{\rm in}(t)$ is the input signal (in charge units) and τ is RC scaled by the frequency. Equation (22) has an exact solution, which is given in Appendix B. We use this solution to calculate the noise on the resistor and obtain the following expressions for its harmonic transfer functions:

$$H_1(0) = \frac{V_0 \tau \omega_0 e^{i(\Delta+\phi_c)}}{2(\tau^2\omega_0^2 + 1)}, \quad H_{n\neq\pm1}(\omega) = 0, \qquad (23)$$

where $\tan\phi_c = 1/(\tau\omega_0)$, and V_0 is proportional to the signal amplitude. This leads to a phase shift between the feedback and the noise of $\Phi_H = \phi_c - \pi/2$, and the phase sensitivity to this noise is $P(\Delta,\phi_c - \pi/2)$. The ability to eliminate it depends on the specific parameter values shown in Fig. 2.

V. CONCLUDING DISCUSSION

To conclude, we have shown that $1/f$ and narrow-bandwidth noise in oscillators can often be eliminated by choosing the appropriate feedback phase. We have demonstrated this by analyzing feedback noise in oscillators based on a high-Q resonator, but we expect it also to apply more generally. An indication of this is given in the results of Demir *et al.* [21], who considered a more general treatment in which the noise vector $\mathbf{v}_n(t)$ and the phase sensitivity vector $\mathbf{v}_\perp(t)$ depend on time. They showed that the phase sensitivity to a noise source with a bandwidth much narrower than the oscillation frequency is proportional to $[\langle \mathbf{v}_n(t)\cdot\mathbf{v}_\perp(t)\rangle]^2$, whereas for white noise it is given by $\langle[\mathbf{v}_n(t)\cdot\mathbf{v}_\perp(t)]^2\rangle$, where $\langle\,\rangle$ denotes an average over a period. The phase diffusion coefficient derived by Nakao *et al.* in [23] also conforms to these different expressions in the appropriate limits. Since generally the scalar product $\mathbf{v}_n(t)\cdot\mathbf{v}_\perp(t)$ would not be 0 at every point along the limit cycle, these results provide another indication that $1/f$ noise can be eliminated by tuning a parameter such as the feedback phase, while wide-bandwidth noise in general cannot.

ACKNOWLEDGMENT

This research was supported by DARPA through the DEFYS program.

APPENDIX A: NOISE IN THE SLOW AMPLITUDE FORMALISM

In this Appendix we calculate the spectra of the slow noise Ξ that appears in Eq. (6). The Fourier transform ($\tilde{\Xi}_x(\Omega) = \mathcal{F}[\Xi_x(T)]$) of the real part of the noise gives the correlation

$$\begin{aligned}
&\langle\tilde{\Xi}_R(\Omega)\tilde{\Xi}_R(\Omega')\rangle \\
&= \int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} dT' \langle \Xi_R(T)\Xi_R(T')\rangle e^{-i\Omega T} e^{-i\Omega' T'} \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} dT' \int_{\epsilon^{-1}T-\pi}^{\epsilon^{-1}T+\pi} dt \int_{\epsilon^{-1}T'-\pi}^{\epsilon^{-1}T'+\pi} dt' \langle\xi(t)\xi(t')\rangle \cos(t + \Omega_0 T + \psi_N)\cos(t' + \Omega_0 T' + \psi_N) e^{-i\Omega T} e^{-i\Omega' T'} \\
&= \frac{\epsilon^2}{\pi^2}\int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt''' \int_{t''-\pi}^{t''+\pi} dt \int_{t'''-\pi}^{t'''+\pi} dt' \langle\xi(t)\xi(t')\rangle \cos(t + \epsilon\Omega_0 t'' + \psi_N)\cos(t' + \epsilon\Omega_0 t''' + \psi_N) e^{-i\Omega\epsilon t''} e^{-i\Omega'\epsilon t'''}.
\end{aligned} \qquad (A1)$$

Changing the order of the t'' and the t integration gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt'' \int_{t''-\pi}^{t''+\pi} dt \cos(t + \epsilon\Omega_0 t'' + \psi_N) e^{-i\Omega\epsilon t''} \\
&= \int_{-\infty}^{\infty} dt \int_{t-\pi}^{t+\pi} dt'' \cos(t + \epsilon\Omega_0 t'' + \psi_N) e^{-i\Omega\epsilon t''} \\
&= \frac{\sin(\epsilon(\Omega-\Omega_0)\pi)}{\epsilon(\Omega-\Omega_0)} \int_{-\infty}^{\infty} dt e^{-i(\Omega-\Omega_0)\epsilon t} e^{i(t+\psi_N)} + \frac{\sin(\epsilon(\Omega+\Omega_0)\pi)}{\epsilon(\Omega+\Omega_0)} \int_{-\infty}^{\infty} dt e^{-i(\Omega+\Omega_0)\epsilon t} e^{-i(t+\psi_N)} \\
&\simeq 2\pi \int_{-\infty}^{\infty} dt \cos(\omega_0 t + \psi_N) e^{-i\Omega\epsilon t},
\end{aligned} \qquad (A2)$$

with $\omega_0 = 1 + \epsilon\Omega_0$ the oscillation frequency. Note that in the last expression we have neglected $O(\epsilon)$ terms outside the integral but retained the small terms $(\Omega \pm \Omega_0)\epsilon t$ in the oscillating functions inside. These terms are relevant for a low-frequency noise source with a spectrum which changes significantly on the frequency scale of $O(\epsilon)$, as we show later. Doing the same with t''' and t' gets us

$$\langle\tilde{\Xi}_R(\Omega)\tilde{\Xi}_R(\Omega')\rangle = 4\epsilon^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle\xi(t)\xi(t')\rangle \cos(\omega_0 t + \psi_N)\cos(\omega_0 t' + \psi_N) e^{-i\Omega\epsilon t} e^{-i\Omega'\epsilon t'}, \qquad (A3)$$

and similarly,

$$\langle\tilde{\Xi}_I(\Omega)\tilde{\Xi}_I(\Omega')\rangle = 4\epsilon^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle\xi(t)\xi(t')\rangle \sin(\omega_0 t + \psi_N)\sin(\omega_0 t' + \psi_N) e^{-i\Omega\epsilon t} e^{-i\Omega'\epsilon t'}, \qquad (A4)$$

and

$$\langle\tilde{\Xi}_R(\Omega)\tilde{\Xi}_I(\Omega')\rangle = -4\epsilon^2\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\langle\xi(t)\xi(t')\rangle\cos(\omega_0 t+\psi_N)\sin(\omega_0 t'+\psi_N)e^{-i\Omega\epsilon t}e^{-i\Omega'\epsilon t'}. \tag{A5}$$

Putting in the cyclostationary noise given by Eq. (10) gives

$$\begin{aligned}\langle\tilde{\Xi}_R(\Omega)\tilde{\Xi}_R(\Omega')\rangle &= 4\epsilon^2\sum_l\left\{\int_{-\infty}^{\infty}ds\,R_{a_l}(s)\cos(\omega_0 s)e^{i\Omega'\epsilon s}\int_{-\infty}^{\infty}dt\cos^2(\omega_0 t+\psi_N)e^{-i((\Omega+\Omega')\epsilon-l)t}\right.\\ &\quad\left.+\int_{-\infty}^{\infty}ds\,R_{a_l}(s)\sin(\omega_0 s)e^{i\Omega'\epsilon s}\int_{-\infty}^{\infty}dt\cos(\omega_0 t+\psi_N)\sin(\omega_0 t+\psi_N)e^{-i((\Omega+\Omega')\epsilon-l)t}\right\}\\ &= 2\epsilon^2\pi\sum_l\left\{\int_{-\infty}^{\infty}ds\,R_{p_l}(s)\cos(\omega_0 s)e^{i\Omega'\epsilon s}\right.\\ &\quad\times[2\delta(l\omega_0-\epsilon(\Omega+\Omega'))+\delta((l+2)\omega_0-\epsilon(\Omega+\Omega'))e^{2i\psi_N}+\delta((l-2)\omega_0-\epsilon(\Omega+\Omega'))e^{-2i\psi_N}]\\ &\quad\left.-i\int_{-\infty}^{\infty}ds\,R_{p_l}(s)\sin(\omega_0 s)e^{i\Omega'\epsilon s}[\delta((l+2)\omega_0-\epsilon(\Omega+\Omega'))e^{2i\psi_N}-\delta((l-2)\omega_0-\epsilon(\Omega+\Omega'))e^{-2i\psi_N}]\right\}.\end{aligned} \tag{A6}$$

Considering small values of ϵ, we can make the replacement for the Dirac δ functions $\delta((l-m)\omega_0-\epsilon(\Omega+\Omega'))\to\delta_{l,m}\delta(\Omega+\Omega')/\epsilon$, to give

$$\langle\tilde{\Xi}_R(\Omega)\tilde{\Xi}_R(\Omega')\rangle = 2\pi\epsilon\delta(\Omega+\Omega')S_{RR}(\Omega), \tag{A7}$$

with

$$S_{RR}(\Omega) = Q_0(\epsilon\Omega-\omega_0)+Q_0(\epsilon\Omega+\omega_0)+Q_{-2}(\epsilon\Omega+\omega_0)e^{2i\psi_N}+Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}. \tag{A8}$$

We have kept the $O(\epsilon)$ terms inside the HPSDs Q_l. This is valid for the case of $1/f$ noise, where the spectrum changes significantly on a frequency scale of order $O(\epsilon)$, which is the case we consider in this paper. Now using the relations

$$Q_n(-\omega) = Q_n(\omega-n\omega_0), Q_{-n}(-\omega) = Q_n^*(\omega), \tag{A9}$$

and including the results for the other two spectra, we get

$$\begin{aligned}S_{RR}(\Omega) &= Q_0(\epsilon\Omega-\omega_0)+Q_0(\epsilon\Omega+\omega_0)+2\mathrm{Re}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}],\\ S_{II}(\Omega) &= Q_0(\epsilon\Omega-\omega_0)+Q_0(\epsilon\Omega+\omega_0)-2\mathrm{Re}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}],\\ S_{RIs}(\Omega) &= S_{RI}(\Omega)+S_{IR}(\Omega) = 4\mathrm{Im}[Q_2(\epsilon\Omega-\omega_0)e^{-2i\psi_N}],\end{aligned} \tag{A10}$$

which are Eqs. (6).

APPENDIX B: HARMONIC TRANSFER FUNCTIONS FOR CAPACITOR NOISE

In this Appendix we calculate the harmonic transfer functions for noise in the capacitor of a series RC circuit. The equation for the current in an RC circuit with a noisy capacitor [Eq. (22)] is

$$\frac{dq}{dt} = -\frac{q-q_{\rm in}}{\tau}+\frac{\xi_s(t)q}{\tau}, \tag{B1}$$

and its solution is

$$q(t) = \frac{e^{-(t/\tau-f(t))}}{\tau}\int_{-\infty}^{t}e^{t'/\tau-f(t')}q_{\rm in}(t')dt', \tag{B2}$$

where $f(t)=\frac{1}{\tau}\int_{-\infty}^{t}\xi_s(t')dt'$, $q_{\rm in}(t)=CV_{\rm in}(t)$, $V_{\rm in}(t)$ is the input signal, and τ is RC scaled by the frequency. In the leading order in the noise strength, the voltage on the resistor can be written as $V_R(t)\simeq V_{R_0}(t)+\xi_v(t)$, with V_{R_0} the voltage with no noise and the noise term ξ_v given by

$$\begin{aligned}\xi_v(t) &= \frac{e^{-t/\tau}\xi_s(t)}{\tau}\int_{-\infty}^{t}e^{x/\tau}V_{\rm in}(x)dx\\ &\quad-\frac{e^{-t/\tau}}{\tau^2}\int_{-\infty}^{t}e^{x/\tau}V_{\rm in}(x)dx\int_{x}^{t}\xi_s(t')dt'\\ &= \frac{\tilde{V}_0}{\sqrt{\tau^2\omega_0^2+1}}\left(\cos(\omega_0 t+\phi_0+\Delta)\xi_s(t)\right.\\ &\quad\left.-\frac{e^{-t/\tau}}{\tau}\int_{-\infty}^{t}dt'\xi_s(t')e^{t'/\tau}\cos(\omega_0 t'+\phi_0+\Delta)\right),\end{aligned} \tag{B3}$$

where we have taken the input signal to be $V_{\rm in}(t)=\tilde{V}_0\cos(\omega_0 t+\phi_0+\Delta+\pi/2-\phi_c)$ with $\tan\phi_c=1/(\tau\omega_0)$, so that the phase of the feedback signal follows our convention $V_{R_0}\sim\cos(\omega_0 t+\phi_0+\Delta+\pi/2)$. In the formalism of Eqs. (8) and (9), the noise given by Eq. (B3) corresponds to $h(t,t')=h_1(t-t')e^{i(\omega_0 t+\phi_0)}+\text{c.c.}$, with

$$h_1(t) = \frac{V_0 e^{i\Delta}}{2\sqrt{\tau^2\omega_0^2+1}}\left(\delta(t)-\Theta(t)\frac{e^{-t/\tau}}{\tau}e^{-i\omega_0 t}\right), \tag{B4}$$

with the scaled voltage V_0 and Heaviside step function Θ. This gives the first harmonic transfer function

$$H_1(\omega) = \mathcal{F}[h_1(t)] = \frac{V_0\tau(\omega_0+\omega)e^{i(\Delta+\phi_c)}}{2\sqrt{\tau^2\omega_0^2+1}\sqrt{\tau^2(\omega_0+\omega)^2+1}}. \tag{B5}$$

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