Hexagonal Uniformly Redundant Arrays for Coded-Aperture Imaging

M. H. Finger and T. A. Prince
California Institute of Technology, Pasadena, CA 91125

1. Introduction.

Uniformly redundant arrays are used in coded-aperture imaging, a technique for forming images without mirrors or lenses. This technique is especially important for the high energy x-ray and γ-ray region above 20 keV. In this technique, a mask consisting of opaque (closed) and transparent (open) areas is placed between the photon sources to be imaged and a position sensitive detector or a detector array. Each source casts a shadow pattern of the mask or aperture onto the detector. This shadow pattern may be viewed as an encoded signal for that source direction. If each possible source code is unique, the detected composite of overlapping shadow patterns may be decoded to produce an image of the source distribution.

Figure 1 shows a mask suitable for imaging. This mask consists of an array of open (white) and closed (gray) cells arranged in a periodic pattern. The unit pattern is outlined. The mask in figure 1 is a *uniformly redundant array*[1,2] (URA). URAs have an especially desirable property: the overlap between two source codes is independent of the source directions as long as the sources are sufficiently separated. Except for periodicity, this guarantees a unique decoding of the composite shadow pattern with a maximal immunity to statistical noise.

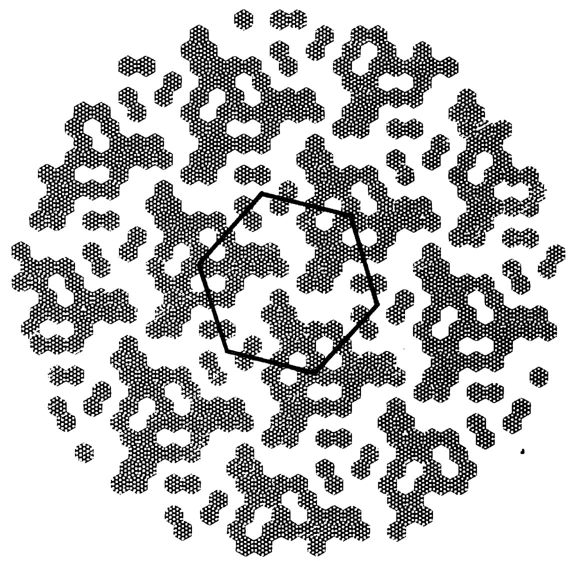

Figure 1. An HURA of order 79.

To date, most work on URAs has concentrated on those constructed on rectangular lattices. In this paper we focus on URAs constructed on hexagonal lattices, although many of the results are independent of the lattice type.

We will present complete details for the construction of a special class of URAs, the *skew-Hadamard URAs*, which have the following properties:

1) They are nearly half open and half closed.
2) They are antisymmetric (exchanging open and closed cells) upon rotation by 180° except for the central cell and its repetitions.

Some of the skew-Hadamard URAs constructed on a hexagonal lattice have additional symmetries. These special URAs that have a hexagonal unit pattern, and are antisymmetric upon rotation by 60°, we call *hexagonal uniformly redundant arrays* (HURAs). The mask in figure 1 is an HURA.

HURAs are particularly suited to our application, γ-ray imaging in high background situations. In a high background situation the best sensitivity is obtained with a half open and half closed mask. Furthermore, systematic variations of the detector background from position to position can be larger than the variations in detected flux due to sources. With a skew-Hadamard URA a simple rotation turns the mask into a near anti-mask, allowing exact position-by-position background subtraction. Also, the hexagonal symmetry of an HURA is more appropriate for a round position-sensitive detector or a close-packed array of detectors than a rectangular symmetry. This is especially true for shielded detector systems where compactness is at a premium.

2. Mathematical Structure of URAs

A URA is defined within a unit pattern which is repeated periodically. The number of cells in this unit pattern is the order v of the URA. Of these cells, k of them are closed and $v-k$ are open. The uniform redundancy property of URAs involves how frequently a given displacement between closed cells occurs. We will consider a cell within a repetition of the unit pattern as equivalent to the corresponding cell in the unit pattern, and will therefore define the difference between two cell centers as the vector displacement between them, folded back into the unit pattern. For a URA, all possible differences occur a uniform number λ times among the pairs of closed cell centers[2]. This property guarantees the uniform overlap of source codes discussed in the introduction[1,2].

The mathematical structure of a URA is that of an Abelian group difference set[3], which is specified by an Abelian (additive) group G of order v, and a set D of k elements of G with the property that any possible nonzero difference occurs exactly λ times between elements of D. For a URA the group G is the lattice translations modulo the periods of the mask pattern, and the set D contains those translations that take the central cell to a closed cell. The simplest examples of group difference sets are one-dimensional sets known as *cyclic difference sets* defined on the group of integers *mod* v. These play an important role in the construction of many URAs.

URAs in the class consided in this paper, the *skew-Hadamard URAs*, are nearly antisymmetric. That is, for any nonzero element in the group G, either it or its negative but not both, are contained in the difference set D[3]. These skew-Hadamard URAs are a subset of the *Hadamard URAs* which are nearly half open and half closed. Hadamard URAs are characterized by the parameters $v=4n-1$, $k=2n-1$, $\lambda=n-1$ for some integer n.

Johnsen[4] has proven two interesting facts about skew-Hadamard URAs :

1) All skew-Hadamard URAs have a *cyclic* group G, and therefore can be constructed from skew-Hadamard cyclic difference sets.
2) All skew-Hadamard cyclic difference sets are of prime order $v = 3$ *mod* 4 and can be generated from the quadratic residues *mod* v .

These facts allow us to present a construction for *all* antisymmetric or skew-Hadamard URAs.

3. Construction of Skew-Hadamard URAs from Quadratic Residues

We now present a simple procedure for generating any skew-Hadamard URA. An example constructed on a hexagonal lattice is shown in figure 2. The procedure consists of the following steps:

1) Choose the lattice on which the URA is to be constructed. The lattice is defined by picking two basis vectors $\vec{e}_0$ and $\vec{e}_1$. For our example we have chosen a hexagonal lattice, which has the basis vectors separated by 60° .
2) Choose as the order of the URA a prime of the form $v = 4n-1$. In our example $v = 23$.
3) Construct the order v skew-Hadamard cyclic difference set from the formula

$$D=\{1^2,2^2,\ldots,(\frac{v-1}{2})^2\} \ mod \ v \qquad (1)$$

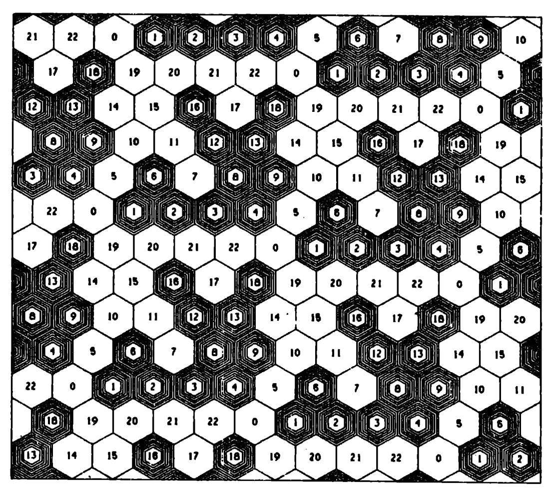

Figure 2. Construction of a skew-Hadamard URA of order 23.

4) Choose an integer r and label all the cells so that the cell centered at $i\vec{e}_0+j\vec{e}_1$ is labeled with

$$l = (i + rj) \ mod \ v \qquad (2)$$

and make all cells with labels in D closed. In our example r=5.

The heart of this procedure is the construction of the skew-Hadamard cyclic difference set in step 3. For a proof that this is a difference set see Baumert[3]. Step 4 transfers the difference set properties onto the lattice. This is done through the labeling, which transforms addition *mod* v to vector addition on the lattice modulo the resulting lattice periods.

The freedom available in this procedure rests in the choice of the lattice, the choice of the order v,and the choice of the multiplier r. The lattice type will determine what symmetries can occur. The possible orders form a rather dense set, the first few choices being v=3,7,11,19,23,31,43,47,59,67,71,79, and 83. The multiplier r determines the periods of the URA, and hence the shape of the unit pattern. Many of the v available choices result in URAs that are related by the symmetries of the lattice.

4. Hexagonal Uniformly Redundant Arrays

Of the large number of skew-Hadamard URAs, all of which can be constructed by the procedure in section 3, we wish to pick out those that have a hexagonal unit pattern when constructed on a hexagonal lattice. These we call *hexagonal uniformly redundant arrays* (HURAs)[5]. For an HURA each period when rotated by 60° is again a period. It can be shown from equation (2) that this is only possible if the multiplier r satisfies

$$r^2 = r-1 \quad mod\ v \tag{3}$$

This property has a simple geometric interpretation: a cell labeled l when rotated by 60° will have the label rl *mod* v. This feature, and the properties of quadratic residues modulo a prime, causes this restricted set of URAs to have a rotational antisymmetry upon rotation by 60° as well as 180°.

It can be shown that HURAs exist for order v=3 and any prime order of the form $12n$+7. If HURAs related by symmetry are considered equivalent, then for each of these orders there is a single HURA. The number of available HURAs is still large; figure 3 shows the number of HURAs with order v or less for v up to 10,000. In figures 4 through 8 we show examples of a few moderate order HURAs.

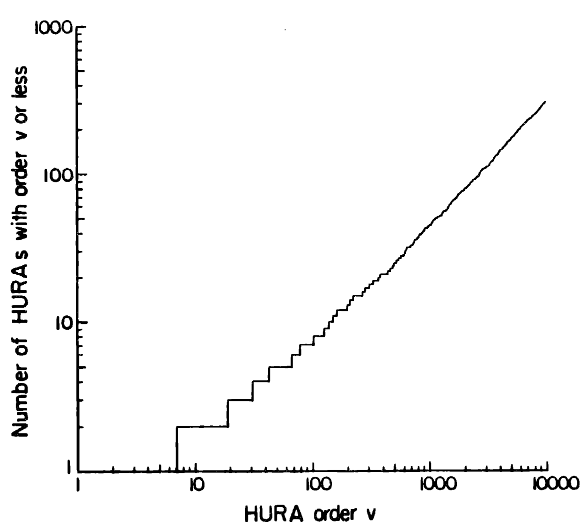

Figure 3. The number of HURAs with a given order or less.

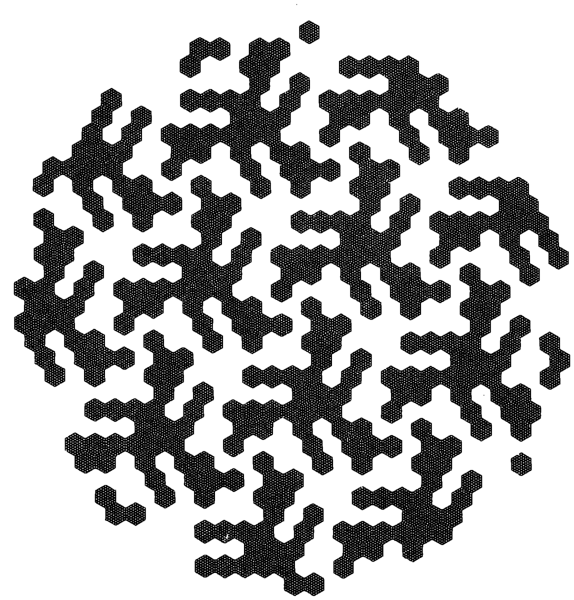

Figure 4. An HURA of order 67.

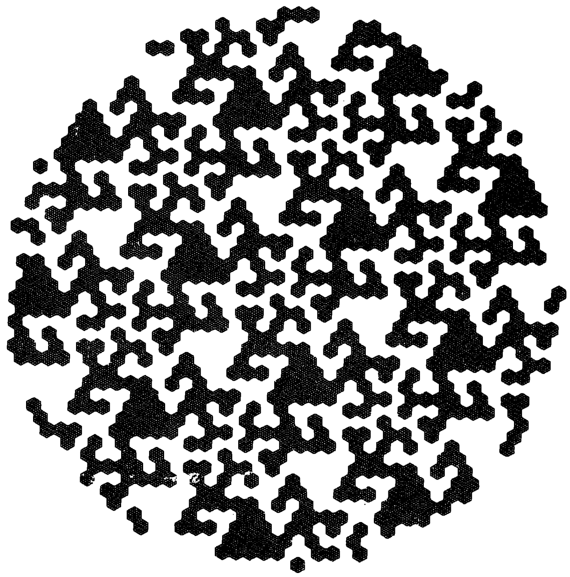

Figure 5. An HURA of order 139.

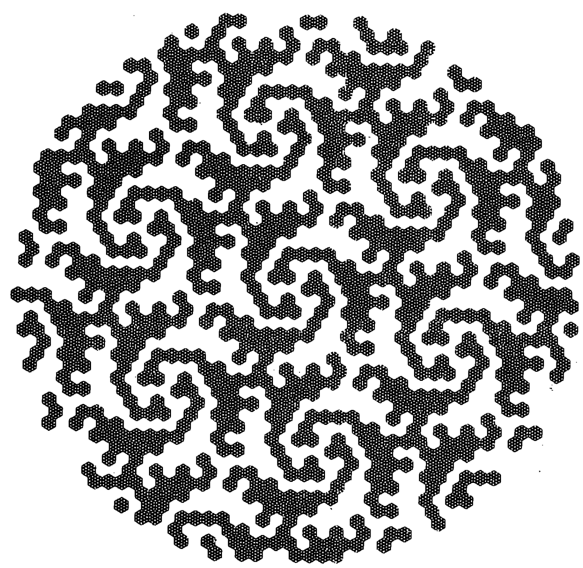

Figure 6. An HURA of order 151.

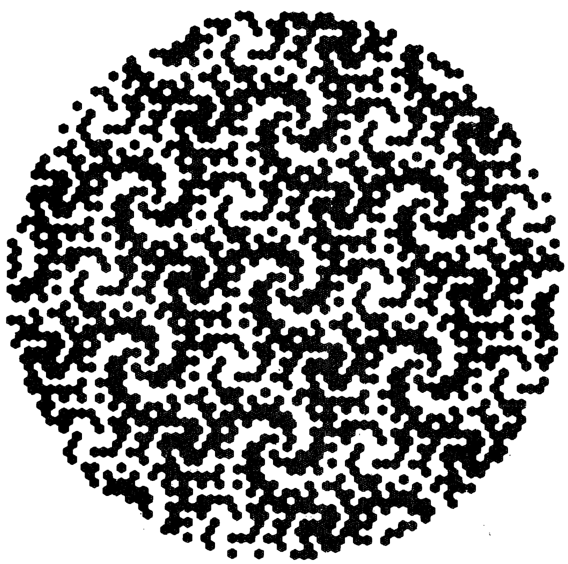

Figure 7. An HURA of order 331.

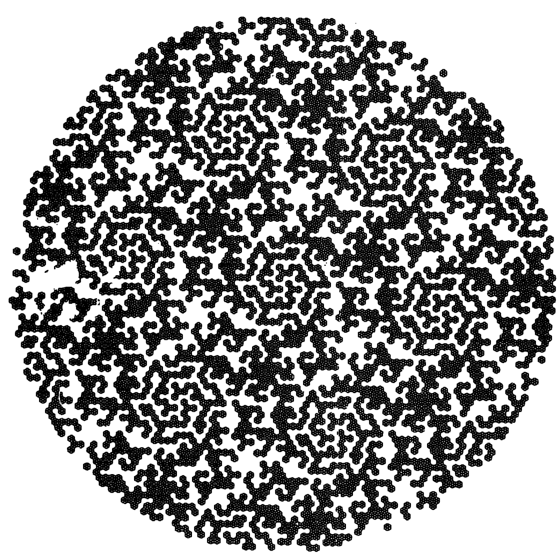

Figure 8. An HURA of order 619.

While HURAs are mathematically interesting constructs, they also have numerous attractive features for applications in astronomy instrumentation. For instance, the HURA of figure 1 is being implemented as a 115 kg lead coded-aperture mask on a Caltech imaging γ-ray telescope (see OG9.2-2).

5. Acknowledgments

We acknowledge useful discussions with W. R. Cook. This work is supported in part by NASA grant NGR 05-002-160.

References

1) Gunson and Polychronopulos,*Mon. Not. R. Astron. Soc.* **177**,485 (1976)
2) Fenimore and Cannon, *Appl. Opt.,* **17**,337 (1978)
3) Baumert, Cyclic difference sets, *Lecture Notes in Mathematics,No. 182,* Springer-Verlag (1971)
4) Johnsen, *J. Algebra,* **4**,388 (1966) [esp. Corollary 3.4 and Theorem 3.5 restricted to two cycles]
5) Cook et al., *IEEE Trans. Nucl. Sci.,* **NS-31**,771 (1984)