factor $(2i)^{-1}$ are

$$\sum_{k=1}^{\infty} \epsilon_k \sum_{l=1}^{N_k} \frac{1}{l} \exp\left[i\pi\left(1 - \frac{l}{2N_k}\right)z\right] \tag{21}$$

and

$$-\sum_{k=1}^{\infty} \epsilon_k \sum_{l}^{N_k} \frac{1}{l} \exp\left[-i\pi\left(1 - \frac{l}{2N_k}\right)z\right]. \tag{22}$$

The total set of Fourier exponents in (21) and (22) is

$$\lambda = \pm\pi\left(1 - \frac{l}{2N_k}\right),\ 1 \leq l \leq N_k,\ 1 \leq k < \infty,$$

so that

$$|\lambda| \geq \frac{\pi}{2}.$$

But, because of this "gap" around $\lambda = 0$, if the series (21) and (22) constitute an expansion jointly, they do so separately. But in (21) all $c(\lambda)$ are ≥ 0, and in this case their sum must converge;[8] which proves the crucial second part of the theorem.

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[1] Boas, R. P., *Entire Functions* (New York: Academic Press, 1954), p. 82.

[2] Ref. 1, page 221.

[3] Besicovitch, A. S., *Almost Periodic Functions* (Cambridge, 1932), p. 17.

[4] For comprehensive versions of this theorem, see also my papers "Abstrakte fastperiodische Funktionen," *Acta Math.*, **61**, 149–184 (1933), and "A new approach to almost periodicity," these PROCEEDINGS, **48**, 2039–2043 (1962).

[5] Ref. 3, pp. 142–143.

[6] *Ibid*, p. 152.

[7] *Ibid.*, p. 158.

[8] *Ibid.*, p. 154.

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*ATTENUATION IN THE MANTLE AND RIGIDITY OF THE CORE FROM MULTIPLY REFLECTED CORE PHASES**

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In 1956 Press[1] reported on single and double reflections of shear waves at near normal incidence from the earth's core and was able to estimate the average dissipation constant of shear waves in the mantle and place an upper bound for the rigidity of the earth's outer core at the core-mantle boundary. However, because of excessive reverberations and the low sensitivity of the seismographs only a few usable seismograms were found in five years of continuous data. In this paper we report on a seismogram showing unusually clear, near-vertical multiple reflections of shear waves from the earth's core. We have been able further to reduce the upper bound on the rigidity of the core, to estimate the average dissipation coefficient for 20–30 sec shear waves in the mantle, and to estimate the dissipation in the regions of the mantle above and below the earthquake focus. In addition, we have recomputed the origin time and depth of focus from information contained in this single seismogram.

Discussion.—On December 8, 1962, an earthquake having an assigned focal depth of 620 km occurred in South America at 25.8° south latitude and 63.4° west longitude. The east-west long-period component seismogram recorded at the United States Coast and Geodetic Survey world-wide network station of Antofagasta, Chile, for this date is shown in Figure 1. The azimuth of approach to Antofagasta is 109°, measured clockwise from north. Unfortunately, the beginning of the earthquake is not available for examination, but the prominent pulses are identified as ScS_2 (twice reflected shear wave), $sScS_2$ (surface image of ScS_2), ScS_3, and $sScS_3$. The epicentral distance to Antofagasta is 6.7° and, because we are dealing with double and triple reflections from the core boundary, the angles of incidence at the core boundary and the earth's surface are near normal incidence (Table 1). Since these pulses are predominant on the radial component seismogram, they are identified as the SV component of the multiply reflected shear waves. Very little motion is evident on the north-south and vertical component seismograms at these arrival times. Motion attributable to ScS_4 and ScS_5 is also apparent on the original record.

TABLE 1

EARTHQUAKE OF DECEMBER 8, 1962, RECORDED AT ANTOFAGASTA, CHILE

Phase	Arrival time (GMT)	Relative amplitude	Period (sec)	Angle of incidence at focus* (deg)	Angle of incidence at core (deg)	Angle of incidence at free surface (deg)
ScS_2	21h 56m 30.0s	104	22–30	2	4	1
	21h 56m 54.0s†					
$sScS_2$	22h 00m 37.0s					
	22h 01m 3.0s†	79	21–28			
ScS_3	22h 12m 7.0s					
	22h 12m 31.0s†	54	22–29	1.5	3	1
$sScS_3$	22h 16m 31.0s					
	22h 16m 43.0s†	39.5	28			

* From charts given by Ritsema[2] based on Jeffreys'[3] velocity distribution.
† Maximum amplitude arrival time.

Arrival times, amplitudes, and other information pertinent to the present study are tabulated in Table 1. The assigned USCGS origin time is 21h 27m 22.2s GMT. The origin time based on the tabulated phases is 21h 27m 18s GMT with an accuracy of about 1 sec. This is one of the few situations in which a seismogram from a single station is sufficient to determine an accurate origin time with no other information or assumptions. The accuracy is limited by the long-period, 20–30 sec, of the waves and the consequent difficulty of accurately picking arrival times.

The transit time for shear waves in the mantle can be calculated from the delay time between successive reflections. From the ScS waves it is determined to be 15m 37s. From the onset times of the sScS waves it is 15m 54s, and from the maximum amplitudes of sScS it is 15m 40s, the latter value being more accurate because of the distortion and low signal to noise ratio of the later arriving sScS signal. These values are appropriate for an epicentral distance

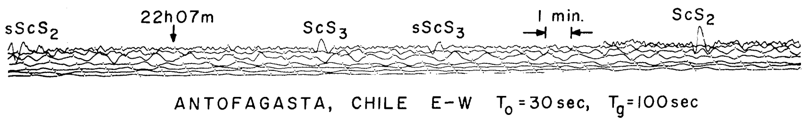

FIG. 1.—Long-period seismogram recorded at Antofagasta, Chile, showing multiply reflected core phases.

of about 3°. A vertical path would involve a correction of less than 0.5 sec, much less than the measuring error. These times can be compared with 15m 35.7s given in the Jeffreys-Bullen tables, 15m 37.1s calculated for a Gutenberg model earth with a 30-km thick crust, and 15m 39.4s determined for the CIT 11A oceanic model of Anderson and Toksoz[4]. The conclusion to be drawn here is that no major revision in the average shear velocity in the mantle seems to be necessary. A tentative conclusion is that the shear velocities averaged over the crust and mantle are the same in continental regions as in oceanic regions. This would indicate that continents are more likely to have been derived from the underlying mantle than to have been superposed on the mantle either from external sources or from adjacent regions of the mantle. The low-density, low-velocity crustal material can be envisaged as a differentiate of mantle material which leaves a higher density, higher velocity residue. The upper mantle under oceans would therefore be less dense and have a lower velocity than the upper mantle under continents. The average shear velocity in the mantle seems to be slightly less than that given by the Jeffreys-Bullen model, and more in agreement with a Gutenberg-type model in which there is a low-velocity zone in the upper mantle.

The two-way travel time between the earth's surface and the focal depth is about 4m 7s, and from this depth to the core-mantle boundary is about 11m 30s. This would place the focal depth between approximately 590 and 600 km, depending on the earth model assumed.

The amplitudes of the successive ScS and sScS phases can be used to place upper bounds on both the rigidity of the earth's core and the attenuation of shear waves in the earth's mantle. Both of these quantities are important in any discussion of the earth's interior.

For convenience we will write the amplitudes of ScS_l and $sScS_l$ as C_l and sC_l. Then

$$C_l = R^l \bar{R}^{l-1} C_o \exp\,[-2k_2 l h_2] \exp\,[-(2l - 1)k_1 h_1]/[2lh_2 + (2l - 1)h_1]$$

$$sC_l = \bar{R}^l C_l \exp\,[-2k_1 l h_1] \left[\frac{2lH - h_1}{2H + h_1}\right]$$

where C_o = amplitude at origin,
R = reflection coefficient at core,
$\bar{R}$ = reflection coefficient at surface,
h_1 = focal depth,
$h_2 = H - h_1$,
H = depth of core,
k_1 = attenuation coefficient above focus, and
k_2 = attenuation coefficient below focus.

For vertically incident shear waves the plane wave reflection coefficient at the core is approximately

$$R = [(\mu_m \rho_m)^{1/2} - (\mu_c \rho_c)^{1/2}][(\mu_m \rho_m)^{1/2} + (\mu_c \rho_c)^{1/2}]^{-1}$$

where μ and ρ are rigidity and density and the subscripts refer to mantle (m) and core (c) in the vicinity of the boundary. An upper bound can be placed on the rigidity of the core by setting $\bar{R}$ equal to one and k_1 and k_2 equal to zero, thereby obtaining

$$\frac{C_{l+n}}{C_l} = R^n \left[\frac{2lH - h_1}{2(l + n)H - h_1}\right].$$

If we have a surface focus ($h_1 = 0$) and compare the amplitudes of $ScS_1(l = 1)$ and $ScS_2(n = 1)$,

$$C_2/C_1 = R/2H$$

which was the expression used by Press[1] for obtaining an upper bound on core rigidity.

Using the observed amplitude ratio of 0.52 for C_3/C_2 and assuming a perfectly elastic mantle we obtain

$$\mu_c < \frac{0.013\mu_m \rho_m}{\rho_c};$$

with $\rho_m/\rho_c = 0.55$ and $\mu_m = 3 \times 10^{12}$ dynes/cm^2 we find $\mu_c < 2 \times 10^{10}$ dynes/cm^2. This can be considered an absolute upper bound for the rigidity of the outer core using the stated assumptions.

From a variety of evidence we can assume that the average attenuation coefficient in the mantle for 20-sec waves is unlikely to be less than 10^{-5}/km. With this value the upper bound on rigidity is reduced by about 30 per cent. In the earlier experiment Press[1] determined that $\mu_c < 10^{11}$ dynes/cm.^{2} Thus, we have been able to reduce the upper bound on core rigidity by about an order of magnitude and strengthen the arguments for a fluid outer core.

The attenuation of seismic waves in the earth is difficult to determine. Path, source, and instrumental effects limit the accuracy of conventional measurements. These difficulties are not present in the method of Press. By setting $R = \bar{R} = 1$ we may obtain an upper bound for the average attenuation coefficient in the mantle by comparing C_l with C_{l+n} or sC_l with sC_{l+n}:

$$k = \log_e \left[\frac{2(l+n) - h_1}{2l - h_1} \cdot \frac{C_{l+n}}{C_l}\right] \Big/ 2H.$$

Using the observed amplitude ratio of 0.52 for the multiple ScS phases we obtain $k = 4.0 \times 10^{-5}$. The amplitude ratio of 0.50 for the multiple sScS phases yields $k = 4.7 \times 10^{-5}$. The dimensionless dissipation parameter Q is given by

$$Q = \pi/\beta T k,$$

where β and T are, respectively, the average shear velocity in the mantle and the period of the waves.

We obtain $Q = 508$ for a period of 25 sec from the ScS data and $Q = 440$ from the sScS data. The former value is considered the more reliable and is essentially identical with the value of 500 obtained in the earlier experiment by Press using 11 sec ScS phases. This indicates that Q is roughly frequency-independent over the period range of 11–25 sec.

The above measurement determines an average Q for the whole mantle. With one additional assumption we can go much further. If the upgoing and downgoing rays have equal amplitude at the source, we can determine an average Q for the regions above and below the focal depth. All simple seismic source models satisfy this criteria. Only an extended source having dimensions comparable to a wavelength and progressing unilaterally with a vertical component of velocity comparable to the shear velocity at the focal depth would have a radiation pattern detrimental to the assumption.

The attenuation coefficient in the upper mantle can be determined from the sC_l/C_l ratios. Using sC_2/C_2 we obtain $k_1 = 1.42 \times 10^{-4}$. With sC_3/C_3 we obtain $k_1 = 1.75 \times 10^{-4}$. The corresponding Q's are 185 and 151. (An independent study on attenuation in the upper mantle which is due to appear in the Carnegie Institute of Washington Yearbook (1962) yields a result substantially in agreement with these values.) Similarly we can obtain the attenuation in the lower mantle from the C_{n+1}/sC_n ratios, or from the relation

$$k_2 = (kH - k_1h_1)/h_2.$$

For k_2 we obtain 1.3×10^{-5}. The corresponding Q is 1430, almost an order of magnitude greater than the Q in the upper 600 km, and about 3 times the average Q of the whole mantle. This is the first direct measurement of the attenuation of seismic waves in the deep mantle. We hope to extend this study by using earthquakes at other focal depths in order to determine Q as a function of depth.

The low order torsional oscillations of the earth give a value for Q of about 400. This represents a weighted average of the Q of the mantle with the upper layers contributing slightly more to the average. Mantle Love waves yield a Q value of about 100. This number is appropriate for the upper several hundred kilometers. All of the above observations are consistent with an increase of Q with depth in the lower mantle.

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