that $\mathfrak{g}^*$ has the same derived algebra as $\mathfrak{g}$ itself and that every ideal in $\mathfrak{g}$ is also an ideal in $\mathfrak{g}^*$.

Let $\mathfrak{g}$ be any algebraic Lie algebra. Denote by $\mathfrak{h}$ the radical of $\mathfrak{g}$ (i.e., the largest solvable ideal in $\mathfrak{g}$) and by $\mathfrak{n}$ the largest ideal of $\mathfrak{g}$ composed only of nilpotent matrices. By Levi's theorem, $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and of a semi-simple subalgebra $\mathfrak{f}$. It can be proved that $\mathfrak{h}$ is the direct sum of $\mathfrak{n}$ and of Abelian algebra $\mathfrak{a}$ whose matrices are semi-simple and commute with those of $\mathfrak{f}$.

Let $\mathfrak{g}$ be any subalgebra of $\mathfrak{gl}(n, K)$; then it can be shown that the derived algebra $\mathfrak{g}'$ is algebraic. Moreover, $\mathfrak{g}'$ can be "defined by its invariants," in the sense that any matrix which admits as its invariants all the common invariants of all matrices in $\mathfrak{g}'$ lies itself in $\mathfrak{g}'$. Our result applies in particular to any semi-simple Lie algebra $\mathfrak{g}$ of $\mathfrak{gl}(n, K)$, which is identical with its derived algebra $\mathfrak{g}'$. Moreover, our method of proof shows more generally that any subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n, K)$ whose radical is composed only of nilpotent matrices is algebraic and is defined by its invariants.

If A is any algebra (associative or not) over the field K, the derivations of A form a Lie algebra which is easily seen to be algebraic.

Finally, let it be mentioned that the notion of algebraic Lie algebras can be used with advantage in the exposition of the theory of semi-simple Lie algebras, notably in establishing Cartan's criterion of semi-simplicity and Lie's theorem on solvable Lie algebras. Barring the recourse to the algebraic closure of the basic field in the proof of Theorem 3 of the paper quoted above,[2] one obtains in this way a rational proof of Cartan's criterion and of Lie's theorem.

[1] Maurer, L., "Zur Theorie der continuerlichen, homogenen und linearen Gruppen," *Sitzungsber. d. Bayerischen Akad. Math. Phys. Classe*, **24**, 297–341 (1894).

[2] Chevalley, C., "A New Kind of Relationship between Matrices," *Amer. J. Math.*, **65**, 521–531 (1943).

TWO INTEGRAL EQUATIONS

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Let $k(s, t) = g(s, t) - c\, g(s, a - t)$ where $g(s, t)$ is a real continuous kernel for $0 \leq s, t \leq a$ and c is an arbitrary constant. The equations to be considered are

$$f(s) = \int_0^a k(s, t)\, F(t)dt \qquad (1)$$

$$f(s) = F(s) - x\int_0^a k(s, t)F(t)dt \qquad (2)$$

and the required function $F(t)$ is to be continuous when $f(s)$ is continuous. When this condition is supposed to be satisfied equation (1) may be written in the equivalent form

$$f(s) = \int_0^a g(s, t)[F(t) - c\,F(a - t)]dt$$

and when a continuous function $h(t)$ exists for which

$$f(s) = \int_0^a g(s, t)h(t)dt$$

a solution of the integral equation may be obtained by solving the functional equation

$$h(t) = F(t) - c\,F(a - t) \qquad 0 \leq t \leq a.$$

Since $h(a - t) = F(a - t) - c\,F(t)$ the solution should be

$$(1 - c^2)F(t) = h(t) + c.h(a - t).$$

If $c^2 = 1$ this last equation fails to determine $F(t)$ and this is just the case when the homogeneous integral equations

$$0 = \int_0^a k(s, t)E(t)dt, \qquad 0 = \int_0^a H(s)k(s, t)ds$$

have innumerable solutions. Thus, when $c = 1$ any solution of the functional equation $E(t) = E(a - t)$ which gives finite integrals is a solution of the first equation and if $L(t) = \int_0^a H(s)k(s, t)ds$, the second equation is satisfied whenever $L(t) = L(a - t)$.

For equation (2) the homogeneous integral equation

$$0 = E(s) - z\int_0^a k(s, t)E(t)dt$$

may be written in the form

$$E(s) = z\int_0^a g(s, t)[E(t) - c\,E(a - t)]dt \qquad 0 \leq s \leq a$$

Changing s into $a - s$ we have

$$c\,E(a - s) = cz\int_0^a g(a - s, t)[E(t) - c\,E(a - t)]dt$$

Hence the function $H(t) = E(t) - c\,E(a - t)$ is a solution of the quasi-adjoint[1] homogeneous integral equation

$$H(s) = z\int_0^a [g(s, t) - c\,g(a - s, t)]H(t)dt$$

When the equation for $E(t)$ is written in the form

$$E(s) - z\int_0^a g(s, t)E(t)dt + cz\int_0^a g(s, a - t)E(t)dt = 0$$

it is seen that

$$E(s) = -cz\int_0^a G(s, a - t)E(t)dt$$

where $G(s, t)$ is the solving kernel associated with $g(s, t)$. Now this is a homogeneous integral equation for $E(t)$ and so $-cz$ is a root of the transcendental equation $\Delta(-cz, z) = 0$, where Δ is the determinantal function associated with the kernel $G(s, a - t)$ and is an entire function of $-cz$.

The kernel $k(s, t) = g(s, t) - c\, g(s, a - t)$ is only of a special type when $c^2 = 1$ for the two equations

$$k(s, t) = g(s, t) - c\, g(s, a - t),\ k(s, a - t) = g(s, a - t) - c\, g(s, t)$$

give the relation

$$k(s, t) + c\, k(s, a - t) = (1 - c^2) g(s, t)$$

from which the function $g(s, t)$ can be determined when c^2 is different from unity.

In the particular case in which

$$\begin{aligned} g(s, t) &= s(1 - t) \qquad s \leq t \qquad a = 1 \\ &= t(1 - s) \qquad t \leq s \end{aligned}$$

the integral equation (2) may be written in the form

$$f(s) = F(s) - x \int_0^1 g(s, t)[F(t) - c\, F(1 - t)]dt.$$

Differentiating twice with respect to s and using the known fact that $g(s, t)$ is the Green's function of the differential expression d^2y/ds^2 we obtain the two equations

$$f''(s) = F''(s) + x[F(s) - c\, F(1 - s)]$$

$$f''(1 - s) = F''(1 - s) + x[F(1 - s) - c\, F(s)].$$

Addition and subtraction gives the equations

$$f''(s) + f''(1 - s) = E''(s) + m^2 E(s) \qquad m^2 = x(1 - c)$$

$$f''(s) - f''(1 - s) = B''(s) + n^2 B(s) \qquad n^2 = x(1 + c)$$

$$E(s) = F(s) + F(1 - s), E(0) = E(1) = F(0) + F(1) = f(0) + f(1)$$

$$B(s) = F(s) - F(1 - s), B(0) = -B(1) = F(0) - F(1) = f(0) - f(1).$$

When $f(s) = 0$ we have simply

$$E''(s) + m^2 E(s) = 0,\ E(0) = E(1) = 0$$

$$B''(s) + n^2 B(s) = 0,\ B(0) = B(1) = 0.$$

Hence

$$E(s) = A \sin(ms) \text{ where } \sin(m) = 0,\ x(1 - c) = p^2\pi^2$$

$$B(s) = B \sin(ns) \text{ where } \sin(n) = 0,\ x(1 + c) = q^2\pi^2$$

where p and q are positive integers. The two equations

$$x(1 - c) = p^2\pi^2, x(1 + c) = q^2\pi^2$$

cannot generally be satisfied simultaneously although this can happen for special values of c. Usually when $x(1 - c) = p^2\pi^2$, $B = 0$ and when $x(1 + c) = q^2\pi^2$, $A = 0$.

In the general case in which $f(s)$ is not zero the solution of the integral equation (2) is

$$F(s) = f(s) + \frac{1}{2}\int_0^1 [U(s, t) + U(s, 1 - t) + V(s, t) - V(s, 1 - t)]f(t)dt$$
$$- \frac{1}{2}\int_0^s [m \sin \{m(s - t)\} + n \sin \{n(s - t)\}]f(t)dt$$
$$+ \frac{1}{2}\int_0^{1-s} [n \sin \{n(s - 1 + t)\} - m \sin \{m(s - 1 + t)\}]f(t)dt$$

where

$$U(s, t) = m \operatorname{cosec} (m) \sin (ms) \sin m(1 - t).$$
$$V(s, t) = n \operatorname{cosec} (n) \sin (ns) \sin n(1 - t).$$

Since

$$F(s) = f(s) + x \int_0^1 K(s, t)f(t)dt$$

where $K(s, t)$ is the solving kernel corresponding to $k(s, t)$, an expression for $K(s, t)$ may be written down.

The present example provides an interesting illustration of a method of solving a homogeneous integral equation of the first kind which was discussed about 36 years ago.[2]

When equation (2) with $f(s) = 0$ is written in the form

$$M(c)E(s) = \int_0^a [g(s, t) - c\, g(s, a - t)]E(t)dt$$

it seems likely that the values of c for which $M(c) = 0$ will be those for which the homogeneous integral equation

$$0 = \int_0^a [g(s, t) - c\, g(s, a - t)]E(t)dt$$

can be satisfied. Now in the present example the possible forms of $M(c)$ are

$$(1 - c)/p^2\pi^2 \qquad \text{and } (1 + c)/q^2\pi^2.$$

When $c = 1$ the function $\sin (ms)$ with an odd integral value of m/π satisfies the functional equation $E(t) = E(1 - t)$ and when $c = -1$ the function $\sin (ns)$ with an even value of n satisfies the functional equation

$E(t) = -E(1 - t)$. The known solutions of the homogeneous integral equation of the first kind are thus actually furnished by this method if it is agreed that a relevant solution of the functional equation $E(t) = \pm E(1 - t)$ can be expressed as a linear combination of the functions of type sin (mt).

[1] When $g(s, t) = g(t, s)$ the equation for $H(s)$ is adjoint to the equation for $E(s)$.

[2] H. Bateman, *Trans. Camb. Phil. Soc.*, **21**, 195 (1909); *Messenger Math.*, **39**, 6–19, 182–191 (1909–1910). Integral equations containing a parameter have been discussed also by G. Barba, *Relat. Soc. Gideniae Catinensis Nat. Soc. Catania*, **66**, 3–10 (1934); N. Gioranescu, *Bull. Sci. Math.*, s. 2, **58**, 270–272 (1934); *Rend. Semin. Mat. Padova*, **5**, 81–98 (1934); C. Miranda, *Palermo Rend.*, **60**, 286–304 (1937); R. Iglisch, *Math. Ann.*, **117**, 129–139 (1939).

VITAL STATISTICS OF THE NATIONAL ACADEMY OF SCIENCES

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Under the title "Vital Statistics of the National Academy of Sciences," Raymond Pearl, about twenty years ago when acting as Chairman of a Committee appointed to consider matters concerned with the rules for election to the Academy, published a series of articles which still deserve attention.[1] He gave figures for the mean and median ages and the standard deviation of the age distribution for different periods in the history of the Academy. I reproduce his results with the addition of those for the 81 persons elected in the three years 1943–1945, and with columns showing the percentage under 40 years of age when elected and the number elected in the period.

Period	Mean Age	Median Age	Standard Deviation	Per Cent Under 40	Number Elected
Charter, 1863	51.7	51.3	11.1	16.7	48
1864–1883	44.5	41.3	10.2	45.5	95
1884–1904	46.5	45.1	8.5	20.0	65
1905–1924	50.5	49.5	8.1	8.4	213
1943–1945	52.1	51.8	7.6	3.7	81

It will be observed that the tendencies toward a higher mean age, a median nearer the mean, a smaller standard deviation about the mean, and a sharply decreasing percentage of really young persons elected, which Pearl so much deplored, have persisted and been intensified.

The average number of persons elected each year in the first twenty years after the charter was 4, in the next twenty years it was 3, in the twenty years 1905–1924 it was 11, and for the most recent three years it