

How Many e -Folds Should We Expect from High-Scale Inflation?

Grant N. Remmen* and Sean M. Carroll†
Walter Burke Institute for Theoretical Physics
California Institute of Technology, Pasadena, CA 91125
 (Dated: May 23, 2014)

We address the issue of how many e -folds we would naturally expect if inflation occurred at an energy scale of order 10^{16} GeV. We use the canonical measure on trajectories in classical phase space, specialized to the case of flat universes with a single scalar field. While there is no exact analytic expression for the measure, we are able to derive conditions that determine its behavior. For a quadratic potential $V(\phi) = m^2\phi^2/2$ with $m = 2 \times 10^{13}$ GeV and cutoff at $M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV, we find an expectation value of 2×10^{10} e -folds on the set of FRW trajectories. For cosine inflation $V(\phi) = \Lambda^4[1 - \cos(\phi/f)]$ with $f = 1.5 \times 10^{19}$ GeV, we find that the expected total number of e -folds is 50, which would just satisfy the observed requirements of our own Universe; if f is larger, more than 50 e -folds are generically attained. We conclude that one should expect a large amount of inflation in large-field models and more limited inflation in small-field (hilltop) scenarios.

PACS numbers: 98.80.Cq, 98.80.Jk, 98.80.Bp

I. INTRODUCTION

The possible detection of tensor perturbations in the cosmic microwave background (CMB) by the BICEP2 experiment [1] suggests that inflation occurred at a high energy scale [2]: $E_I = 2 \times 10^{16}$ GeV, just two orders of magnitude below the reduced Planck scale $M_{\text{Pl}} = 1/\sqrt{8\pi G} = 2.4 \times 10^{18}$ GeV. Knowing this parameter with some confidence allows both for much more focused inflationary model-building and for quantitative exploration of some of the conceptual issues underlying the inflationary paradigm. In this paper we address one of the latter: given an inflaton potential that is able to reproduce the measured cosmological parameters, how much inflation is likely to have occurred? In this paper we answer this question, finding that the expected number of e -folds of inflation depends dramatically on the general type of inflaton potential chosen.

The amount of inflation that occurs is measured by the number of e -folds,

$$N = \int_{a_i}^{a_f} d \ln a = \int_{t_i}^{t_f} H dt. \quad (1)$$

Here, a_i and a_f are the values of the scale factor at the beginning and end of inflation, while t_i and t_f are the corresponding proper times. We can define the period during which inflation is occurring as that for which the Universe is accelerating, $\ddot{a} > 0$. In conventional inflationary models, it is necessary to achieve at least 50 e -folds to successfully address the horizon problem. It is generally accepted that this requirement can be met by a wide variety of potentials.

We would like to know not only whether a certain potential can possibly produce sufficient amounts of in-

flation, but whether such an outcome is actually likely. Presumably, a complete theory of cosmological initial conditions in the context of quantum gravity would provide a unique answer to this question, but we don't have such a theory at present. What we do have are classical models of inflaton dynamics coupled to general relativity. Any classical theory comes with a natural measure on phase space, the Liouville measure. Gibbons, Hawking, and Stewart (GHS) showed how to use this measure to define a canonical measure on cosmological *trajectories* (rather than individual points in phase space) [3]. In this measure, we can calculate the fraction of universes with given properties, such as “more than 50 e -folds of inflation.” Given the current state of the art, this is the best we can do to decide whether such solutions are likely or not.

The GHS measure comes with a technical problem when applied to (homogeneous, isotropic) Friedmann–Robertson–Walker (FRW) cosmologies: it diverges as the spatial curvature approaches zero, assigning almost all measure to flat universes. Different proposals have been advanced for dealing with this divergence, including removing the region of infinite measure by hand [4]. As noted in Refs. [5] and [6], the divergence for flat universes is an indication that, in the canonical measure, almost all cosmological spacetimes are flat. For this reason, and also given the physical relevance of spatially flat solutions [7], it is on these that we concentrate our efforts. In a previous paper [6], we developed a formalism for defining the Hamiltonian-conserved measure on the effective two-dimensional phase space for a canonical scalar field with a potential in a flat FRW cosmology. Although we did not prove the uniqueness of this measure in arbitrary theories, we could establish it for quadratic potentials and expect it to hold for well-behaved potentials more generally.

In this paper we employ the formalism developed in Ref. [6] to study high-scale inflation. We focus on two representative models: quadratic inflation and cosine

* gremmen@theory.caltech.edu

† seancarroll@gmail.com

(“natural”) inflation. We find dramatically different quantitative results for the two cases. In quadratic inflation, given that the potential is chosen to fit observed cosmological parameters, we find that large amounts of inflation are favored by the canonical measure — billions of e -folds of inflation — provided we extrapolate the quadratic potential up to the Planck scale $H = M_{\text{Pl}}$ and allow the inflaton field ϕ to run over a super-Planckian range $\sim 10^5 M_{\text{Pl}}$. Moreover, we find that almost all trajectories experience well more than 50 e -folds. For cosine potentials, by contrast, the expected amount of inflation under the canonical measure is relatively small: if the symmetry-breaking parameter f is set to the reduced Planck scale, $M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV, we expect of order one e -fold, with the probability of attaining as many as 50 e -folds being exponentially small. These numbers depending sensitively on f ; once it is above 10^{19} GeV, as favored by the BICEP2 result [1, 8], the probability of getting more than 50 e -folds rises above 50%.

This last result is interesting, since cosine potentials feature “hilltops” from which trajectories with arbitrarily large numbers of e -folds can originate. Our analysis demonstrates that, while such lingering solutions are allowed, they contribute a relatively small amount to the measure on the space of trajectories. We conjecture that this behavior reflects a more general difference between potentials that rise up to the Planck scale, in which we expect large amounts of inflation, and models with potential maxima below the Planck scale, where the expected number of e -folds will be comparatively small.

Any analysis of this form necessarily comes with caveats. As noted, we are using a classical measure, whereas a particular theory of initial conditions (*e.g.*, a proposal for the wave function of the Universe) will presumably make its own predictions. More seriously, our analysis applies only to universes that are assumed to be homogeneous from the start. Once perturbations are included, it is clear that most universes should be wildly inhomogeneous; the existence of the sufficiently smooth initial conditions necessary for inflation to begin is highly non-generic [5, 9]. Given the evidence that inflation did happen, we consider the expected number of e -folds according to the canonical measure to be a useful diagnostic of which models are robust and which are more delicate. An ultimate justification for why inflation occurs in the first place awaits further insight.

This paper is organized as follows. In Sec. II we first review the formalism of Refs. [3] and [6] for finding the canonical measure on phase space, as well as the sense in which phase space becomes effectively only two-dimensional for flat FRW cosmologies. The connection between the measure on effective phase space and the measure on the space of possible trajectories of evolution of a FRW universe is presented. Next, in Sec. III we derive some general properties of the measure for arbitrary slow-roll and hilltop potentials. Finally, we examine representative models of each class, quadratic

inflation and cosine inflation, in Secs. IV and V, making statistical calculations on the ensemble of all FRW universes and finding the expected number of e -folds of inflation attained.

II. THE PROBABILITY DISTRIBUTION ON THE SET OF UNIVERSES

A. The Hamiltonian-conserved measure

We are interested in the theory of a homogeneous scalar field in an expanding FRW universe. The action is

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \quad (2)$$

The metric can be written

$$ds^2 = -N^2(t) dt^2 + a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right), \quad (3)$$

where N is the lapse function and the curvature parameter κ is an arbitrary real parameter with mass dimension 2. The number κ is fixed for a given FRW universe and we can write $k = \kappa R_0^2 \in \{-1, 0, 1\}$, where R_0 is the radius of curvature of the universe at unit scale factor. Taking $\phi(t)$ to depend only on time, the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= -3a^3 N M_{\text{Pl}}^2 \left\{ \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{1}{3M_{\text{Pl}}^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \right\} \\ &= N \left[-\frac{p_a^2}{12aM_{\text{Pl}}^2} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) - 3a\kappa M_{\text{Pl}}^2 \right], \end{aligned} \quad (4)$$

where p_a and p_ϕ are the momenta conjugate to the scale factor and scalar field, respectively. The scalar equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (5)$$

where $V'(\phi) = dV/d\phi$. The Hamiltonian constraint, which comes from varying with respect to N , is the Friedmann equation,

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] - \frac{\kappa}{a^2}, \quad (6)$$

where $H = \dot{a}/a$ is the Hubble parameter.

Any classical theory comes with a preferred choice of measure on phase space: the Liouville measure, which is preserved under time evolution. In cosmology our interest is less in a measure on individual points in phase space and more in a measure on trajectories through time or specific cosmological evolutions. Gibbons, Hawking, and Stewart [3] showed how to construct such a measure for a scalar field coupled to general relativity. The phase space is naïvely four-dimensional, with coordinates

given by a and ϕ and their conjugate momenta. But the Hamiltonian constraint, implemented by the Friedmann equation, cuts this down to three dimensions. The space of trajectories (equivalent under the equations of motion to the space of initial conditions) is one lower, leaving us with a two-dimensional space. GHS were able to construct a unique measure on this space that is positive and invariant under time evolution (for further discussion see Refs. [4–6]).

As Ref. [5] shows, the GHS measure [3] has an interesting property: on a transverse surface in phase space defined by fixed Hubble parameter, the measure diverges for small curvature κ as $|\Omega_k|^{-5/2}$. This behavior has the good feature that it implies that the collection of non-flat FRW universes is a set of measure zero under the GHS measure; that is, the flatness problem in cosmology is solved by the GHS measure, since almost all trajectories are flat. However, from the point of view of understanding the set of flat FRW universes itself, this behavior poses a technical challenge. It is difficult to regularize the divergence in the GHS measure to construct a well-defined measure within the space of flat universes.

In our previous paper we showed how to find a measure on the space of flat universes by constructing it by hand, subject to the requirement that it be conserved under time evolution [6]. We note from Eqs. (5) and (6) that the scale factor a disappears from the equations of motion when $\kappa = 0$. The effective phase space is therefore only two-dimensional; specifying the two quantities ϕ and $\dot{\phi}$ completely determines the solution (although they are not conjugate variables). The set of trajectories in effective phase space is therefore one-dimensional. In Ref. [6] we formalized the notion of an effective phase space via the property of vector field invariance between two manifolds. We argued that there exists a unique measure on this space that is conserved under Hamiltonian flow, in analogy with the conventional Liouville measure, and used that to construct a measure on the space of flat universes.

The time evolution given by Eq. (5) can be characterized by a vector field \mathbf{v} on ϕ - $\dot{\phi}$ space, with components

$$\mathbf{v} = \left(\dot{\phi}, -V'(\phi) - 3H\dot{\phi} \right). \quad (7)$$

The Hubble parameter (and thus the scale factor, up to an irrelevant scaling) is then fixed by Eq. (6). We seek a two-form

$$\boldsymbol{\sigma} = \sigma(\phi, \dot{\phi}) d\dot{\phi} \wedge d\phi \quad (8)$$

that is conserved under evolution,

$$\mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma} = 0, \quad (9)$$

or equivalently,

$$\nabla \cdot (\boldsymbol{\sigma}\mathbf{v}) = 0. \quad (10)$$

A two-form $\boldsymbol{\sigma}$ for which σ satisfies the Hamiltonian-conservation constraint (10) — the same as the Euler

equation for stationary fluid flow — is the natural measure on the effective phase space, exactly in analogy with the Liouville measure. We will call the function σ , which forms the probability distribution on effective phase space in a given coordinate system, the *measure density*.

At this point, it is natural to ask whether there is a Lagrangian description \mathcal{L}_{Φ} of the trajectories on the effective phase space Φ . Using Douglas’ theorem and the Helmholtz conditions, we showed [6] that there exists a time-independent Lagrangian description of the equation of motion (5) on effective phase space if and only if there exists a Hamiltonian conserved measure: in fact, finding the Lagrangian gives a measure satisfying Eq. (10) and vice versa. Further, defining $\pi_{\dot{\phi}} = \partial\mathcal{L}_{\Phi}/\partial\dot{\phi}$ as the conjugate momentum on Φ , one finds that the Liouville measure $d\pi_{\dot{\phi}} \wedge d\dot{\phi}$ on effective phase space under \mathcal{L}_{Φ} is just equal to $\sigma d\dot{\phi} \wedge d\phi$, obtained merely by demanding conservation under Hamiltonian evolution.¹ For the specific example of $m^2\phi^2$ inflation, we proved that such a measure exists and is unique; such an existence/uniqueness result likely holds for any reasonably well-behaved potential $V(\phi)$.

B. The space of trajectories

Given the appropriate measure on our effective phase space, one can use this to determine the natural measure on the space of trajectories. In general, given some arbitrary measure density on a two-dimensional manifold and a one-parameter collection of curves that cover the manifold, there is not a well-defined probability distribution on the set of curves. However, the Hamiltonian-conserved measure density on effective phase space is not an arbitrary function vis-à-vis the family of trajectories. Following Refs. [3, 6], we can construct a measure on the space of trajectories in terms of a one-dimensional measure on any curve transverse to those trajectories, by demanding that the physical result be independent of our choice of transverse curve.

We begin by choosing some curve in the ϕ - $\dot{\phi}$ effective phase space on which to evaluate the measure density $\sigma(\phi, \dot{\phi})$. For simplicity, we’ll imagine choosing $H = \text{constant}$ surfaces, but any other slicing transverse to the trajectories that evolves monotonically in time would work just as well.² We can reparametrize ϕ - $\dot{\phi}$ space in terms of H and another coordinate, which we

¹ The corresponding Hamiltonian on effective phase space, $\mathcal{H}_{\Phi} = \pi_{\dot{\phi}}\dot{\phi} - \mathcal{L}_{\Phi}$, is of course not subject to any additional Hamiltonian constraint as in the full phase space; that is, the Friedmann equation is merely a redefinition of coordinates on Φ and does not constrain \mathcal{H}_{Φ} . With this definition, the measure can be written as $d\mathcal{H}_{\Phi} \wedge dt$.

² Note that, regardless of the potential, the scalar equation assures that H evolves monotonically in time, with $\dot{H} = -\dot{\phi}^2/2M_{\text{Pl}}^2$.

will call θ . For the bundle of trajectories that, on the H_1 surface, is centered at θ_1 and spans $d\theta_1$, write the measure as $P(\theta_1)|_{H_1} d\theta_1$. Suppose this bundle of trajectories evolves to $H = H_2$, on which surface it is centered at θ_2 and spans $d\theta_2$. We could equivalently write its probability measure as $P(\theta_2)|_{H_2} d\theta_2$. Of course, the functional forms of $P(\theta_1)|_{H_1}$ and $P(\theta_2)|_{H_2}$ and can be very different. However, this is the same bundle of trajectories, so for the measure on the space of trajectories to be well defined, we require

$$P(\theta_1)|_{H_1} d\theta_1 = P(\theta_2)|_{H_2} d\theta_2. \quad (11)$$

Now, we note that, given a parcel on effective phase space covering the region $d\theta_1 dH_1$ that evolves to $d\theta_2 dH_2$, we have

$$\sigma(H_1, \theta_1) d\theta_1 dH_1 = \sigma(H_2, \theta_2) d\theta_2 dH_2. \quad (12)$$

This is just the statement of Liouville's theorem for effective phase space, *i.e.*, the requirement that σ satisfy (10). Hence, the correct way to compute $P(\theta)|_H$, the probability distribution on the space of trajectories, parametrized by the coordinate θ with which the trajectory intersects the H surface, is

$$P(\theta)|_H \propto \sigma(H, \theta) dH. \quad (13)$$

We can divide through by dt , since t evolves uniformly for all trajectories. We therefore have

$$P(\theta)|_H = \frac{\sigma(H, \theta) |\dot{H}|}{\int \sigma(H, \theta') |\dot{H}| d\theta'}. \quad (14)$$

Note that we suppressed the arguments (H, θ) of \dot{H} .

Eq. (14) is the important expression for this work. The measure on the space of trajectories is constructed by finding a conserved measure density σ on the effective phase space and evaluating $|\dot{H}|$ times this measure along a surface of constant H .

As a consistency check, we can derive Eq. (14) in a slightly different way. If we had written the effective phase space measure in the coordinates (t, θ) as $\tilde{\sigma}(t, \theta) d\theta \wedge dt = -\sigma(H, \theta) d\theta \wedge dH$ (with the minus sign compensating for the fact that H decreases with t , so that σ and $\tilde{\sigma}$ are positive) we could have equivalently defined the measure on the space of trajectories by explicitly performing the integration over t :

$$P(\theta_0)|_{H_0} \propto \int_0^\infty \tilde{\sigma}(t, \theta(t)) dt, \quad (15)$$

where the path $(t, \theta(t))$ is chosen such that $\theta(t_0) = \theta_0$ and $H(t_0) = H_0$ for some t_0 . Since t evolves uniformly for all trajectories, we have

$$P(\theta_0)|_{H_0} \propto \tilde{\sigma}(t_0, \theta_0)|_{H(t_0)=H_0} = \sigma(H_0, \theta_0) |\dot{H}|, \quad (16)$$

in agreement with Eq. (14).

We are now equipped to make quantitative statements about probabilities of different FRW trajectories for universes with zero curvature and compare these predictions for different models of inflation.

III. THE EFFECTIVE PHASE SPACE MEASURE FOR GENERIC POTENTIALS

Before examining specific models of inflation, it will first be informative to examine the behavior of the effective phase space measure σ , without assuming an explicit functional form of the potential, in two representative classes of inflation: slow roll down a potential and quasi-de Sitter inflation near a local maximum in a potential, *i.e.*, a hilltop. The cases are distinct because the fixed point in effective phase space corresponding to a stationary field at a potential maximum is a distinguished trajectory by itself and must be treated carefully.

For this analysis it will be useful to define dimensionless coordinates

$$z_1 = \frac{\phi}{M_{\text{Pl}}} \quad \text{and} \quad z_2 = \frac{\dot{\phi}}{M_{\text{Pl}}^2}. \quad (17)$$

and a dimensionless speed in effective phase space

$$\tilde{\mathbf{v}} \equiv \frac{\dot{\mathbf{z}}}{M_{\text{Pl}}} = \left(z_2, -\tilde{V}'(z_1) - \sqrt{3}z_2 \sqrt{z_2^2/2 + \tilde{V}(z_1)} \right), \quad (18)$$

defining $\tilde{V}(z_1) \equiv V(\phi(z_1))/M_{\text{Pl}}^4$ as a dimensionless potential and notation $\tilde{V}'(z_1) \equiv d\tilde{V}/dz_1$. Define the first potential slow-roll parameter

$$\epsilon_V \equiv \frac{M_{\text{Pl}}^2}{2} \left[\frac{V'(\phi)}{V(\phi)} \right]^2 = \frac{1}{2} \left[\frac{\tilde{V}'(z_1)}{\tilde{V}(z_1)} \right]^2 \quad (19)$$

and the first Hubble slow-roll parameter:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2 M_{\text{Pl}}^2} = 3 \frac{z_2^2}{z_2^2 + 2\tilde{V}(z_1)}. \quad (20)$$

Substituting (20) into (18) and rearranging, one finds

$$\frac{|\tilde{\mathbf{v}}|^2}{\tilde{V}(z_1)^2} = \frac{4\epsilon^2 z_2^{-2} + 18\epsilon}{(3 - \epsilon)^2} + 2\epsilon_V + 6s \frac{\sqrt{2\epsilon_V} \sqrt{2\epsilon}}{3 - \epsilon}. \quad (21)$$

Here, $s \equiv \text{sgn}[z_2 \tilde{V}'(z_1)] = \pm 1$ indicates whether the potential is increasing ($s = +1$) or decreasing ($s = -1$) in the direction along which the field is evolving; we will generally have $s = -1$ during inflation. Furthermore, after simplifying with Eq. (6), we have

$$\frac{\nabla \cdot \tilde{\mathbf{v}}}{|z_2|} = -\frac{3}{\sqrt{2\epsilon}} - \sqrt{\frac{\epsilon}{2}}. \quad (22)$$

Note that Eqs. (21) and (22) are exact expressions: no slow-roll approximation has yet been made.

A. Slow roll down a potential

As we discussed in Ref. [6], slow-roll behavior corresponds to apparent attractors in effective phase space

— places where the conserved measure grows large. In this subsection we consider monotonic slow-roll behavior, characterized by two conditions imposed on Eqs. (5) and (6):

$$\begin{aligned} \dot{\phi}^2 \ll |V(\phi)| \quad \text{so} \quad H^2 \simeq \frac{1}{3M_{\text{Pl}}^2} V(\phi) \\ |\ddot{\phi}| \ll |H\dot{\phi}|, |V'(\phi)| \quad \text{so} \quad 3H\dot{\phi} \simeq -V'(\phi). \end{aligned} \quad (23)$$

Then $\varepsilon \approx \varepsilon_V \equiv \epsilon \ll 1$ and we have from Eq. (21):

$$\frac{|\tilde{\mathbf{v}}|^2}{\tilde{V}(z_1)^2} \simeq \frac{4\epsilon^2}{9z_2^2} + 4(1+s)\epsilon. \quad (24)$$

Further, imposing $H^2 \ll M_{\text{Pl}}^2$, so $\varepsilon \gg z_2^2$,

$$|\tilde{\mathbf{v}}| \simeq |z_2| \simeq \sqrt{\frac{2}{3}\epsilon\tilde{V}(z_1)}. \quad (25)$$

Similarly, in the slow-roll regime,

$$\nabla \cdot \tilde{\mathbf{v}} \simeq -\frac{3|z_2|}{\sqrt{2\epsilon}} \simeq -\sqrt{3\tilde{V}(z_1)}. \quad (26)$$

Note that the second slow-roll condition in Eq. (23) does not necessarily apply near a hilltop, as $|H\dot{\phi}| \gg |\ddot{\phi}|$ can fail. This is an important distinction; as we will see, behavior of the measure density near a hilltop in effective phase space is very different than what we find for trajectories that are uniformly slowly rolling down a potential. Our slow-roll conditions in this subsection are most compatible with potentials with $V''(\phi) > 0$, such as monomial models, in which hilltop behavior is manifestly absent.

Now, we examine what implications our analysis has for the form of the measure $\sigma = \sigma(z_1, z_2) dz_2 \wedge dz_1$ on effective phase space. From Eq. (10) we have $(\nabla\sigma) \cdot \tilde{\mathbf{v}} + \sigma(\nabla \cdot \tilde{\mathbf{v}}) = 0$, so by the Cauchy–Schwarz inequality,

$$|\nabla \ln \sigma| = \frac{|\nabla\sigma|}{\sigma} \geq \frac{|\nabla \cdot \tilde{\mathbf{v}}|}{|\tilde{\mathbf{v}}|}. \quad (27)$$

Hence, near the slow-roll regime and for $H^2 \ll M_{\text{Pl}}^2$, we find, using Eqs. (25) and (26):

$$|\nabla \ln \sigma| \gtrsim \frac{3}{\sqrt{2\epsilon}}. \quad (28)$$

Thus, the closer to the slow-roll regime we approach, the bigger $|\nabla \ln \sigma|$ has to be, so σ becomes very large. We expect that $\nabla\sigma$ is in general roughly parallel with $\tilde{\mathbf{v}}$ during slow roll, so for two points $\mathbf{z}(t_1)$ and $\mathbf{z}(t_2)$ on a single trajectory parametrized by t , we should have

$$\begin{aligned} \frac{\sigma(\mathbf{z}(t_2))}{\sigma(\mathbf{z}(t_1))} &\gtrsim \exp \left[3 \int_{\mathbf{z}(t_1)}^{\mathbf{z}(t_2)} \frac{d\mathbf{z}}{\sqrt{2\epsilon(\mathbf{z})}} \right] \\ &= \exp \left[3 \int_{t_1}^{t_2} \frac{\dot{\mathbf{z}} dt}{\sqrt{2\epsilon(\mathbf{z}(t))}} \right]. \end{aligned} \quad \left[\begin{array}{l} \text{slow roll,} \\ H \ll M_{\text{Pl}} \end{array} \right] \quad (29)$$

Hence, we generically expect that any region in effective phase space satisfying our slow-roll conditions (23) will have large measure density σ on effective phase space, relative to nearby regions. Of course, the measure density on effective phase space can be large in regions that fail the slow-roll conditions, such as during reheating, in which apparent attractor solutions traverse long paths in compact regions of effective phase space and trajectories appear to converge. All of these statements are made with regard to the phase space measure, not the measure on the space of trajectories; the factor of $|\dot{H}|$ in Eq. (14) makes this an important distinction.

B. Inflation on a hilltop potential

In a model of inflation governed by a potential with a hilltop (a local maximum), there are two types of classical solutions that differ qualitatively from the usual picture the inflaton field of rolling down the potential and reheating: 1) fixed point trajectories, *i.e.*, exactly de Sitter solutions, which start with $\dot{\phi} = 0$ at the top of the potential and inflate forever; and 2) roll-up trajectories, which start from somewhere on the slope of the potential and asymptotically approach the fixed point. The *fixed point* is the location $(\phi, \dot{\phi} = 0)$, equivalently $(z_1, z_2 = 0) \equiv \mathbf{z}_0$ in effective phase space, for which ϕ is at the hilltop of $V(\phi)$. We would like to elucidate the behavior of the phase space measure density $\sigma(z_1, z_2)$ in the region of effective phase space near the fixed point.

Near the fixed point, $\dot{\phi}^2 \ll V(\phi)$, so Eq. (22) implies $|\nabla \cdot \tilde{\mathbf{v}}| \rightarrow (3\tilde{V}(z_1))^{1/2}$, as in Eq. (26); moreover, $\tilde{\mathbf{v}} \rightarrow 0$. Imposing the requirement (10) that the measure be conserved under Hamiltonian evolution, noting that $\nabla \cdot \tilde{\mathbf{v}} < 0$ and $\sigma > 0$, and using the Cauchy–Schwarz inequality, we find

$$|\nabla \cdot \tilde{\mathbf{v}}|\sigma \leq |\nabla\sigma||\tilde{\mathbf{v}}|. \quad (30)$$

Since $|\nabla \cdot \tilde{\mathbf{v}}|$ is finite and $|\tilde{\mathbf{v}}| \rightarrow 0$, requiring that σ be smooth implies

$$\sigma(\mathbf{z}) \rightarrow 0 \quad \text{as} \quad \mathbf{z} \rightarrow \mathbf{z}_0. \quad (31)$$

Even if we relaxed the assumption of regularity, we can still show that σ is small near the fixed point. We observe, given a smooth, slowly-varying potential $V(\phi)$, that any fixed point in effective phase space will be at the terminus of an apparent attractor, that is, a region where both slow-roll conditions (23) are met. As trajectories flow from near the fixed point along the apparent attractor, the first condition is always met, while the second becomes an increasingly good approximation. Hence, our conclusion from the slow-roll regime becomes applicable and so Eq. (29) implies that the effective phase space measure near the fixed point is exponentially suppressed compared with the measure further down the slow-roll apparent attractor. Other than the fixed point

trajectory itself — which is irrelevant to inflation, since the field does not evolve — there is, relative to the slow-roll regime, very little measure near the hilltop. Recall that slicing effective phase space into sets of constant H to parametrize the space of trajectories incurs an additional factor of $|\dot{H}|$ to convert the phase space measure density into the probability distribution on the trajectories; this suppresses the measure assigned to the roll-up trajectories even more. However, we have shown here that roll-up trajectories are suppressed in the canonical measure on effective phase space, even without the help of this additional factor. Since the measure is conserved under Hamiltonian evolution, any roll-up trajectory, *i.e.*, the FRW evolution that comes arbitrarily close to de Sitter, is a set of measure zero.

IV. QUADRATIC INFLATION

A. Preliminaries

As a representative example of slow-roll inflation with $V''(\phi) > 0$, we consider monomial inflation with a quadratic potential,

$$V(\phi) = \frac{1}{2}m^2\phi^2. \quad (32)$$

If the recent BICEP2 discovery of B -mode polarization [1] is the result of primordial gravitational waves, then this simple model is in good agreement with the observed tensor perturbations. A canonical model in the inflationary literature [10, 11] and one of theoretical interest [12], the set of quadratic and related potentials is an important area of current investigation [13–15], given the status of observations [1, 7].

It will eventually be useful to redefine our dimensionless coordinates differently than in Eq. (17):

$$z_1 = \frac{\phi}{\sqrt{6}M_{\text{Pl}}} \quad \text{and} \quad z_2 = \frac{\dot{\phi}}{\sqrt{6}mM_{\text{Pl}}}, \quad (33)$$

so

$$z \equiv \sqrt{z_1^2 + z_2^2} = \frac{H}{m}. \quad (34)$$

Also, we define an angular variable

$$\tan \theta = \frac{z_2}{z_1} = \frac{\dot{\phi}}{m\phi}, \quad (35)$$

so $\dot{\phi} = \sqrt{6}mM_{\text{Pl}}z \sin \theta$ and $\phi = \sqrt{6}M_{\text{Pl}}z \cos \theta$.

Using Eq. (5), we can plot trajectories in the ϕ - $\dot{\phi}$ plane and see explicitly the effective phase space behavior, as shown in Fig. 1. In particular, note the apparent attractor solutions that appear at $z_2 = \pm 1/3$, corresponding to $\dot{\phi} = \pm\sqrt{2/3}mM_{\text{Pl}}$. Of course, in a strict phase-space sense, these “attractors” are illusory [6]. In the Liouville

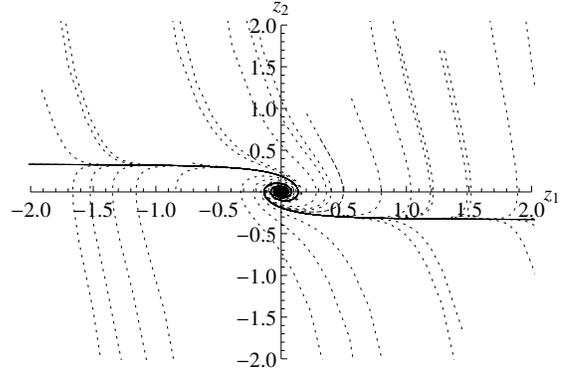


Figure 1. Trajectories in effective phase space for quadratic inflation. The field value and velocity are parameterized by the variables z_1 and z_2 , defined in Eq. (33). The dark nearly-horizontal lines indicate the apparent attractors, where the conserved measure grows large. For clarity we used the (unrealistic) value of $m = 0.2M_{\text{Pl}}$ to make this plot.

measure, phase-space density is conserved. The apparent attractor behavior actually indicates that the measure density grows large in that region.

The apparent attractor solution at $\dot{\phi} = \pm\sqrt{2/3}mM_{\text{Pl}}$ intersects the $H = \text{constant}$ ellipse at

$$|\sin \theta| = \frac{m}{3H}. \quad (36)$$

In the early universe ($H \gg m$), we therefore have $\theta \simeq 0$ or π on the apparent attractor.

B. Counting e -folds

For the quadratic potential (32), one has the potential slow-roll parameter from Eq. (19):

$$\epsilon_V = 2 \left(\frac{M_{\text{Pl}}}{\phi} \right)^2. \quad (37)$$

Inflation (and counting of e -folds N) ends when $\epsilon_V = 1$, which occurs at $\phi_f = \sqrt{2}M_{\text{Pl}}$.

For slow roll, $H^2 \simeq V/3M_{\text{Pl}}^2$, the scalar equation (5) becomes $3H\dot{\phi} \simeq -V'$ and so $Hdt \simeq \pm d\phi/\sqrt{2\epsilon_V}M_{\text{Pl}}$. Thus, when the field value is ϕ , the number of e -folds remaining before the end of inflation is

$$N(\phi) = \left| \int_{|\phi|}^{\sqrt{2}M_{\text{Pl}}} \frac{1}{\sqrt{2\epsilon_V}} \frac{d\phi'}{M_{\text{Pl}}} \right| = \frac{1}{4} \left(\frac{\phi}{M_{\text{Pl}}} \right)^2 - \frac{1}{2}. \quad (38)$$

Consider a trajectory that starts at angle θ on the surface where $H = M_{\text{Pl}}$. We will call this the Planck surface; of course, one could choose a different ultraviolet cutoff $\Lambda_{\text{UV}} \gg m$ for the effective field theory, on which to start evaluating trajectories at time $t = 0$. In that case, one could simply replace M_{Pl} by Λ_{UV} as appropriate in all of our e -fold counting. For simplicity we will

choose $\Lambda_{UV} = M_{\text{Pl}}$. The initial field value for ϕ is then $\sqrt{6}(\cos\theta)M_{\text{Pl}}^2/m$.

In the $H \gg m$ region of ϕ - $\dot{\phi}$ space, trajectories snap quickly to the apparent attractor, with $\dot{\phi}$ changing much faster than $m\phi$. That is, writing $\mathbf{z} = (z_1, z_2)$, we have, using the scalar equation,

$$\frac{\dot{\mathbf{z}}}{m} = \left(z_2, -z_1 - 3z_2\sqrt{z_1^2 + z_2^2} \right). \quad (39)$$

Thus, when $z = \sqrt{z_1^2 + z_2^2} \gg 1$, we have $\dot{z}_2 \gg \dot{z}_1$, as claimed. Hence, $z_1(t=0)$, to a very good approximation, is equal to z_1 at the time when the trajectory starts the slow-roll process. Therefore, we can write the *total* number of e -folds that this trajectory (parametrized by θ on the Planck surface) undergoes as

$$N_{\text{tot}} = \frac{3}{2} \left(\frac{M_{\text{Pl}}}{m} \right)^2 \cos^2 \theta - \frac{1}{2} \simeq \frac{3}{2} \left(\frac{M_{\text{Pl}}}{m} \right)^2 \cos^2 \theta. \quad (40)$$

Maximal inflation occurs when $\theta \simeq 0$ or π , *i.e.*, the trajectory starts out near the apparent attractor at the Planck scale, which gives

$$N_{\text{max}} = \frac{3}{2} \left(\frac{M_{\text{Pl}}}{m} \right)^2 \quad (41)$$

e -folds of inflation. Comparing the analytical prediction (40) with numerical simulation, we find very good agreement.

C. How many e -folds should we expect in quadratic inflation?

We know from Eq. (40) how to predict the total number of e -folds of inflation a trajectory will undergo based on a particular parametrization of the family of trajectories, namely, by the angular coordinate θ with which the trajectory intersects a surface of particular energy density, in this case $H = M_{\text{Pl}}$. We would now like to ask the question of how many e -folds we should expect, using the prescription for finding the appropriate measure (14) on the space of trajectories, as described in Sec. II B.

First, we need to find the measure σ on effective phase space for quadratic inflation. In the (z_1, z_2) coordinates defined in Eqs. (33) through (35), we can write the velocity (7) of trajectories in effective phase space as

$$\mathbf{v} = \dot{\mathbf{z}} = -3mz^2 \sin^2 \theta \hat{\mathbf{z}} - m(z + 3z^2 \sin \theta \cos \theta) \hat{\boldsymbol{\theta}}, \quad (42)$$

where

$$\begin{pmatrix} \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \end{pmatrix}. \quad (43)$$

In the early universe, when $H \gg m$ (such as on the Planck ellipse $H = M_{\text{Pl}}$), we have $z \gg 1$. Thus, \mathbf{v} becomes approximately

$$\mathbf{v} \simeq -3mz^2 \sin^2 \theta \hat{\mathbf{z}} - 3mz^2 \sin \theta \cos \theta \hat{\boldsymbol{\theta}}, \quad (44)$$

and so the requirement (10) for σ to be conserved under Hamiltonian evolution becomes

$$\partial_\theta \sigma = -z \tan \theta \partial_z \sigma - (2 \tan \theta + \cot \theta) \sigma. \quad (45)$$

The general solutions for σ take the form [6]

$$\sigma = \sum_\gamma C_\gamma z^{\gamma-3} \left| \frac{\cos^{\gamma-1} \theta}{\sin \theta} \right|, \quad (46)$$

for $z \gg 1$, where $\gamma, C_\gamma \in \mathbb{R}$. We note that σ diverges along the $\sin \theta = 0$ axis, corresponding to the buildup of trajectories along the apparent attractor; in an exact numerical solution, the distribution σ would become large on the apparent attractor solution, as is clear from Fig. 1, using the fluid flow analogy. For the potential (32), we proved in Ref. [6] that the measure σ has a unique solution; hence, many possible solutions in (46) are spurious or unphysical.

As we can see from flow of the vector field shown in Fig. 1, we should require that σ be finite everywhere except on the apparent attractor solution; imposing this condition requires $\gamma \geq 1$. Further, at fixed θ , trajectories become more squeezed together as z decreases, since more and more time evolution is compressed into a smaller and smaller range of H . Hence, we should require σ to be a non-increasing function of z at fixed θ , so $\gamma \leq 3$. Imposing the further requirement that σ be infinitely differentiable everywhere except the apparent attractor solution selects $\gamma = 3$ as the physical solution, so we end up with

$$\sigma(H = M_{\text{Pl}}, \theta) \propto \left| \frac{\cos^2 \theta}{\sin \theta} \right|. \quad (47)$$

In Sec. II B we demonstrated how to obtain the measure on the space of trajectories from σ , *cf.* Ref. [3]. Specifically, the probability distribution on the space of trajectories, parametrized by θ on some surface of constant H , is, up to normalization, given by $\sigma(H, \theta) |\dot{H}|$. Using Eq. (5), we have in the z coordinates:

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{\text{Pl}}^2} = -3m^2 z^2 = -3m^2 z^2 \sin^2 \theta. \quad (48)$$

Thus, the probability distribution on the space of trajectories on the Planck surface (where $z = M_{\text{Pl}}/m = \text{constant}$), parametrized by the coordinate θ , is

$$P(\theta)|_{H=M_{\text{Pl}}} = \frac{3}{4} |\cos^2 \theta \sin \theta|. \quad (49)$$

The overall normalization has been fixed by requiring $\int d\theta P(\theta) = 1$.

Finally, we can now compute the expected total number of e -folds of inflation, using the canonical measure

(49) and our e -fold counting (40):

$$\begin{aligned}
\langle N_{\text{tot}} \rangle &= \int_0^{2\pi} N(\theta) P(\theta)|_{H=M_{\text{Pl}}} d\theta \\
&= \frac{9}{8} \left(\frac{M_{\text{Pl}}}{m} \right)^2 \int_0^{2\pi} \cos^4 \theta |\sin \theta| d\theta \\
&= \frac{9}{10} \left(\frac{M_{\text{Pl}}}{m} \right)^2 \\
&= \frac{3}{5} N_{\text{max}}.
\end{aligned} \tag{50}$$

Now, assuming a quadratic potential (32), the amplitude of observed CMB scalar perturbations is

$$\Delta_s^2(k_{\text{CMB}}) = \frac{1}{6\pi^2} \left(\frac{m}{M_{\text{Pl}}} \right)^2 N_{\text{CMB}}^2, \tag{51}$$

where $N_{\text{CMB}} \approx 50$ is the number of e -folds between horizon exit of CMB scales and the end of inflation. Using the Planck observations [7] for the amplitude of scalar perturbations, we have $m = 7 \times 10^{-6} M_{\text{Pl}} = 2 \times 10^{13}$ GeV, which implies that for quadratic inflation we expect

$$\langle N_{\text{tot}} \rangle = 2 \times 10^{10}. \tag{52}$$

That is, typical universes under the canonical measure (49) with the inflaton mass we obtain by positing a quadratic potential (32) and requiring consistency with Planck [7] undergo much more than the required number of e -folds needed to solve the horizon problem; hence, one can view our observed Universe as natural in the theory of quadratic inflation, with regard to the canonical measure (with the caveats about inhomogeneities noted in the Introduction).

Looking at the conclusion another way, we note that the probability for N_{tot} to be greater than some particular value N_0 is just the probability that $\cos^2 \theta > (2/3)(m/M_{\text{Pl}})^2 N_0 \equiv \cos^2 \zeta$. Thus,

$$\begin{aligned}
\text{Pr}(N_{\text{tot}} > N_0) &= 4 \times \frac{3}{4} \int_0^\zeta \cos^2 \theta \sin \theta d\theta \\
&= 1 - \left(\frac{2}{3} \right)^{3/2} \left(\frac{m}{M_{\text{Pl}}} \right)^3 N_0^{3/2} \\
&= 1 - \left(\frac{N_0}{N_{\text{max}}} \right)^{3/2},
\end{aligned} \tag{53}$$

where N_{max} is defined in Eq. (41). That is, if $m = 2 \times 10^{13}$ GeV, the probability of having fewer than 50 e -folds of inflation is of order 10^{-13} . Differentiating Eq. (53), the measure on the space of trajectories can be written in terms of the number N_{tot} of e -folds ultimately achieved, between zero and N_{max} :

$$P(N_{\text{tot}}) dN_{\text{tot}} = \frac{3}{2N_{\text{max}}^{3/2}} N_{\text{tot}}^{1/2} dN_{\text{tot}}. \tag{54}$$

Universes that undergo 50 or more e -folds of inflation, like our own, are overwhelmingly generic from

the perspective of the canonical measure for high-scale quadratic inflation.

The specific number $\langle N_{\text{tot}} \rangle = 2 \times 10^{10}$ is suggestive, but it shouldn't be taken too literally. In quadratic inflation, the field has a value $\phi \sim 10 M_{\text{Pl}}$ at the epoch when currently observable large-scale perturbations are being generated; our calculation fearlessly extrapolates the functional form of the potential to values of order $10^5 M_{\text{Pl}}$, where there is little reason for it to be trusted. Nevertheless, we expect that our result has a robust physical interpretation for more general potentials: in large-field inflation, when the potential increases to the Planck limit, it is natural to achieve a large amount of inflation. There are certainly some trajectories that spend little or no time on the apparent attractor, remaining dominated by kinetic energy all the way up to Planck densities. Our results suggest that, in large-field inflation, such trajectories are extremely unlikely, as generic evolution quickly snaps to the apparent attractor, yielding many e -folds of inflation.

V. COSINE (NATURAL) INFLATION

A. Preliminaries

We now turn to the model of cosine or ‘‘natural’’ inflation [16, 17], in which the inflaton ϕ could be a pseudo-Nambu–Goldstone boson $\theta = \phi/f$ with a global shift symmetry broken at scale f . The global symmetry is explicitly broken at scale Λ , giving the boson a mass, via a potential

$$V(\phi) = \Lambda^4 [1 - \cos(\phi/f)]. \tag{55}$$

Cosine inflation is representative of the general class of hilltop inflation models: the inflaton potential has a region where $V''(\phi) < 0$. Qualitatively, this model has similarities and differences with monomial inflation models. Like monomial inflation, it can exhibit slow-roll behavior. Unlike monomial models, however, hilltop models have trajectories in which the inflaton stays near the top of the potential for a parametrically long time and pure de Sitter space is allowed if the field sits exactly at the potential maximum. Such trajectories would seem to allow hilltop models to achieve a very large number of e -foldings without the concomitant large excursion in field values endemic to monomial models and potentially troublesome from the effective field theory perspective. Ultraviolet completions of cosine inflation models have been investigated [18–20], which improve the applicability of effective field theoretic reasoning. Cosine inflation models are of significant current interest [8] and generically have regions of parameter space that can achieve agreement with observations from BICEP2 [1] and Planck [7]. In cosine inflation, the field can without loss of generality be restricted to the interval between

$\pm\pi f$, with the periodic identification $\phi \sim \phi + 2\pi f$ as an equivalence class.

As for quadratic inflation, we will find dimensionless coordinates useful (different than Eqs. (17) and (33)):

$$z_1 = \sqrt{\frac{2}{3}} \frac{f}{M_{\text{Pl}}} \sin(\phi/2f) \quad \text{and} \quad z_2 = \frac{f\dot{\phi}}{\sqrt{6}\Lambda^2 M_{\text{Pl}}}. \quad (56)$$

Because of the restricted range of the field, z_1 is isomorphic to ϕ , so our discussion about vector field invariance from Ref. [6] applies and (z_1, z_2) forms an effective phase space. Note that $z_1 \in [-\sqrt{2/3}f/M_{\text{Pl}}, \sqrt{2/3}f/M_{\text{Pl}}]$, with the identification $z_1 \sim z_1 + 2\sqrt{2/3}f/M_{\text{Pl}}$. As before, define polar coordinates

$$z \equiv \sqrt{z_1^2 + z_2^2} = \frac{f}{\Lambda^2} H \quad (57)$$

and

$$\tan \theta = \frac{z_2}{z_1} = \frac{\dot{\phi}}{2\Lambda^2 \sin(\phi/2f)}. \quad (58)$$

Because z_1 can only take values between $\pm\sqrt{2/3}f/M_{\text{Pl}}$, the Planck surface $H = M_{\text{Pl}}$ subtends a finite set of angles $[\theta_0, \pi - \theta_0] \cup [\pi + \theta_0, 2\pi - \theta_0]$, where

$$\cos \theta_0 = \frac{\sqrt{\frac{2}{3}} \frac{f}{M_{\text{Pl}}}}{\frac{f}{\Lambda^2} M_{\text{Pl}}} = \sqrt{\frac{2}{3}} \frac{\Lambda^2}{M_{\text{Pl}}^2} \ll 1, \quad (59)$$

i.e., θ_0 is close to $\pi/2$ or $3\pi/2$.

In these coordinates, the velocity vector $\mathbf{v} = \dot{\mathbf{z}}$, using Eq. (5), is

$$\begin{aligned} \frac{f\mathbf{v}}{\Lambda^2} &= z_2 \sqrt{1 - \frac{3}{2} \frac{M_{\text{Pl}}^2}{f^2} z_1^2} \hat{\mathbf{z}}_1 \\ &\quad - \left(3z_2 \sqrt{z_1^2 + z_2^2} + z_1 \sqrt{1 - \frac{3}{2} \frac{M_{\text{Pl}}^2}{f^2} z_1^2} \right) \hat{\mathbf{z}}_2 \\ &= -3z^2 \sin^2 \theta \hat{\mathbf{z}} \\ &\quad - \left(z \sqrt{1 - \frac{3}{2} \frac{M_{\text{Pl}}^2}{f^2} z^2 \cos^2 \theta} + 3z^2 \sin \theta \cos \theta \right) \hat{\boldsymbol{\theta}}, \end{aligned} \quad (60)$$

where $\hat{\mathbf{z}}_1$, $\hat{\mathbf{z}}_2$, $\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\theta}}$ are related as in Eq. (43). Plotting integral curves of this vector field, one can visualize trajectories in effective phase space, shown in Fig. 2. As for quadratic inflation, there is an apparent attractor, but for ϕ near $\pm f$, lingering behavior near the hilltop is also possible.

B. Counting e -folds

With the potential slow-roll parameter ϵ_V defined as in Eq. (19), for the cosine inflation potential (55) one

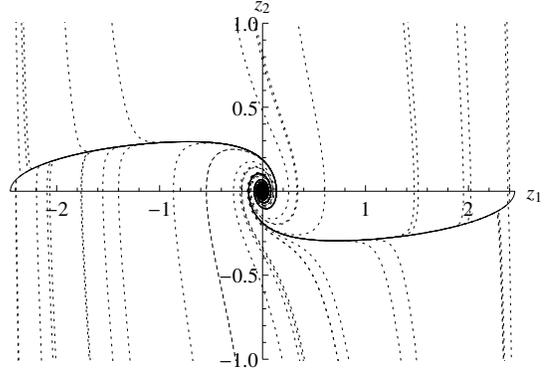


Figure 2. Trajectories in effective phase space for cosine inflation. The field value and velocity are parameterized by the variables z_1 and z_2 , defined in Eq. (56). The dark spirals indicate the apparent attractors, where the conserved measure grows large. For this plot we used $f = 3M_{\text{Pl}}$, $\Lambda = 0.1M_{\text{Pl}}$.

has

$$\epsilon_V = \frac{M_{\text{Pl}}^2}{2f^2} \frac{\sin^2(\phi/f)}{[1 - \cos(\phi/f)]^2} = \frac{1 - b^2 z_1^2}{3z_1^2}, \quad (61)$$

for convenience defining a constant

$$b \equiv \sqrt{3/2} M_{\text{Pl}}/f. \quad (62)$$

Inflation — and counting of e -folds — ends when $\epsilon_V = 1$, which occurs at $|z_1| = (3 + b^2)^{-1/2}$.

For slow roll and assuming $\ddot{\phi}$ is small compared to other terms in the scalar equation (5), we have $H^2 \simeq V/3M_{\text{Pl}}^2$ and $3H\dot{\phi} \simeq -V'$, so

$$H dt \simeq \pm \frac{d\phi}{\sqrt{2\epsilon_V} M_{\text{Pl}}} = \pm \frac{3|z_1| dz_1}{1 - b^2 z_1^2}, \quad (63)$$

after using Eqs. (56) and (61).³ Thus, when the field value is z_1 , the number of e -folds remaining before the end of inflation is

$$\begin{aligned} N(z_1) &= \left| \int_{|z_1|}^{(3+b^2)^{-1/2}} \frac{3z_1' dz_1'}{1 - b^2 z_1'^2} \right| \\ &= \frac{3}{2b^2} \ln \left[\frac{1}{(1 - b^2 z_1^2) (1 + \frac{1}{3} b^2)} \right]. \end{aligned} \quad (64)$$

We would like to parametrize the number of e -folds a trajectory undergoes based upon its coordinate θ on the Planck surface, *not* its coordinate z_1 when it enters

³ Though in general a hilltop trajectory can violate the condition that $3H\dot{\phi} \simeq -V'$, one can show that, for the potential (55), the total number of e -folds we compute is accurate even without this assumption. See footnote 5.

the slow-roll regime. From the vector field (60), we see that when $z = \sqrt{z_1^2 + z_2^2} \gg 1$ and $z_2 \gg z_1$ (which is true on the Planck surface) we have $\dot{z}_2 \gg \dot{z}_1$. Therefore, as for quadratic inflation, we are able to approximate z_1 (Planck surface) $\simeq z_1$ (enter slow roll) for a given trajectory.⁴ The total number of e -folds attained by a trajectory that starts out at angle θ on the Planck surface is then

$$N_{\text{tot}}(\theta) = \frac{3}{2b^2} \ln \left[\frac{1}{\left(1 - \frac{\cos^2 \theta}{\cos^2 \theta_0}\right) \left(1 + \frac{1}{3}b^2\right)} \right], \quad (65)$$

where θ_0 is defined in Eq. (59). Note that when z_1 approaches $1/b$, *i.e.*, when θ approaches θ_0 , N_{tot} diverges, as we would expect. In our approximation that z_1 (Planck surface) $\simeq z_1$ (enter slow roll), z_1 (Planck surface) $= 1/b$ is identified as the roll-up trajectory discussed in Sec. III B.

C. How many e -folds should we expect in cosine inflation?

From Eq. (65), we know, given a trajectory that intersects the Planck surface with angular coordinate θ , how many e -folds that trajectory will ultimately undergo. As in Sec. IV C, we now turn to the question of how many e -folds we should expect under the canonical measure (14) on the space of trajectories. As shown in Sec. II B, we must first find the Hamiltonian-conserved measure — a measure whose density satisfies the condition (10) — on effective phase space.

In our z coordinates (56), the $H = M_{\text{Pl}}$ surface corresponds to

$$z = \frac{M_{\text{Pl}} f}{\Lambda^2} \gg 1. \quad (66)$$

As we have previously noted, had we used a different ultraviolet cutoff Λ_{UV} , other than M_{Pl} , all of the results that follow would be the same, with M_{Pl} replaced by Λ_{UV} , so our conclusions would not qualitatively change. Taking the large- z limit of Eq. (60), we have

$$\mathbf{v} \simeq -3 \frac{\Lambda^2}{f} z^2 \sin^2 \theta \hat{\mathbf{z}} - 3 \frac{\Lambda^2}{f} z^2 \sin \theta \cos \theta \hat{\boldsymbol{\theta}}, \quad (67)$$

which is identical to Eq. (44) up to a multiplicative factor. That is, we have turned the large- H behavior of

cosine inflation into the large- H behavior of quadratic inflation, through the judicious choice of coordinates (56). The effective phase space measure density therefore takes the same general form (46) and on physical grounds we can restrict to the $\gamma = 3$ case for the reasons discussed in Sec. IV C, so that the measure density becomes as in Eq. (47).

There is nevertheless an important difference between the quadratic and cosine inflation scenarios, since ϕ is restricted to a small window in the latter. This implies that the normalization of Eq. (47) will be different. As we have noted, this restriction translates into a restriction of values of θ on the Planck surface to a small range near $\pi/2$ and near $3\pi/2$, with width given in Eq. (59). Hence, σ does not diverge on the $H = M_{\text{Pl}}$ surface for cosine inflation. In the z coordinates, we have as in Eq. (48)

$$\dot{H} = -3 \frac{\Lambda^4}{f^2} z^2 \sin^2 \theta. \quad (68)$$

The probability distribution over the space of trajectories, parametrized by the angle θ on the $H = M_{\text{Pl}}$ surface, is therefore

$$P(\theta)|_{H=M_{\text{Pl}}} = \frac{3}{4 \cos^3 \theta_0} |\cos^2 \theta \sin \theta|, \quad (69)$$

with θ_0 defined in Eq. (59) as before. The normalization once again comes from demanding that the total probability equal unity.

Having found the canonical measure (69) on the space of trajectories, we can now use our e -fold counting (65) to compute the expectation value for the total number of e -folds attained by a FRW universe in the cosine inflation model:

$$\begin{aligned} \langle N_{\text{tot}} \rangle &= 2 \int_{\theta_0}^{\pi-\theta_0} N_{\text{tot}}(\theta) P(\theta)|_{H=M_{\text{Pl}}} d\theta \\ &= \frac{9}{2b^2} \int_0^1 \ln \left[\frac{1}{(1-u^2) \left(1 + \frac{1}{3}b^2\right)} \right] u^2 du \\ &= \frac{f^2}{3M_{\text{Pl}}^2} \left[8 - 6 \ln 2 - 3 \ln \left(1 + \frac{M_{\text{Pl}}^2}{2f^2} \right) \right] \\ &\simeq \left(\frac{8}{3} - 2 \ln 2 \right) \frac{f^2}{M_{\text{Pl}}^2}, \end{aligned} \quad (70)$$

which is plotted in Fig. 3; the constant b was defined in Eq. (62).⁵ For example, setting $f = 1.2 \times 10^{19}$ GeV (the

⁴ Note that this approximation leads us to assign nonzero measure to the set of trajectories that come arbitrarily close to the fixed point. It therefore assigns nonzero measure where the roll-up trajectory intersects the Planck surface, which we argued in Sec. III is not strictly correct. However, if anything, this assumption should overestimate the expected total number of e -folds.

⁵ An exact expression for Hdt (63) would have ε in place of ε_V . Taking into account relaxation of the slow-roll conditions near the hilltop, one can show that, if $f \ll M_{\text{Pl}}$, then the total e -fold count we estimate should be increased by a factor of at most $M_{\text{Pl}}/\sqrt{6}f$. However, this would still lead to less than one e -fold of inflation expected under the canonical measure in the $f \lesssim M_{\text{Pl}}$ case. Moreover, one can show that, even near the hilltop, the approximation $\varepsilon_V \simeq \varepsilon$ is very accurate in the $f \gtrsim M_{\text{Pl}}$ case.

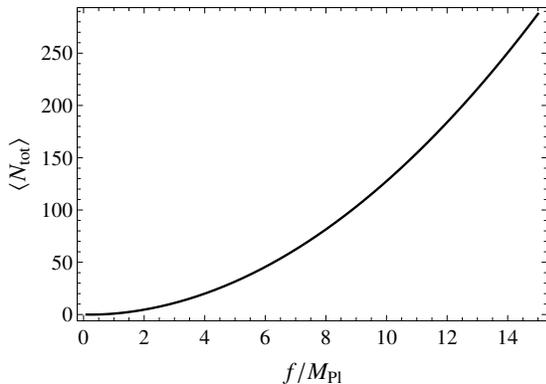


Figure 3. Expected number of e -folds $\langle N_{\text{tot}} \rangle$, as computed in Eq. (70) using the canonical measure on the space of trajectories, for cosine inflation with potential $V(\phi) = \Lambda^4 [1 - \cos(\phi/f)]$.

unreduced Planck mass) gives

$$\langle N_{\text{tot}} \rangle = 32. \quad (71)$$

This implies an insufficient amount of inflation to address the horizon problem, but clearly $\langle N_{\text{tot}} \rangle$ can be increased by a small boost in f .

Interestingly, $\langle N_{\text{tot}} \rangle$ is independent of Λ , only depending on f . This can be understood as follows: for small $\phi \ll f$, cosine inflation is equivalent to quadratic inflation, with mass Λ^2/f taking the place of m . Then the expected number of e -folds should be of order N_{max} (41), multiplied by a factor of $\cos^2 \theta_0$, as given in Eq. (59), to account for the limited allowed range of ϕ , cf. Eq. (40); this reasoning would lead one to expect $\langle N_{\text{tot}} \rangle \sim f^2/M_{\text{Pl}}^2$, which is indeed what we find. We find that $f > 6.3 M_{\text{Pl}} = 1.5 \times 10^{19}$ GeV is needed in order to have $\langle N_{\text{total}} \rangle > 50$.

Recently, in light of results from Refs. [1, 7], much attention has been devoted to cosine inflation. By varying f/M_{Pl} , a one-parameter family of predictions is obtained that is able to achieve agreement with either the Planck or BICEP2 results [8]. In particular, $f \sim 5 - 10 \times M_{\text{Pl}}$ was found to be in better agreement⁶ with the Planck observations [7], while larger f (which brings the predictions closer to those of quadratic inflation) is in better agreement with BICEP2.

What we have found is that smaller values of f are, in the sense of the canonical measure, highly unlikely to give a universe consistent with the observed uniformity of the CMB. In particular, if $f \leq M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV, we have less than one e -fold of inflation. More quantitatively, we can compute the probability of attaining

a given number N_0 of e -folds as a function of f/M_{Pl} . From our expression (65) for N_{tot} as a function of θ on the Planck surface, we find that this is just the probability that

$$\cos^2 \theta > \cos^2 \theta_0 \left[1 - \frac{3}{3+b^2} \exp\left(-\frac{2b^2}{3} N_0\right) \right] \equiv \cos^2 \delta. \quad (72)$$

That is, evaluating the integral $\text{Pr}(N_{\text{tot}} > N_0) = 4 \times (3 \sec^3 \theta_0 / 4) \int_{\theta_0}^{\delta} \cos^2 \theta \sin \theta d\theta$, we find

$$\begin{aligned} \text{Pr}(N_{\text{tot}} > N_0) \\ = 1 - \left[1 - \left(1 + \frac{M_{\text{Pl}}^2}{2f^2} \right)^{-1} \exp\left(-\frac{M_{\text{Pl}}^2}{f^2} N_0\right) \right]^{3/2}. \end{aligned} \quad (73)$$

The result is plotted in Fig. 4 for $N_0 = 50$. We find that, if $f \leq 2M_{\text{Pl}} = 4.9 \times 10^{18}$ GeV, the probability under the canonical measure of attaining 50 or more e -folds of inflation is less than 10^{-5} . While the details of Eq. (73) break down if $f \lesssim M_{\text{Pl}}$ due to corrections to the slow-roll approximation near the hilltop, the expected total number of e -folds (70) remains valid and the probability of attaining more than 50 e -folds for $f \lesssim M_{\text{Pl}}$ remains infinitesimal. That is, for $f \lesssim M_{\text{Pl}}$, the overwhelming majority of universes will under-inflate. On the other hand, if $f = 100M_{\text{Pl}} = 2.4 \times 10^{20}$ GeV, we find that the probability of a universe attaining at least 50 e -folds of inflation is approximately 0.99964. Hence, the probability of a FRW universe undergoing sufficient inflation to explain the observed uniformity of the CMB is sensitively dependent on f/M_{Pl} in cosine inflation, with larger values of $f \gtrsim \mathcal{O}(\text{few}) \times M_{\text{Pl}} \sim 10^{19}$ GeV much preferred.

If f is too small in cosine inflation, then our Universe is finely tuned from the perspective of the canonical measure. Hence, models of cosine inflation with f on the order of 10^{18} GeV or less do not solve the cosmological fine-tuning problems that are the original purpose of inflationary theory. For cosine inflation to truly be natural in the cosmological sense, f must be above 10^{19} GeV. On the other hand, this result helps motivate the possibility that our Universe did experience just the right amount of inflation but not too much, suggesting that there may be observable relics of the pre-inflationary Universe that might be observable on very large angular scales. Interestingly, in the large- f limit that is favored by the canonical measure, the observational predictions of cosine inflation merge with those of quadratic inflation.

As with the quadratic case, we expect our results for cosine inflation to be indicative of a more general lesson for hilltop (small-field) models. Unlike the large-field case, where the potential rises all the way to the Planck density, in cosine inflation the maximum is well below that scale. There are trajectories that linger for an arbitrarily large number of e -folds in the slow-roll regime near the top of the hill, but there are also trajectories that exhibit a kinetic-dominated phase of evolution prior

⁶ Note that this range of f could also be written as $f \sim 1-2 \times m_{\text{Pl}}$, where $m_{\text{Pl}} = 1/\sqrt{G} = 1.2 \times 10^{19}$ GeV is the unreduced Planck mass.

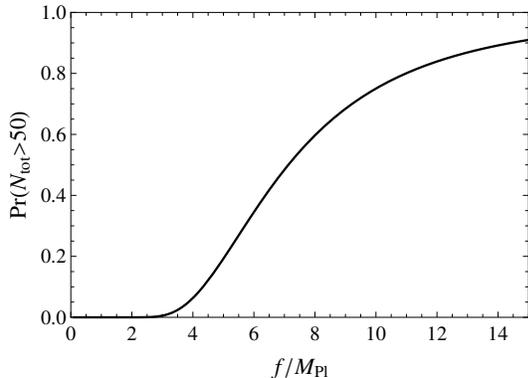


Figure 4. The probability of obtaining 50 or more e -folds of inflation as a function of f for cosine inflation with potential $V(\phi) = \Lambda^4[1 - \cos(\phi/f)]$, as computed using the canonical measure on the space of trajectories and starting on the $H = M_{\text{Pl}}$ surface.

to a finite period of slow roll. Our result shows that it is the latter category that are most likely, as quantified by the conserved measure on effective phase space.

VI. CONCLUSIONS

The recent BICEP2 discovery, if verified, suggests that high-scale cosmic inflation is the correct theory of the very early Universe. With characteristic energy of order 10^{16} GeV, observational signatures of inflation open the door to physics on the threshold of the Planck scale. Many models of inflation are currently being investigated for their ability to fit precision CMB observations. The current success of relatively simple models of inflation, driven by a single scalar field with a potential and a canonical kinetic term, is impressive. Given the large set of possible inflaton potentials, it is of vital importance to develop useful theoretical tools that enable observations to discriminate among competing models.

The theory of cosmic inflation was originally posited to solve problems of fine-tuning of initial conditions, such as the uniformity of the CMB temperature, lack of observed monopoles, and smallness of curvature. Given the current wealth of precise cosmological measurements, it is well-motivated to apply the same question of genericness to various proposed models of inflation. That is, given a particular model, does it generically produce the observed properties of our Universe? In particular, does it typically produce the requisite number of e -folds (40–60) to account for the uniformity of the CMB? Inherent in such questions is the idea of a measure: a probability distribution on the set of all possible FRW universes. Following GHS [3], in Ref. [6] we developed a formalism for constructing such a measure on the subset of flat universes (on which the GHS measure diverges).

In the present work, we investigated the behavior of

the effective phase space measure for two general classes of potentials important for single-field inflation: slow roll down a potential and lingering behavior near a potential hilltop. In the former case, we showed that the effective phase space measure generically becomes large, while in the latter case it generically becomes small. That is, trajectories that linger arbitrarily long near quasi-de Sitter space at a potential hilltop are disfavored by the canonical measure, while trajectories that slow roll and eventually reheat are favored. We next quantitatively examined the statistical conclusions offered by the canonical measure for two representative inflaton potentials: quadratic inflation and cosine inflation. Interestingly, the statistical expectation for the amount of inflation experienced in these two cases differed dramatically. For quadratic inflation, we found that, given an inflaton mass consistent with the observed amplitude of scalar perturbations [7], nearly all trajectories undergo 50 e -folds of inflation. In fact, generic trajectories experience billions of e -folds. On the other hand, for cosine inflation with symmetry-breaking parameter f , typical trajectories under-inflate unless $f \gtrsim 10^{19}$ GeV. Above this scale, 50 e -folds are generically attainable and the observational cosmology predictions of cosine inflation merge with those of the quadratic potential.

From our demonstration with these two examples, we illustrated the utility of the canonical measure in elucidating differences in physical predictions among models of inflation. While a given potential may have *some* trajectory — some possible history of a FRW universe — that undergoes enough inflation to correspond to our observed Universe, that does not mean that this trajectory is generic. Indeed, in some models, such as cosine potentials with $f \lesssim M_{\text{Pl}}$, the vast majority of trajectories, as weighted by the canonical measure, do not undergo sufficient inflation, despite the existence of a small subset of finely-tuned trajectories that do. The canonical measure allows one to quantify the amount of tuning required in a given model to reproduce our Universe.⁷ The degree of tuning required on the space of trajectories to produce at least 50 e -folds of inflation (or whatever other observed quantity one is computing) should correspond inversely with the degree of credence given a particular model, modulo theoretical bias. That is, given two potentials, one generically attaining many e -folds and another in which only a small subset (as computed in the canonical measure) of trajectories attain 50 e -folds, the former model should be favored: one could say that such a model is more “natural,” in the sense that it requires less fine-tuning to match observations. This approach is an interesting parallel to current discussions in particle physics regarding naturalness of the electroweak scale

⁷ All of these statements are made under the assumption of homogeneity, ignoring perturbations. Given a universe in which inflation occurs at all, this is a very good approximation.

and the amount of tuning required in various models, such as supersymmetry.

The contrapositive of this line of thinking is also illuminating. If future cosmological observations point to a particular inflaton potential for which our Universe is not generic under the canonical measure, that would shed light on even higher-scale physics. Such a circumstance would tell us that our Universe is tuned — on a non-generic trajectory — from the point of view of the classical measure. This would indicate the importance of intrinsically quantum gravitational processes or some ultimate theory of initial conditions.

As we enter an era of precision inflationary cosmology, models of inflation will be subjected to increasingly refined measurement. In the effort to determine which models best reflect reality, the notion of naturalness, in the sense of genericness under the canonical measure

on the space of trajectories, can be very useful. The methods developed in this work provide for quantitative probabilistic comparison among models of inflation, providing a new means of shedding light on the earliest moments of our Universe.

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1144469, by DOE Grant No. DE-SC0011632, and by the Gordon and Betty Moore Foundation through Grant No. 776 to the Caltech Moore Center for Theoretical Cosmology and Physics. G.N.R. is supported by a Hertz Graduate Fellowship and a NSF Graduate Research Fellowship.

-
- [1] P. Ade *et al.* (BICEP2 Collaboration) (2014), [arXiv:1403.3985 \[astro-ph.CO\]](#).
 - [2] D. Baumann (2009), [arXiv:0907.5424 \[hep-th\]](#).
 - [3] G. Gibbons, S. Hawking, and J. Stewart, *Nucl.Phys.* **B281**, 736 (1987).
 - [4] G. Gibbons and N. Turok, *Phys.Rev.* **D77**, 063516 (2008), [arXiv:hep-th/0609095 \[hep-th\]](#).
 - [5] S. M. Carroll and H. Tam (2010), [arXiv:1007.1417 \[hep-th\]](#).
 - [6] G. N. Remmen and S. M. Carroll, *Phys.Rev.* **D88**, 083518 (2013), [arXiv:1309.2611 \[gr-qc\]](#).
 - [7] P. Ade *et al.* (Planck Collaboration) (2013), [arXiv:1303.5076 \[astro-ph.CO\]](#).
 - [8] K. Freese and W. H. Kinney (2014), [arXiv:1403.5277 \[astro-ph.CO\]](#).
 - [9] R. Penrose, *Annals N.Y.Acad.Sci.* **571**, 249 (1989).
 - [10] A. D. Linde, *Nucl.Phys.* **B252**, 153 (1985).
 - [11] M. Madsen and P. Coles, *Nucl.Phys.* **B298**, 701 (1988).
 - [12] M. Kawasaki, M. Yamaguchi, and T. Yanagida, *Phys.Rev.Lett.* **85**, 3572 (2000), [arXiv:hep-ph/0004243 \[hep-ph\]](#).
 - [13] J. Ellis, M. Fairbairn, and M. Sueiro, *JCAP* **1402**, 044 (2014), [arXiv:1312.1353 \[astro-ph.CO\]](#).
 - [14] J. Ellis, M. Fairbairn, and M. Sueiro, *JCAP* **1402**, 044 (2014), [arXiv:1312.1353 \[astro-ph.CO\]](#).
 - [15] F. Marchesano, G. Shiu, and A. M. Uranga (2014), [arXiv:1404.3040 \[hep-th\]](#).
 - [16] F. C. Adams, J. R. Bond, K. Freese, J. A. Frieman, and A. V. Olinto, *Phys.Rev.* **D47**, 426 (1993), [arXiv:hep-ph/9207245 \[hep-ph\]](#).
 - [17] K. Freese, J. A. Frieman, and A. V. Olinto, *Phys.Rev.Lett.* **65**, 3233 (1990).
 - [18] N. Arkani-Hamed, H.-C. Cheng, P. Creminelli, and L. Randall, *Phys.Rev.Lett.* **90**, 221302 (2003), [arXiv:hep-th/0301218 \[hep-th\]](#).
 - [19] C. Germani and A. Kehagias, *Phys.Rev.Lett.* **106**, 161302 (2011), [arXiv:1012.0853 \[hep-ph\]](#).
 - [20] R. Kallosh, A. Linde, and B. Vercnocke (2014), [arXiv:1404.6244 \[hep-th\]](#).