

On the Delay Advantage of Coding in Packet Erasure Networks

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Abstract—We consider the delay of network coding compared to routing for a family of simple networks with parallel links. We investigate the sub-linear term in the block delay required for unicasting n packets and show that there is an unbounded gap between network coding and routing. In particular, we show that delay benefit of network coding is scaling at least as fast as \sqrt{n} . The main technical contribution involves showing that the delay function for the routing retransmission strategy is unbounded. This problem is equivalent to computing the expected maximum of two negative binomial random variables. This problem has also been addressed previously and we derive the first exact characterization which might be of independent interest.

I. INTRODUCTION

This paper considers the block delay for unicasting a file consisting of n packets over a packet erasure network with probabilistic erasures. Such networks have been extensively studied from the standpoint of capacity. Various schemes involving coding or retransmissions have been shown to be capacity-achieving for unicasting in networks with packet erasures [1], [2], [3], [4]. For a capacity-achieving strategy, the expected block delay for transmitting n packets is $\frac{n}{C} + D(n)$ where C is the minimum cut capacity and the delay function $D(n)$ is sublinear in n but differs in general for different strategies. In general networks, the optimal $D(n)$ is achieved by random linear network coding¹, in that decoding succeeds with high probability for any realization of packet erasure events for which the corresponding minimum cut capacity is n . However, the delay function $D(n)$ has not been characterized in general. Note that this term is going to be significant if the number of packets communicated is not very large. In our previous work [5] we showed that for multi-hop line networks, $D(n)$ is bounded and nondecreasing for both network coding and routing.

In this paper, we consider networks with parallel paths and compare the delay function $D(n)$ for coding versus a retransmission strategy where only one copy of each packet is kept in intermediate node buffers. Schemes such as [6], [4] ensure that there is only one copy of each packet in the network; without substantial non-local coordination or feedback, it is complicated for an uncoded topology-independent scheme to keep track of multiple copies of packets at intermediate nodes. Coding allows redundant packets to be transmitted efficiently in a topology-independent manner, without feedback or coordination, except for an acknowledgment from the destination

¹Random linear network coding is capacity-achieving if the overhead of specifying the random coding vectors can be neglected.

when it has received the entire file. This results in an advantage in delay function $D(n)$ which, as we show below, can be unbounded with increasing n .

Note that the main technical difficulty involves showing that the delay function for the routing retransmission strategy is unbounded. This problem turns out to be equivalent to computing the expected value of the maximum of two negative binomial random variables. This problem has also been addressed in [7], where the authors explain in detail why it is fairly complicated² and derive an approximate solution to the problem. Our analysis addresses this open problem by finding an exact expression and showing that it grows to infinity at least as fast as the square root of n .

The remainder of this paper is organized as follows: Section II presents the precise model we use for packet communication. Section III presents the analysis for the two different transmission schemes considered in this paper, finally Section IV contains a discussion of the results presented in this paper along with comments for possible extensions.

II. MODEL

We define the *two parallel multi-hop line network* as the network depicted in Fig. 1. This network consists of two parallel multi-hop line networks with 2ℓ nodes and 2ℓ links, *i.e.* ℓ links in each line (our results are readily extended to networks with different number of links in each line). Nodes \mathcal{S} , \mathcal{T} are the source and the destination respectively, whereas nodes N_{ij} , $i \in \{1, 2\}$ and $0 < j < \ell$ belong to the two line networks. All nodes $N_{i(j-1)}$ are connected to the node N_{ij} on their right by link L_{ij} , for $i \in \{1, 2\}$ and $1 \leq j \leq \ell$ (for consistency we will assume that the source \mathcal{S} and the destination \mathcal{T} are defined as nodes N_{i0} and $N_{i\ell}$, $i \in \{1, 2\}$, respectively).

Source \mathcal{S} wishes to transmit n packets to destination \mathcal{T} and we assume a discrete time model. At each time step, node $N_{i(j-1)}$ can transmit one packet through link L_{ij} to node N_{ij} , $i \in \{1, 2\}$ and $1 \leq j \leq \ell$. The transmission succeeds with probability $1 - p_{ij}$ or the packet gets erased with probability p_{ij} . Erasures across different links and time steps are assumed to be independent.

For reasons that will become evident later, we assume that both line networks have a single worst link with the

²Authors in [7] deal with the expected maximum of any number of negative binomial distributions but the difficulty remains even for two negative binomial distributions.

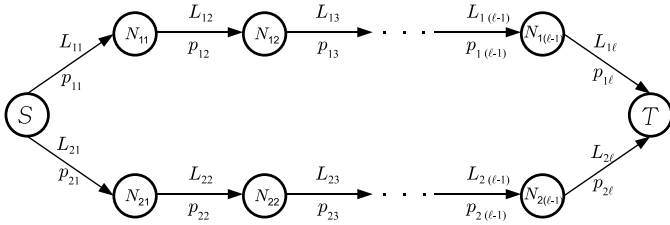


Fig. 1. Two parallel multi-hop line networks having links with different erasure probabilities

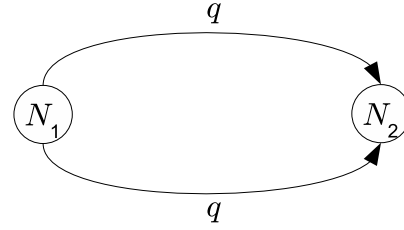


Fig. 2. Two parallel erasure links with erasure probabilities q connecting the source N_1 to the destination N_2 .

same probability of erasure. As shown in [5], [8], without loss of generality, we can regard that in both line networks the worst link is the first link, *i.e.*, $p = p_{11} = p_{21} = \max\{p_{11}, \dots, p_{1\ell}\} = \max\{p_{21}, \dots, p_{2\ell}\}$ and $p_{ij} < p$ for $i \in \{1, 2\}$ and $2 \leq j \leq \ell$. We also assume that no links fail with probability 1 ($p_{ij} < 1$) or else the problem becomes trivial since no packets can be transmitted from source S to the destination T .

We want to compare the expected time taken to send the n packets through the network from source S to the destination T using two different transmission schemes. On the first scheme all nodes perform *coding* and transmit random linear combinations of all previously received packets. Source S in particular combines all the packets together and keeps sending to both N_{11} and N_{21} random linear combinations of the initial n packets (for a large field the random combinations will be linearly independent with high probability). The destination T will decode once it receives n independent linear combinations of the initial packets. On the second transmission scheme all nodes perform *routing* and source S in particular sends some of the n packets using only the upper part of the network (links L_{1j}) and the rest of the packets using only the lower part of the network (links L_{2j}). The destination T will decode the initial information when it receives all the packets both from the upper part and the lower part of the network.

Since the two line networks have the same capacity due to the fact that their worst links have identical erasure probabilities it would only make sense for the source S to send the same number of packets from the upper and the lower line network. If source S sends a different number of packets through the upper and lower line networks, it will only perform worse (in expectation). Therefore from now on, source S is assumed to send half of the n packets from the upper part and the half from the lower part of the network. To simplify the notation and without loss of generality we will assume that n is an even number so that it can get divided into half.³

In our model, in case of a successful transmission, the packet is assumed to be transmitted to the next node instantaneously, *i.e.* we ignore the transmission delay along the links. Moreover there is no restriction on the number of packets n , and there is no requirement for the network to reach the steady state.

³Our results hold even in the case that n is odd.

III. TWO PARALLEL MULTI-HOP LINE NETWORK

A. Coding Strategy

Before we analyze the expected time $\mathbb{E}T_n^c$ taken to send n packets through the network in Fig.1 using *coding* (where the c superscript stands for coding), we need to prove the following proposition that holds for the simplified network of two parallel erasure links connecting the source to the destination like in Fig. 2.

Proposition 1. *The expected time $\mathbb{E}\hat{T}_n^c$ taken to send by coding n packets from the source to the destination through two parallel erasure links is*

$$\mathbb{E}\hat{T}_n^c = \frac{n}{2-2q} + B_n$$

where B_n is a bounded term (non-monotonic) and q is the erasure probability at the two links connecting the source and the destination.

Proof: Omitted due to space constraints, see [9]. ■

Now we are ready to prove the following theorem:

Theorem 1. *The expected time $\mathbb{E}T_n^c$ taken to send by coding n packets through a two parallel multi-hop line network is*

$$\mathbb{E}T_n^c = \frac{n}{2-2p} + D_n^c$$

where p is the erasure probability of the two worst links in the network and the delay function D_n^c depends on all the erasure probabilities p_{ij} , for $i \in \{1, 2\}$, $1 \leq j \leq n$ and is bounded.

Proof: Omitted due to space constraints, see [9]. ■

B. Routing Strategy

Before we analyze the expected time $\mathbb{E}T_n^r$ taken to send n packets through the network in Fig.1 using *routing* (where the r superscript stands for routing), we need to prove the following two propositions

Proposition 2. *For $a, b, n \in \mathbb{N}^+$ with $a < b$ the sum $\sum_{k=a}^b \frac{n-k}{n+k}$ is equal to:*

$$\sum_{k=a}^b \frac{n-k}{n+k} = a - b - 1 + 2n(H_{b+n} - H_{a+n-1}) \quad (1)$$

where H_n is the n^{th} Harmonic number, i.e. $H_n = \sum_{i=1}^n \frac{1}{i}$.

Proof: Omitted due to space constraints, see [9]. ■

Proposition 3. *The expected time $\mathbb{E}\hat{T}_{2k}^r$ taken to send by routing $2k$ packets from the source to the destination through two parallel erasure links (k packets from the upper link and k packets from the lower link) is*

$$\mathbb{E}\hat{T}_{2k}^r = \frac{k}{1-q} + U_{2k}$$

where U_{2k} is an unbounded term that grows at least as square root of k and q is the erasure probabilities at the two links connecting the source and the destination.

Proof: We will denote as $A_{i,j}$ the expected time to send i packets from the upper link and j packets from the lower link of Fig. 2. Clearly $\mathbb{E}\hat{T}_{2k}^r = A_{k,k}$ and $A_{i,j}$ satisfies the following two dimensional recursion formula:

$$\left\{ \begin{array}{l} A_{i,j} = q^2(A_{i,j} + 1) + q(1-q)(A_{i-1,j} + 1) + \\ + q(1-q)(A_{i,j-1} + 1) + (1-q)^2(A_{i-1,j-1} + 1) \\ A_{0,j} = \frac{j}{1-q}, \quad A_{i,0} = \frac{i}{1-q}, \quad A_{0,0} = 0 \end{array} \right\}$$

or equivalently

$$\left\{ \begin{array}{l} A_{i,j} = \frac{q}{1+q}(A_{i-1,j} + A_{i,j-1}) + \frac{1-q}{1+q}A_{i-1,j-1} + \frac{1}{1-q^2} \\ A_{0,j} = \frac{j}{1-q}, \quad A_{i,0} = \frac{i}{1-q}, \quad A_{0,0} = 0 \end{array} \right\}. \quad (2)$$

The two dimensional recursion formula in (2) has a specific solution $\frac{i+j}{2(1-q)}$ and a general solution $B_{i,j}$ where

$$\left\{ \begin{array}{l} B_{i,j} = \frac{q}{1+q}(B_{i-1,j} + B_{i,j-1}) + \frac{1-q}{1+q}B_{i-1,j-1}, i, j \geq 1 \\ B_{0,j} = \frac{j}{2(1-q)}, \quad B_{i,0} = \frac{i}{2(1-q)}, \quad B_{0,0} = 0 \end{array} \right\}. \quad (3)$$

In order to solve equation (3) we will use the Z -transform with respect to i . More specifically we define the Z -transform as:

$$\hat{B}_{z,j} = \sum_{i=0}^{\infty} B_{i,j} \cdot z^i \quad (4)$$

and by multiplying all terms in equation (3) by z^i and summing everything we get:

$$\begin{aligned} \sum_{i=1}^{\infty} B_{i,j} \cdot z^i &= \frac{q}{1+q} \sum_{i=1}^{\infty} B_{i-1,j} \cdot z^i + \frac{q}{1+q} \sum_{i=1}^{\infty} B_{i,j-1} \cdot z^i \\ &\quad + \frac{1-q}{1+q} \sum_{i=1}^{\infty} B_{i-1,j-1} \cdot z^i \\ \Leftrightarrow \hat{B}_{z,j} - B_{0,j} &= z \cdot \frac{q}{1+q} \hat{B}_{z,j} + \frac{q}{1+q} (\hat{B}_{z,j-1} - B_{0,j-1}) \\ &\quad + z \cdot \frac{1-q}{1+q} \hat{B}_{z,j-1} \end{aligned}$$

Since $B_{0,j} = \frac{j}{1-q}$ the above equation becomes:

$$\left\{ \begin{array}{l} \hat{B}_{z,j} \cdot \left(1 - z \cdot \frac{q}{1+q}\right) - \hat{B}_{z,j-1} \cdot \left(\frac{q}{1+q} + z \cdot \frac{1-q}{1+q}\right) = \\ = \frac{j+q}{2(1-q^2)} \\ \hat{B}_{z,0} = \sum_{i=0}^{\infty} B_{i,0} z^i = \sum_{i=0}^{\infty} \frac{i}{2(1-q)} z^i \equiv \frac{z}{2(1-q)(1-z)^2} \end{array} \right\} \quad (5)$$

TABLE I
SOME PAIRS OF FUNCTIONS ALONG WITH THEIR Z -TRANSFORMS

Sequence	Z -transform
1	$\frac{1}{1-z}$
i	$\frac{z^i}{(1-z)^2}$
$\binom{i+j-t-1}{j-1}$	$\frac{z^t}{(b-z)^j}$, for $t \leq j$

where equation (5) is an one dimensional recursion formula with the following general solution:

$$\begin{aligned} \hat{B}_{z,j} &= \frac{z}{(1-q)(1-z)^2} \cdot \left(\frac{q+z(1-q)}{1+q(1-z)}\right)^j + \\ &\quad + \frac{j}{2(1-q)(1-z)} - \frac{z}{2(1-q)(1-z)^2}. \end{aligned} \quad (6)$$

and equation (6) can be written in a compact form

$$\hat{B}_{z,j} = \hat{a}(z) \cdot \hat{b}(j, z) + \hat{d}(j, z) \quad (7)$$

by defining the functions $\hat{a}(z)$, $\hat{b}(z, j)$ and $\hat{d}(z, j)$ as follows:

$$\begin{aligned} \hat{a}(z) &= \frac{z}{(1-q) \cdot (1-z)^2} \\ \hat{b}(z, j) &= \left(\frac{q+(1-q) \cdot z}{1+q \cdot (1-z)}\right)^j \\ \hat{d}(z, j) &= \frac{j}{2(1-q)(1-z)} - \frac{z}{2(1-q)(1-z)^2}. \end{aligned}$$

Now we are ready to compute the inverse Z -transform of $\hat{B}_{z,j}$. Clearly from Table I along with equation (7):

$$\begin{aligned} B_{i,j} &= Z^{-1} \left\{ \hat{a}(z) \cdot \hat{b}(z, j) \right\} + Z^{-1} \left\{ \hat{d}(z, j) \right\} \\ \Leftrightarrow B_{i,j} &= \sum_{m=0}^i a(i-m) \cdot b(m, j) + \frac{j-i}{2(1-p)} \end{aligned} \quad (8)$$

where $a(i)$ and $b(i, j)$ are the inverse Z -transforms of $\hat{a}(z)$ and $\hat{b}(z, j)$ respectively. It is clear from Table I that $a(i) = \frac{i}{1-q}$ so the missing step is to evaluate $b(i, j)$:

$$\begin{aligned} b(i, j) &= Z^{-1} \left\{ \left(\frac{q+(1-q) \cdot z}{1+q \cdot (1-z)}\right)^j \right\} \\ \Leftrightarrow b(i, j) &= \frac{1}{q^j} \cdot Z^{-1} \left\{ \frac{\sum_{t=0}^j \binom{j}{t} (1-q)^t \cdot z^t \cdot q^{j-t}}{\left(\frac{1+q}{q} - z\right)^j} \right\} \\ \Leftrightarrow b(i, j) &= \sum_{t=0}^j \binom{j}{t} \cdot \left(\frac{1-q}{q}\right)^t \cdot Z^{-1} \left\{ \frac{z^t}{\left(\frac{1+q}{q} - z\right)^j} \right\} \\ \Leftrightarrow b(i, j) &= C^{i+j} \cdot \sum_{t=0}^j \binom{j}{t} \cdot \binom{i+j-t-1}{j-1} F^t \end{aligned}$$

where $C = \frac{q}{1+q}$ and $F = \frac{1-q^2}{q^2}$. Therefore equation (8) becomes

$$B_{i,j} = \sum_{m=0}^i \sum_{t=0}^j \frac{i-m}{1-q} C^{m+j} \binom{j}{t} \binom{m+j-t-1}{j-1} F^t + \frac{j-i}{2(1-q)} \quad (9)$$

We are interested in evaluating $\mathbb{E}\hat{T}_{2k}^r = A_{k,k}$ and since $A_{i,j} = B_{i,j} + \frac{i+j}{2(1-q)}$ equation (9) gives:

$$\mathbb{E}\hat{T}_{2k}^r = \frac{k}{1-q} + U_{2k} \quad (10)$$

where

$$U_{2k} = \frac{C^k}{1-q} \sum_{m,t=0}^k (k-m) \binom{k}{t} \binom{k-1+m-t}{k-1} C^m F^t \quad (11)$$

with $\binom{m}{w} = 0$ if $m < w$.

In order to prove that function U_{2k} is unbounded we will prove that U_{2k} is larger than another simpler to analyze function that goes to infinity and therefore U_{2k} also increases to infinity. Indeed equation (11) can be written as:

$$\begin{aligned} U_{2k} &= \frac{C^k}{1-q} \sum_{m,t=0}^k \binom{k}{t} \binom{k+m-t}{k} \frac{k(k-m)}{k+m-t} C^m F^t \\ \Rightarrow U_{2k} &> \frac{kC^k}{1-q} \sum_{m,t=0}^k \binom{k}{t} \binom{k+m-t}{k} \frac{k-m}{k+m} C^m F^t \end{aligned}$$

and since all terms in the above double sum are non-negative we can disregard as many terms as we wish without violating direction of the inequality, specifically:

$$U_{2k} > \frac{kC^k}{1-q} \sum_{m \in E, t \in G} \binom{k}{t} \binom{k+m-t}{k} \frac{k-m}{k+m} C^m F^t \quad (12)$$

where $E = \{[k - \sqrt{k}], \dots, k\}$, $G = \{[(1-q)k - \sqrt{k}], \dots, [(1-p)k]\}$ and $[x]$, $\lceil x \rceil$ are the floor and the ceiling functions respectively.

By using the lower and upper Stirling-based bound [10]:

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}, \quad n \geq 1$$

one can find that

$$\binom{n}{\beta n} > \frac{1}{\sqrt{2\pi\beta(1-\beta)n}} \cdot 2^{nH(\beta)} \cdot e^{-\frac{1}{12n\beta(1-\beta)}}, \quad \beta \in (0, 1)$$

where $H(\beta) = -\beta \log_2(\beta) - (1-\beta) \log_2(1-\beta)$ is the entropy function and therefore using inequality (12) we can derive:

$$U_{2k} > \frac{1}{2\pi(1-q)} \sum_{m \in E, t \in G} \frac{k-m}{k+m} f\left(\frac{m}{k}, \frac{t}{k}\right) e^{-\frac{1}{12k} h\left(\frac{m}{k}, \frac{t}{k}\right)} 2^{ng\left(\frac{m}{k}, \frac{t}{k}\right)} \quad (13)$$

where $f(\alpha, \beta) = \sqrt{\frac{1+\alpha-\beta}{\beta(1-\beta)(\alpha-\beta)}}$, $h(\alpha, \beta) = \frac{1+\alpha-\beta}{\alpha-\beta} + \frac{1}{\beta(1-\beta)}$ and

$$\begin{aligned} g(\alpha, \beta) &= (1+\alpha) \log_2(C) + H(\beta) + \\ &+ (1+\alpha-\beta) H\left(\frac{1}{1+\alpha-\beta}\right) + \beta \log_2(F). \end{aligned} \quad (14)$$

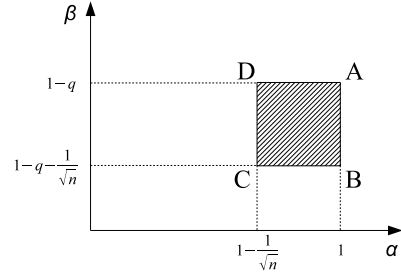


Fig. 3. The region K where function $g(\alpha, \beta)$ is defined on.

Since $1 - \frac{1}{\sqrt{n}} \leq \frac{m}{k} \leq 1$ and $(1-q) - \frac{1}{\sqrt{n}} \leq \frac{t}{k} \leq (1-q)$ we define functions $f(\alpha, \beta)$, $h(\alpha, \beta)$ and $g(\alpha, \beta)$ within the region $K = \left[1 - \frac{1}{\sqrt{k}}, 1\right] \times \left[1 - q - \frac{1}{\sqrt{k}}, 1 - q\right]$. Moreover we are only concerned with large enough k so that $0 < \beta < \alpha$ and region K looks like the one in Fig. 3. For large values of k , $f(\alpha, \beta) > \sqrt{\frac{1+q}{q}}$ and $g(\alpha, \beta) < 1 + \frac{4-2q}{q(1-q)}$ within region K and therefore from inequality (13) we get:

$$\begin{aligned} U_{2k} &> \frac{\sqrt{1+q}}{2\pi(1-q)\sqrt{q}} e^{-\frac{1}{12k}(1+\frac{4-2q}{q(1-q)})} \sum_{m \in E, t \in G} \frac{k-m}{k+m} 2^{ng\left(\frac{m}{k}, \frac{t}{k}\right)} \\ U_{2k} &> \frac{\sqrt{1+q}}{2\pi(1-q)\sqrt{q}} e^{-1} \sum_{m \in E, t \in G} \frac{k-m}{k+m} 2^{ng\left(\frac{m}{k}, \frac{t}{k}\right)} \end{aligned} \quad (15)$$

for large enough k .

Function $g(\alpha, \beta)$ satisfies the following three conditions:

- 1) $\frac{\partial g}{\partial \alpha} = \log_2\left(\frac{C(1+\alpha-\beta)}{(\alpha-\beta)}\right)$ and $\frac{\partial g}{\partial \beta} = \log_2\left(\frac{F(1-\beta)(\alpha-\beta)}{\beta(1+\alpha-\beta)}\right)$
- 2) $\frac{\partial^2 g}{\partial \alpha^2} = -\frac{1}{(\alpha-\beta)(1+\alpha-\beta) \ln 2} < 0$
- 3) $\frac{\partial^2 g}{\partial \alpha^2} \cdot \frac{\partial^2 g}{\partial \beta^2} - \frac{\partial^2 g}{\partial \alpha \partial \beta} \cdot \frac{\partial^2 g}{\partial \beta \partial \alpha} = \frac{1}{\beta(1-\beta)(\alpha-\beta)(\alpha-\beta+1) \ln 2} > 0$

It's very easy to see from condition 1, that $\frac{\partial g(\alpha, \beta)}{\partial \alpha} \Big|_{(1, 1-q)} = 0$ and $\frac{\partial g(\alpha, \beta)}{\partial \beta} \Big|_{(1, 1-q)} = 0$. Moreover conditions 2 and 3 show the concavity of $g(\alpha, \beta)$ within region K and along with condition 1 it is proved that function $g(\alpha, \beta)$ achieves a maximum at point $(\alpha, \beta) = (1, 1-q)$, where that maximum is equal to 0. Since region K is compact (closed and convex) and function $g(\alpha, \beta)$ is concave, it will achieve its minimum on the boundary of K . It's not difficult to show that $\frac{\partial g(\alpha, 1-q)}{\partial \alpha} \geq 0$ for $\alpha \leq 1$ and therefore function $g(\alpha, 1-q)$ decreases in value from point A to point D . Similarly $\frac{\partial g(1, \beta)}{\partial \beta} \geq 0$ for $\beta \leq 1-q$ and therefore function $g(1, \beta)$ decreases in value from point A to point B . Since $\frac{\partial g(\alpha, 1-q-1/\sqrt{n})}{\partial \alpha} \geq 0$ for $\alpha \leq 1$ and $\frac{\partial g(1-1/\sqrt{n}, \beta)}{\partial \beta} \geq 0$ for $\beta \leq 1-q$ with similar arguments as above we show that the minimum value for $g(\alpha, \beta)$ within K is achieved at point $C \equiv (\alpha_m, \beta_m) = (1 - \frac{1}{\sqrt{n}}, 1 - q - \frac{1}{\sqrt{n}})$. Therefore $g\left(\frac{k}{n}, \frac{i}{n}\right) \geq g(\alpha_m, \beta_m)$ or else from equation (15):

$$U_{2k} > \frac{e^{-1} \sqrt{k(1+q)}}{2\pi(1-q)\sqrt{q}} 2^{ng(\alpha_m, \beta_m)} \sum_{m \in E} \frac{k-m}{k+m}$$

Using the Taylor expansion of function $r(x) = g(1-x, 1-p-x)$ around $x = 0$ we get the following expres-

sion:

$$f(x) = -\frac{x^2}{(1-q)q} + O(x^3)$$

or else for $x = \frac{1}{\sqrt{k}}$ we have $n \cdot g(\alpha_m, \beta_m) = -\frac{1}{(1-q)q} + O\left(\frac{1}{\sqrt{k}}\right)$ along with Proposition 2 we get for $k = \rho^2$:

$$U_{2\rho^2} > \frac{e^{-1}\rho\sqrt{(1+q)}}{2\pi(1-q)\sqrt{q}} 2^{-\frac{1}{(1-q)q} + \frac{c}{\rho}} t(\rho) \quad (16)$$

where $t(\rho) = 2\rho^2 (H_{2\rho^2} - H_{2\rho^2-\rho-1}) - \rho - 1$. The above expression can be simplified by using the bounds proved by Young in [11]:

$$\ln n + \gamma + \frac{1}{2(n+1)} < H_n < \ln n + \gamma + \frac{1}{2n}$$

where γ is the Euler's constant and finally get from (16):

$$U_{2\rho^2} > \frac{e^{-1}\rho\sqrt{(1+q)}}{2\pi(1-q)\sqrt{q}} 2^{-\frac{1}{(1-q)q} + \frac{c}{\rho}} \phi(\rho)$$

where $\phi(\rho) = 2\rho^2 \ln\left(\frac{2\rho^2}{2\rho^2-\rho-1}\right) - \rho - 1 - \frac{\rho^2(\rho+2)}{(2\rho^2+1)(2\rho^2-\rho-1)}$.

It's proved in [9] that function $2\rho^2 \ln\left(\frac{2\rho^2}{2\rho^2-\rho-1}\right) - \rho - 1$ is greater than $\frac{1}{4}$ and therefore the last equation becomes

$$U_{2\rho^2} > \frac{e^{-1}\rho\sqrt{(1+q)}}{2\pi(1-q)\sqrt{q}} 2^{-\frac{1}{(1-q)q} + \frac{c}{\rho}} \left(\frac{1}{4} - \frac{\rho^2(\rho+2)}{(2\rho^2+1)(2\rho^2-\rho-1)}\right)$$

Clearly the above function is unbounded and $U_{2\rho^2}$ increases at least linearly with ρ or U_{2k} increases at list as \sqrt{k} . ■

Now we have all the necessary tools to prove the following theorem.

Theorem 2. *The expected time $\mathbb{E}T_n^r$ taken to send by routing n packets through a two parallel multi-hop line network is*

$$\mathbb{E}T_n^r = \frac{n}{2-2p} + D_n^r \quad (17)$$

where p is the erasure probability of the two worst links in the network and the delay function D_n^r depends on all the erasure probabilities p_{ij} , for $i \in \{1, 2\}$, $1 \leq j \leq n$ and is unbounded.

Proof: The first term in equation (17) is due to the capacity of the two parallel multi-hop line network. Term D_n^r is clearly sublinear in n since if function D_n^r was growing faster than n then the capacity of the parallel multi-hop line network would have been equal to 0. Therefore what is left to prove is that term D_n^r is not bounded. This is given by Proposition 3. Indeed, time T_n^r is always greater than the time $\hat{T}_{\frac{n}{2}}^r$ taken for half of the packets ($n/2$) to reach node N_{11} and the other half packets to reach node N_{21} . Therefore

$$\mathbb{E}T_n^r > \mathbb{E}\hat{T}_{\frac{n}{2}}^r \Rightarrow \mathbb{E}T_n^r > \frac{n}{2-2p} + U_n.$$

And since U_n is unbounded D_n^r is also unbounded and this concludes our proof. ■

IV. CONCLUSIONS

In this paper we compared the expected time it takes to communicate n packets over a network of two parallel multi-hop paths. In our previous work we had shown that for a multi-hop line network, the delay function of both routing and network coding are bounded by absolute constants. Therefore the two-parallel path network seems to be the simplest case where there is a gap between the routing and network coding delay. This is intuitive because when there are parallel paths, decisions have to be made on which path to select for each packet. If the random erasures happen to be atypically bad for some paths and atypically good for others, network coding can opportunistically exploit these deviations, contrary to routing when only a single copy of each packet is present in the network. This gives an intuitive explanation of the derived gap, since a random walk typically has deviations of $O(\sqrt{n})$ from expectation. Our results can be generalized to multiple parallel paths and when the worst links are different. More generally we conjecture that as the number of possible routing choices increases, the delay gap between network coding and routing becomes larger.

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