It is evident that if M is cyclicly connected, then Lim $M(x) = M$ for *every* point x of M and conversely.

THEOREM 13.—*If M is a continuous curve, either* (1) *Lim* $M(x) = x$ *for every poin. x of M, or* (2) *Lim* $M(x) = M$ *for every point x of M, or* (3) *M contains a point x such that the limiting arc-curve of M at x does not exist.*

Proof. If (1) does not hold, then M contains a simple closed curve by theorem 11. If (2) is not true, M is not cyclicly connected by theorem 12. Hence M contains a cut-point.[8] By theorem 5, M contains a cut-point which lies on a simple closed curve of M. By theorem 8, the limiting arc-curve of M does not exist at this point. Thus, if neither (1) nor (2) is true, (3) must be true.

[1] Presented to the American Mathematical Society, May 7, 1927.

[2] This paper has been submitted for publication in *Trans. Amer. Math. Soc.*

[3] A point P of a set H is said to separate two points x and y in H if H-P is the sum of two mutually separated sets, one containing x and the other containing y.

[4] Wilder, R. L., *Fund. Math.*, **7**, 1925 (342).

[5] Cf. my paper, "Note on a Theorem Concerning Continuous Curves," *Annals Math.*, **28**, 1927 (501–2).

[6] Ayres, W. L., *Annals Math.*, **28**, 1927 (396–418).

[7] A continuous curve H is said to be cyclicly connected if and only if every two points of H lie together on some simple closed curve which is a subset of H. Cf. Whyburn, G. T., these PROCEEDINGS, **13**, 1927 (31–38).

[8] Loc. cit., theorem 1.

[9] A continuous curve is said to be acyclic if it contains no simple closed curve. This term has been introduced recently by H. M. Gehman.

ON THE ARITHMETIC OF ABELINA FUNCTIONS

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1. Systematic application of abelian functions to the theory of numbers discloses many new arithmetical phenomena of considerable interest. Particularly is this the case when the periods of the functions are connected by one or more singular relations in the usual sense. An example is afforded by the novel type of arithmetical invariance made precise in §2, and illustrated in §3, which, roughly, is as follows.

Consider a set of $n > 1$ quadratic forms in the same s indeterminates. The sets of rational integral values of the s indeterminates which represent an arbitrary set of n integers representable simultaneously in the n forms are separated into sets of residue classes with respect to a modulus α. If a particular set of h of these classes is such that the number of representations of the specified set of n integers which fall into each of the h classes is finite, and if further the respective numbers of representations in the h classes are w, k_2w, ..., k_2w, where the k's are absolute constants, the set of positive integers 1, k_2, ..., k_3 may legitimately be considered as an arithmetical invariant of the set of n forms with respect to α. Next, all of the sets of n arbitrary integers simultaneously representable in the n forms are separated into sets of residue classes with respect to a modulus μ, and in general μ is taken different from α. Each of certain of such sets yields, as above, an invariant set of integers with respect to α; a set of any number of such invariant sets is an invariant set of the n forms with respect to α and μ. Obviously the number of invariant sets is finite, if any exist; not every set of n forms generates an invariant set, nor does the existence of a set for a particular pair of values of α, μ imply the existence of a set for another pair of values.

The remarkable situation just described typifies a cardinal distinction between the arithmetic implicit in the abelian functions and that in the elliptic: simultaneous representations and their properties are the rule in the abelian case, while in the elliptic they are the exception.

The examples in §3 are among the simplest illustrations of §2. The first was deduced from the rational algebraic relations connecting singular functions of two arguments, for both arguments zero, when the singular relation between the periods has the invariant 5; the second is equivalent to one of the classic biquadratic relations.

2. A one-rowed matrix $(n_1, \ldots, n_p)$ of p rational integers will be called integral of order p. If δ is a constant integer different from zero, and r_j is the least positive residue modulo δ of n_j $(j = 1, \ldots, p)$, we write

$$(n_1, \ldots, n_p) \equiv (r_1, \ldots, r_p) \bmod \delta,$$

and call $(r_1, \ldots, r_p)$ the residue mod δ of $(n_1, \ldots, n_p)$. Such residues being matrices, their equality is defined. The residue class $C(r_1, \ldots, r_p)$ mod δ is the set of all integral matrices of order p each of which has the residue $(r_1, \ldots, r_p)$ mod δ; the number of these classes is δ^p. It is convenient presently to use both ( ) and { } to designate integral matrices.

Henceforth α, μ are constant integers different from zero, and the n distinct quadratic forms Q_j,

$$Q_j \equiv Q_j(u_1, \ldots, u_s) \qquad (j = 1, \ldots, n),$$

in the s indeterminates $u_1, \ldots, u_s$, with rational integral coefficients, are given and fixed.

If for $(m_1, \ldots, m_n)$ the general element (integral matrix of order n) in $C(\mu_1, \ldots, \mu_n)$ mod μ there exist s rational integers $a_1, \ldots, a_s$ such that

$$Q_j(a_1, \ldots, a_s) = m_j \quad (j = 1, \ldots, n),$$

we say that $(m_1, \ldots, m_n)$ is equivalent to $\{a_1, \ldots, a_s\}$ with reference to $[Q_1, \ldots, Q_n]$, and write

$$(m_1, \ldots, m_n) \sim \{a_1, \ldots, a_s\} \text{ ref. } [Q_1, \ldots, Q_n] \tag{1}$$

Keep $(m_1, \ldots, m_n)$ fixed, and therefore also $(\mu_1, \ldots, \mu_n)$, its residue mod μ, and consider the set of all $\{a_1, \ldots, a_s\}$ satisfying (1). If in this set there are only a finite number of matrics, necessarily all unequal, they will fall into $h \leqq \alpha^s$ distinct residue classes K_i mod α, such that K_i contains precisely w_i values of $\{a_1, \ldots, a_s\}$ satisfying (1), where w_i is finite $(i = 1, \ldots, h)$. If now $h - 1$ of the integers w_i $(i = 1, \ldots, h)$ are constant integral multiples of the remaining one, which, without loss of generality, may be taken to be w_i, so that

$$w_i = k_i w_1 \ (i = 1, \ldots, h), \quad k_1 = 1,$$

where the k's are absolute constants, we shall say that $M \equiv (\mu_1, \ldots, \mu_n)$, the residue of the general element $(m_1, \ldots, m_n)$ of $C(\mu_1, \ldots, \mu_n)$ mod μ, is finite in the class matrix $K \equiv ((K_1, \ldots, K_h))$ mod α of order h, and has with respect to K the relative weight $W \equiv | 1, k_2, \ldots, k_h |$. We shall symbolize this (exceptional) situation thus,

$$M \text{ resp. } K = W. \tag{2}$$

Next, in (1), let $(m_1, \ldots, m_n)$ range over the set of general elements in the respective μ^n residue classes mod μ of integral matrices of order n. Now it may happen, and in general does for the $[Q_1, \ldots, Q_n]$ arising from singular abelian functions, that several of the residues mod μ are finite in certain class matrices mod α, and that for each such residue mod μ there is a weight relation of the type defined by (2). Let

$$M^{(j)} \text{ resp. } K^{(j)} = W^{(j)} \ (j = 1, \ldots t) \tag{3}$$

be any set of these relations for the $M^{(j)}$ all different. Then, subsuming the t relations (3) under a single matric relation, we write

$$(M^{(1)}, \ldots, M^{(t)}) \text{ resp. } ((K^{(1)}, \ldots, K^{(t)})) = | W^{(1)}, \ldots, W^{(t)} | \tag{4}$$

which may be read: the residue matrix $(M^{(1)}, \ldots, M^{(t)})$ mod μ is finite in the class matrix $((K^{(1)}, \ldots, K^{(t)}))$, mod α and has with respect to it the relative weight $| W^{(1)}, \ldots, W^{(t)} |$. When $t = 1$, (4) becomes (2). In (3) the orders of $K^{(j)}$ $(j = 1, \ldots, t)$ are not necessarily equal, although in the examples of §3 they are.

From the above definitions it is clear that a relative weight is a set of absolute constants (positive integers) connected with the simultaneous representation of sets of n arbitrary integers in the given set of n quadratic forms in s indeterminates; in this sense a relative weight is an invariant of $[Q_1, \ldots, Q_n]$ determined by α, μ alone.

3. Take $s = 6$, $n = 2$, $\alpha = 2$, $\mu = 8$,

$$Q_1 \equiv u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2,$$

$$Q_2 \equiv 2(u_1u_6 + u_2u_5 + u_3u_4) + u_4^2 + u_5^2 + u_6^2,$$

and consider (1), (2) of §2 in this case for the following 6 values M_j of the possible 64 of $M \equiv (\mu_1, \mu_2)$,

$$M_1 = (3,3), \quad M_3 = (3,0), \quad M_5 = (6,1),$$
$$M_2 = (7,7), \quad M_4 = (7,0), \quad M_6 = (6,5),$$

with respect to the 6 residue classes, I, II..., VI, of the 64 possible $C\{u_1, \ldots, u_6\}$ mod 2,

I : (0, 1, 1, 1, 0, 0),
II : (0, 0, 0, 1, 1, 1),
III : (0, 0, 1, 1, 1, 0),
IV : (1, 1, 1, 0, 0, 0),
V : (0, 0, 1, 0, 1, 0),
VI : (1, 1, 1, 1, 1, 1).

Corresponding to (3), we have the following table, in which (i, j, k) is either (1,3,5) or (2,4,6).

(μ_1, μ_2)	:	Class matrix	:	Relative weight
M_i	:	((I, II))	:	$\mid 1, 2 \mid$
M_j	:	((III, IV))	:	$\mid 1, 2 \mid$
M_k	:	((V, VI))	:	$\mid 1, 2 \mid$,

which can be combined into a single relation of the type (4). For example, the third row of the table states that if m_1, m_2 are any rational integers such that either $m_1 \equiv 6$, $m_2 \equiv 1 \bmod 8$, or $m_1 \equiv 6$, $m_2 \equiv 5 \bmod 8$, then the number of sets of integers $\{u_1, \ldots, u_6\}$ satisfying $Q_1 = m_1$, $Q_2 = m_2$, in which all values of $u_1, \ldots, u_6$ are odd (see VI) is twice that in which only the values of u_3, u_5 are odd (see V).

As a second example, let c be an arbitrary constant integer which is not a square. Take $n = 2$, $s = 8$, $\alpha = 2$, $\mu = 4$. Of the 256 residue classes $C\{u_1, \ldots, u_8\}$ mod 2
the pair

I : (1, 1, 1, 1, 0, 0, 0, 0),
II : (0, 0, 0, 0, 1, 1, 1, 1),

gives for either of $M \equiv (\mu_1, \mu_2) = (0, 0), (0, 2)$, and the pair of forms

$$Q_1 \equiv u_1^2 + u_2^2 + u_3^2 + u_4^2 + c(u_5^2 + u_6^2 + u_7^2 + u_8^2),$$
$$Q_2 \equiv u_1u_8 + u_2u_7 + u_3u_6 + u_4u_5,$$

the following instance of (2),

$$M \text{ resp. } ((\text{I, II})) = | 1, 1 |,$$

That is, the number of sets $\{u_1, \ldots, u_8\}$ satisfying $Q_1 = m_1$, $Q_2 = m_2$, in which $u_1, \ldots, u_8$ have the respective residues mod 2 indicated in I, is equal to the number of sets in which the residues are as in II, for either $m_1 \equiv m_2 \equiv 0 \bmod 4$ or $m_1 \equiv 0$, $m_2 \equiv 2 \bmod 4$.

Both examples illustrate the following for a pair ($n = 2$) of quadratic forms in $2r(= s)$ indeterminates. Let p,q be rational integers such that $p^2 + 4q$ is not a square. Then, $\sum$ referring to $j = 1, \ldots, r$, a typical pair of forms for which relations of the type (4) subsist, when α, μ are powers of 2, is

$$Q_1(u_1, \ldots, u_{2r}) = \sum(u_j^2 + qu_{2r-j+1}^2),$$
$$Q_2(u_1, \ldots, u_{2r}) = \sum u_{2r-j+1} \quad (2u_j + pu_{2r-j+1}).$$

In the first example $r = 3$, $p = q = 1$; in the second, $r = 4$, $p = 0$, $q = c$, since evidently $Q_2 = m_2$ is equivalent to $2Q_2 = 2m_2$.

GROUPS WHOSE OPERATORS ARE OF THE FORM $s^p t^q$

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A. Cayley called attention to the fact that if a group G is generated by two non-commutative operators s and t which satisfy the equation $st = t^2s^2$ then it is not possible to represent all of the operators of G by the expression s^pt^q.[1] This tends to augment the interest in the question what properties a group which is generated by the two operators s and t must have in order that all the operators of this group are of the form s^pt^q, where p and q are integers which do not exceed the orders of s and t, respectively. In what follows the symbol G will represent such a group and the subgroups generated by s and t will be represented by S and T, respectively. If a subgroup of S is invariant under t it is obviously invariant under G, and if a subgroup of T is invariant under s it is also invariant under G. We shall, therefore, represent by S_1 and T_1 the maximal subgroups of S and T which are invariant under t and s, respectively. The index of S_1 under S will be represented by m and the index of T_1 under T will be represented by n.

Since all the operators of G can be represented by s^pt^q they can also be represented by t^qs^p, for the inverses of all the operators of a finite group