effect on the form, quantitatively considered, of one highly important phase of the life cycle, namely, growth. More precisely stated, the relation between attained size and relative time in the development of the canteloup plant, is, to a first approximation, identical whether the environment in which it has its being is highly variable in respect of temperature and all other physical and chemical particulars, or is, to a high degree, constant. The pattern of the events which constitute the life cycle of the organism is primarily and basically determined by the physico-chemical organization or pattern of the organism itself.

[1] From the Institute for Biological Research of the Johns Hopkins University.

[2] Pearl, R., Agnes A. Winsor and J. R. Miner, "The Growth of Seedlings of the Canteloup *Cucumis melo*, in the Absence of Exogenous Food and Light," *Proc. Nat. Acad. Sci.*, **14**, pp. 1–4, 1928.

[3] Pearl, R., and L. J. Reed, "Skew Growth Curves," *Proc. Nat. Acad. Sci.*, **11**, pp. 16–22, 1925.

[4] Reed, L. J., and R. Pearl, "On the Summation of Logistic Curves," *Jour. Roy. Stat. Soc.*, **90**, pp. 729–746, 1927.

[5] Cf. Pearl, R., "On the Distribution of Differences in Vitality among Individuals," *Amer. Nat.*, **61**, pp. 113–131, 1927; and "The Rate of Living," New York (Alfred A. Knopf), 1928, pp. 14 unnumbered +185.

[6] Whitehead, A. N., "Science and the Modern World," New York (Macmillan), 1925. Pp. xi + 296.

INVARIANT SEQUENCES

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1. *Introduction.*—Let $f_n(x)$ $(n = 0, 1, \ldots)$ be uniform functions of the real or complex variable x; let a, b, $\tau(n)$ be independent of x. Then if a, b, $\tau(n)$ exist such that

$$f_n'(x) = f_{n-1}(x), \quad f_n(ax + b) = \tau(n) f_n(x), \quad \left[f_n'(x) \equiv \frac{d}{dx} f_n(x) \right],$$

for $n = 0, 1, 2, \ldots$, we shall say that the *invariant sequence* $f_n(x)$ $(n = 0, 1, \ldots)$ has the *characteristic* $(a, b, \tau(n))$. It is known that the general solution of the first of the above functional equations is $f_n(x) = k(x + \varphi)^n/n!$, where φ^n (including $n = 0$) is to be replaced by φ_n after expansion by the binomial theorem, and k is independent of both n and x. Hence *the nth element of an invariant sequence is a polynomial of degree n.* We call φ as above defined the *base* of $f_n(x)$ $(n = 0, 1, \ldots)$, and, if $s \geqq 0$ is the least integer such that $\varphi_s \neq 0$, we say that s is the *index* of φ. *We regard φ as the quaesitum.* In terms of φ, s, n we shall find a, b and $\tau(n)$.

That is, *we shall determine all bases φ, all indices s, and all transformations* $[x, ax + b]$, *which yield invariant sequences.*

2. *Theorems.*—In a paper to appear fully elsewhere I have proved, among many others, the following.

THEOREM 1.—*There exists precisely one nontrivial characteristic* $(a, b, \tau(n))$ *such that the invariant sequence* $f_n(x)$ $(n = 0, 1, \ldots)$ *with the base* φ *has the property*

$$f_n(ax + b) = \tau(n)f_n(x);$$

if s is the index of the base φ *of the* $f_n(x)$, *the characteristic is*

$$(a, b, \tau(n)) = (-1, -2\varphi_{s+1}/(s + 1)\varphi_s, (-1)^{n+s}),$$

and φ_n *is the coefficient of* $z^n/n!$ *in* $R(z, e^z)$, *where* $R(u, v)$ *is any solution of the functional equation*

$$v^b R(u, v) = (-1)^s R(-u, v^{-1}),$$

in which u, v are independent variables and b is as above.

THEOREM 2.—*The functional equation of Theorem 1, in which b is an arbitrary constant, s an arbitrary constant integer* ≥ 0, *has the elementary solution involving both u and v or only v*

$$R(u, v) \equiv v^{rb+t} F(u) + v^{-(r+1)b-t} (-1)^s F(-u),$$

in which r, t are arbitrary constants, and $F(u)$ *is an arbitrary function of u alone, (including* $F(u) \equiv$ *constant).*

THEOREM 3.—*All solutions of the functional equation of Theorem 1 of the type* $A(u)$, $B(v)$, *where* $A(u)$, $B(v)$ *are functions of u alone, v alone, are given by*

$$A(u) = F(u) + (-1)^{\eta s} F(-u), \quad B(v) = G(v) \ (\eta = 0, 1),$$

where $F(u)$ *is an arbitrary function of u, and* $G(v)$ *is any solution of*

$$G(v^{-1}) = (-1)^{(\eta+1)s} v^b G(v),$$

where the same value of η *is to be used in both of* $A(u)$, $B(v)$. *The G equation has the elementary solution*

$$G(v) \equiv v^{rb+t} + (-1)^{(\eta+1)s} v^{-(r+1)b-t},$$

where r, t are arbitrary constants; all rational functions of given solutions that are again solutions are constructible according to the operations indicated in

$$k\,G(v),\ G_1(v) + G_2(v),\ (-1)^{(\eta+1)s} v^{\epsilon b/2} G_1(v) [G_2(v)]^{\epsilon},$$

where $\epsilon = 1, -1$, *k is an arbitrary constant, and* $G_1(v)$, $G_2(v)$, $G(v)$ *are given solutions.*

In another theorem, not here reproduced, the chain of possible types is closed by the discussion of solutions

$A(u)\, R(u, v)$, $B(v)\, R(u, v)$, where $R(u, v)$ is any solution involving both u and v, and $A(u)$, $B(v)$ are solutions involving only u or only v.

It is to be noticed particularly that the foregoing includes the complete solution of the following: given a, b in the transformation $[x, ax + b]$, *to construct* φ. In the light of the foregoing theorems this apparent problem is trivial.

3. *Equivalent invariant sequences.*—Invariant sequences which, by a linear transformation on the rank n and a linear transformation on the variable x, can be exhibited as instances of one and the same invariant sequence will be called *equivalent.*

THEOREM 4.—*Let* λ, μ, *of indices* s, t, *be the respective bases of* $F_n(x)$, $H_n(x)$ $(n = 0, 1, \ldots)$, *so that*

$$F_n'(x) = F_{n-1}(x), \quad F_n(-x + b) = (-1)^{n+s} F_n(x),$$

$$H_n'(x) = H_{n+1}(x), \; H_n(-x + c) = (-1)^{n+t} H_n(x),$$

$$b \equiv -2\lambda_{s+1}/(s + 1)\lambda_s, \quad c \equiv -2\mu_{t+1}/(t + 1)\mu_t.$$

Let i, j, m *be arbitrary constant integers* $\geqq 0$. *Then*

$$X_n(x) = G_n(x) \equiv F_{n+2i+s+m}\left(x + \frac{b - a}{2}\right),$$

$$X_n(x) = K_n(x) \equiv H_{n+2j+t+m}\left(x + \frac{c - a}{2}\right)$$

are solutions of

$$X_n'(x) = X_{n-1}(x), \quad X_n(-x + a) = (-1)^{n+m} X_n(x),$$

where a *is an arbitrary constant, being equal to* $-2\sigma^{m+1}/(m + 1)\sigma_m$, *where* σ *is the base of the general Appell polynomial in* x *and* m *is the index of* σ.

Any number of invariant sequences can be unified by this theorem, in precisely the same way that two are therein unified, by an obvious repetition of the process.

4. *Rational invariant sequences.*—If $R(u, v)$ which defines φ in Theorem 1 is a rational function of u, v, the invariant sequence $f_n(x)$ $(n = 0, 1, \ldots)$ is called *rational.* Obviously φ_0, φ_1, $\ldots$ are not necessarily rational numbers. In the paper to be published, *I completely determine all rational sequences in explicit form.* These naturally involve arbitrary constants. I note particularly that this corollary to the general solutions already stated, includes as a very special case the situation italicized at the end of §2.

5. *Applications.*—There are four classic instances of such sequences, namely, those in which the base φ represents the numbers of Bernoulli, Euler, Genocchi, Lucas, respectively. In any instance it is necessary

to know only the index s and the first $s + 2$ terms of the φ sequence in order to write down the functional equations for the polynomials concerned. The foregoing adds a double infinity of such sequences to those already known.

In conclusion we remark that the so-called "elementary method" of N. Nielsen and others is abstractly identical with the symbolic method of Blissard used in this paper. This fact has been misunderstood by some who, incidentally, erroneously attribute the method to Lucas. It is proved in my paper cited above that each method implies and is implied by the other.

NORMAL DIVISION ALGEBRAS SATISFYING MILD ASSUMPTIONS

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Let D be a normal division algebra in n^2 units over an infinite field F. Every element of D satisfies some equation of degree n, with leading coefficient unity and further coefficients in F. There exists an infinity of elements of D satisfying an equation of this kind, irreducible in F.

1. *Definitions.*

An element x of A shall be called of grade r if its minimum equation has degree r.

An equation

$$\phi(\omega) \equiv \omega^r + \alpha_1\omega^{r-1} + \ldots + \alpha_p = 0$$

shall be called of type R_k if $k - 1$ distinct complex roots of $\phi(\omega) = 0$ are expressible as rational functions, with coefficients in F, of a kth distinct root.

An element x of a normal division algebra D is said to be of type R_k if its minimum equation is of type R_k. A normal division algebra D in n^2 units is said to be of type R_k if it contains an element x of grade n and type R_k.

2. *Known Division Algebras.*

Let Σ be any associative division algebra over F. Let to every element A of Σ correspond a unique element A' of Σ, and let $s \neq 0$ be a self corresponding element of Σ. Let $A'' \equiv (A')'$, $A''' \equiv (A'')'$, ... Let $A^{(\rho)} = sAs^{-1}$ for every A of Σ. Let $\alpha' = \alpha$, $(AB)' = A'B'$, $(A + B)' =$