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The H^1-compact global attractor for the solutions to the Navier–Stokes equations in two-dimensional unbounded domains

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Abstract. We extend previous results obtained by Rosa (1998 *Nonlinear Anal.* **32** 71–85) on the existence of the global attractor for the two-dimensional Navier–Stokes equations on some unbounded domains. We show that if the forcing term is in the natural space H, then the global attractor is compact not only in the L^2 norm but also in the H^1 norm, and it attracts all bounded sets in H in the metric of V. The proof is based on the concept of asymptotic compactness and the use of the enstrophy equation. As compared with the work of Rosa, which proved the compactness and the attraction in the L^2 norm, the new difficulty comes from the fact that the nonlinear term of the Navier–Stokes equations does not disappear from the enstrophy equation, while it does disappear in the energy equation due to its antisymmetry property.

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1. Introduction

The global attractor for the two-dimensional (2D) Navier–Stokes equations was first obtained in [L1, FT] for bounded domains. In [FT], the finite dimensionality of the attractor in the sense of the Hausdorff dimension is also shown (see also [CF2, CFMT, T2]). For the case of unbounded domains, [A1, Bb] worked on the problem with the forcing term being required to be in some weighted space. However, their estimate of the dimension of the attractor in this case was independent of the weighted norm of the forcing term, which suggested a natural expectation of the existence of the global attractor for more general forces. For the unbounded cases, see also [A2, BbV, FLST].

Recently, it was proved by [R] that for the 2D unbounded, non-smooth domain, provided that the Poincaré inequality is verified, the semigroup generated by solutions of the Navier–Stokes equations in the phase space H has a global attractor $\mathcal{A}$ when the external forcing term f is in V', the natural dual space of the theory of the Navier–Stokes equations (see the notation given in section 2). The attractor is found to be bounded in V and compact in H, i.e. $\mathcal{A}$ attracts all bounded sets in H, is a compact invariant set in H, is connected in H and is maximal for the inclusion relation among all the functional invariant sets bounded in H. An estimate of the dimensions of the attractor was also obtained.

An interesting question naturally arises here, namely whether or not the semigroup generated by solutions of the Navier–Stokes equations in the phase space V has a global attractor $\mathcal{A}$ compact in V when f is in H. In this paper, we give a positive answer to this question. We are able to show that the global attractor attracts all bounded sets in V. It is connected in V and maximal for the inclusion relation among all the functional invariant sets bounded in V. Due to the regularity effect, the attractor obtained here and the attractor obtained in [R] in fact coincide (see remark 3.1). Thus the attractor is compact in V and attracts all bounded sets in H in the metric of V.

Note that the above result is known for the case of bounded domains if f is in H. This can be obtained with the appropriate *a priori* estimate for $|Au|$ using the time analyticity of the solutions, which gives a bound on $|u_t|$ (see, e.g., [T2]). Thus, the only novelty here is for the case of unbounded domains.

The requirement for the domains is such that $D(A)$, the domain of the Stokes operator, can be characterized as $H^2(\Omega) \bigcap V$ and that the Poincaré inequality is satisfied. Hence it requires some smoothness of the boundary of the domains, and that the domains be bounded in one direction, e.g. the channel-like domains.

The results of [R] were obtained using an asymptotic compactness argument applied to the energy equation. This idea was successfully implemented to some weakly damped hyperbolic equations first by [Bl] , and then by [G, W] (see also [MRW] and the references cited therein). The concept of asymptotic compactness had already been used by [A1, A2, L2], and is implicit in [T2], theorem I.1.1, (1.13). See also the concept of asymptotic smoothness in [Hal, Har]. [R] realized that these proofs do not make essential use of the compactness of the Sobolev embeddings and thus can naturally be extended to equations on unbounded domains, provided that the Poincaré inequality is verified. Here we apply the same argument to the *enstrophy* equation instead of the *energy* equation. Note that the nonlinear term of the NSEs disappears in the energy equation due to its antisymmetry, while the corresponding term does *not* disappear in the enstrophy equation. This nonlinearity presents a new difficulty. In this paper we are able to overcome this difficulty with a careful analysis.

The rest of the paper is organized as follows. In section 2, we first recall some notation and preliminary results about the 2D NSEs with the no-slip boundary condition. In section 3, we study the H^1-compact global attractor and derive our main result.

2. The 2D NSEs in an unbounded domain

Suppose that $\Omega \subset R^d$ $(d = 2)$ is an open bounded or unbounded set with the boundary $\partial\Omega$ smooth enough such that (2.5) and (2.12) hold.

Consider the following non-stationary Navier–Stokes equations describing the flow of a viscous incompressible fluid confined in Ω:

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^{d} u_i \frac{\partial u}{\partial x_i} + \nabla p = f \qquad \text{for} \quad t > 0 \tag{2.1}$$

$$\nabla \cdot u = 0 \qquad \text{for} \quad t > 0 \tag{2.2}$$

which are supplemented with the no-slip boundary condition

$$u|_{\partial\Omega} = 0 \tag{2.3}$$

and the initial condition

$$u|_{t=0} = u_0 \tag{2.4}$$

where $u(x,t) \in R^2$, $x \in R^2$ and $p(x,t) \in R^1$. Assume that there exists a $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 \,\mathrm{d}x \leqslant \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 \,\mathrm{d}x \qquad \forall \phi \in H_0^1(\Omega). \tag{2.5}$$

It is well known that for Lipschitz domains bounded in one direction, the above Poincaré inequality holds.

Let $\mathbb{L}^2(\Omega) := (L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega) := (H_0^1(\Omega))^2$ with the inner products $(\cdot,\cdot)$ and $((\cdot,\cdot))$ and the norms $|\cdot| := (\cdot,\cdot)^{\frac{1}{2}}$ and $\|\cdot\| := ((\cdot,\cdot))^{\frac{1}{2}}$ where

$$(u,v) := \int_\Omega u \cdot v \,\mathrm{d}x \qquad \text{for} \quad u, v \in \mathbb{L}^2(\Omega)$$

$$((u,v)) := \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j \,\mathrm{d}x \qquad \text{for} \quad u, v \in \mathbb{H}_0^1(\Omega).$$

Thanks to the Poincaré inequality, $\|\cdot\|$ is a norm of $\mathbb{H}_0^1(\Omega)$. Set

$$\mathcal{V} := \{v \in (\mathcal{D}(\Omega))^2 \mid \nabla \cdot v = 0 \text{ in } \Omega\}$$

$$H := \overline{\mathcal{V}}^{\mathbb{H}_0^1(\Omega)} \qquad V := \overline{\mathcal{V}}^{\mathbb{L}^2(\Omega)}.$$

Define $A : V \mapsto V'$ (the dual of V) as

$$\langle Au, v\rangle := ((u,v)) \qquad \forall u, v \in V$$

where $\langle\cdot,\cdot\rangle$ is the duality product between V' and V. We identify $H' = H$.

Using integration by parts,

$$|\nabla v|^2 = -\int_\Omega \Delta v \cdot v \,\mathrm{d}x = -\int_\Omega \Delta v \cdot Pv \,\mathrm{d}x \qquad \forall v \in D(A)$$

where P is the Stokes projector. So,

$$\|v\|^2 = ((v,v)) = (Av, v) \leqslant |Av| \cdot |v| \leqslant \frac{1}{\lambda_1^{1/2}} |Av| \|v\| \qquad \forall v \in D(A).$$

Hence

$$\|v\| \leqslant \frac{1}{\lambda_1^{1/2}} |Av| \qquad \forall v \in D(A). \tag{2.6}$$

The bilinear operator $B : V \times V \mapsto V'$ is defined as

$$\langle B(u,v), w\rangle := b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x \qquad \forall u, v, w \in V.$$

Denote $B(u) := B(u,u)$. It is well known that

$$b(u,v,w) = -b(u,w,v) \qquad \forall u \in V,\ v, w \in \mathbb{H}_0^1(\Omega). \tag{2.7}$$

By [T1], lemma III.3.4, we also have

$$|b(u,v,w)| \leqslant \sqrt{2} |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}} \qquad \forall u, v, w \in \mathbb{H}_0^1(\Omega). \tag{2.8}$$

Therefore,

$$\|B(u)\|_{V'} \leqslant \sqrt{2} |u| \|u\| \qquad \forall u \in V. \tag{2.9}$$

Moreover, if $|A\cdot|$ is a norm on $V\bigcap(H^2(\Omega))^2$, which is equivalent to the norm induced by $(H^2(\Omega))^2$, then

$$|B(u)| \leqslant c|u|^{\frac{1}{2}}\|u\||Au|^{\frac{1}{2}} \qquad \forall u \in D(A). \tag{2.10}$$

See, for example, [T1], chapter III, section 3.7.1, for the idea of the proof, though the discussion there is for the bounded domains of class $\mathcal{C}^2$. Similarly, one can also show that

$$|b(u,v,w)| \leqslant c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \qquad \forall u \in V \quad v \in D(A) \quad w \in H \tag{2.11}$$

where c is a positive constant independent of u, v and w.

Using (2.5) and (2.6), it can be shown that for Ω unbounded with $\partial\Omega$ being uniformly of $\mathcal{C}^3$ there is a constant $c > 0$ independent of v such that

$$|D^2 v| \leqslant c|Av| \qquad \forall v \in D(A). \tag{2.12}$$

See [H] for details. Thus, in this case, $|A\cdot|$ is a norm on $V\bigcap(H^2(\Omega))^2$, which is equivalent to the norm induced by $(H^2(\Omega))^2$, and therefore (2.10) and (2.11) hold.

Consider the following weak formulation. Find $u \in L^\infty(0,T;H)\bigcap L^2(0,T;V)$ for $T > 0$, such that in the sense of distribution on $(0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v) + \nu((u,v)) + b(u,u,v) = \langle f, v\rangle \qquad \forall v \in V \tag{2.13}$$

with initial condition (2.4). This is equivalent to the following functional equation:

$$u' + \nu Au + B(u) = f \qquad u(0) = u_0 \tag{2.14}$$

in V', where $u' = \frac{\mathrm{d}}{\mathrm{d}t}u$.

Now we have the following existence and regularity results.

Theorem 2.1. *Assume that Ω is such that (2.5) holds. Given that $f \in V'$ and $u_0 \in H$, there exists a unique u solving (2.14) (hence (2.13) and (2.4) and (2.1)–(2.3)). Moreover, $u \in L^\infty(R_+,H)\bigcap L^2(0,T;V)$, $u \in \mathcal{C}(R_+,H)$ and $u' \in L^2(0,T;V')$ for all $T > 0$.*

Further, assume that Ω is such that (2.12) also holds. Then $f \in H$ and $u_0 \in V$, then $u \in \mathcal{C}(R_+;V)$, $u \in L^\infty(R_+,V)\bigcap L^2(0,T;D(A))$ and $u' \in L^2(0,T;H)$ for $T > 0$.

The proof of the first part of the above theorem can be found in [T1] along with the following important *a priori* estimates (2.15) and (2.16).

Let $v = u$ in (2.13). Then, by (2.7),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 = (f,u) \leqslant \frac{\|f\|_{V'}^2}{2\nu} + \frac{\nu}{2}\|u\|^2$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 \leqslant \frac{\|f\|_{V'}^2}{\nu} \qquad \forall t > 0.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\lambda_1|u|^2 \leqslant \frac{\|f\|_{V'}^2}{\nu} \qquad \forall t > 0.$$

Thus, there is a constant $C_0 = C_0(|u_0|, \|f\|_{V'}) \geqslant 0$ such that for all $t > 0$

$$|u(t)|^2 \leqslant |u_0|^2 \mathrm{e}^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1}\|f\|_{V'}^2. \tag{2.15}$$

Moreover,

$$\int_0^t \|u(s)\|^2 \, \mathrm{d}s \leqslant \frac{1}{|u_0|^2} + \frac{t}{\nu^2}\|f\|_{V'}^2. \tag{2.16}$$

The second part of theorem 2.1 can be obtained using the inequalities (2.5), (2.6) and (2.10) and the following *a priori* estimates (2.18) and (2.19).

Let $v = Au$ in (2.13). Then, by (2.5), (2.6) and (2.10),

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|^2 &= (f, Au) - b(u, u, Au)\\ &\leqslant |f||Au| + c|u|^{\frac{1}{2}}\|u\||Au|^{\frac{3}{2}}\\ &\leqslant \frac{|f|^2}{\nu} + \tfrac{1}{4}\nu|Au|^2 + c_\nu|u|^2\|u\|^4 + \tfrac{1}{4}\nu|Au|^2.\end{aligned}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|^2 \leqslant \frac{2}{\nu}|f|^2 + 2c_\nu|u|^2\|u\|^4. \tag{2.17}$$

Thus, by the uniform Gronwall lemma [T2], there exists a constant $C_1 = C_1(\|u_0\|, |f|) \geqslant 0$ such that

$$\|u(t)\| \leqslant C_1 \qquad \forall t > 0. \tag{2.18}$$

For the detailed derivation of (2.18) from (2.17) using the uniform Gronwall lemma, one can refer to [T2]. Moreover, it easy to see that there exists $C_\nu > 0$ such that

$$\int_0^t |Au(s)|^2 \, \mathrm{d}s \leqslant \frac{2t}{\nu^2}|f|^2 + \frac{\|u_0\|^2}{\nu} + 2C_\nu(C_0C_1)^2 \int_0^t \|u(s)\|^2 \, \mathrm{d}s. \tag{2.19}$$

Theorem 2.1 defines

$$S(t) : u_0 \in H \mapsto S(t)u_0 = u(t) \in H, \forall t > 0.$$

$\{S(t)\}_{t\geqslant 0}$ is a continuous semigroup in the H norm. By (2.18) and (2.19), it can be shown that $\{S(t)\}_{t\geqslant 0}$ is also a continuous semigroup in the V norm.

Lemma 2.1. *$S(t) : V \mapsto V$ is a Lipschitz continuous map (operator) on V for $t \geqslant 0$.*

Proof. Suppose u, v are two solutions with initial values u_0 and v_0 in V. Let $w = u - v$ and $w_0 = u_0 - v_0$. Then, by (2.7) and (2.11),

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu|Aw|^2 &= -b(u, u, Aw) + b(v, v, Aw)\\ &= b(u, Aw, w) + b(w, Aw, v)\\ &\leqslant c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}|Aw|^{\frac{3}{2}} + c|w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}|Av|^{\frac{1}{2}}|Aw|\\ &\leqslant c|u|^2\|u\|^2\|w\|^2 + \tfrac{1}{4}\nu|Aw|^2 + c|w|\|w\|\|v\||Av| + \tfrac{1}{4}\nu|Aw|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\lambda_1\|w\|^2 &\leqslant C(|u|^2\|u\|^2 + \|v\||Av|)\|w\|^2\\ &\leqslant C(\|u\|^2 + \|v\||Av|)\|w\|^2.\end{aligned}$$

So,

$$\|w(t)\|^2 \leqslant \|w_0\| \exp\left\{c\int_0^t (\|u(s)\|^2 + \|v(s)\||Av(s)|)\, \mathrm{d}s\right\}. \qquad \square$$

The Lipschitz continuity in the H norm can be proved in a similar way.

Remark 2.1. By (2.18), we know there is a set $\mathcal{B} \subset V$, which is bounded in V and absorbing in V for the semigroup $\{S(t)\}_{t\geqslant 0}$. For the convenience of later discussion, we may assume without loss of generality that $\mathcal{B} \subset V$ is an absorbing ball of $\{S(t)\}_{t\geqslant 0}$.

Remark 2.2. It can be shown that $u' \in L^2(0, T; H)$, for $T > 0$. In fact, setting $v = u'$ in (2.13), we have, by using (2.10), that

$$
\begin{aligned}
|u'|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &= (f, u') - b(u, u, u') \\
&\leqslant |f||u'| + c|u|^{\frac{1}{2}}\|u\||Au|^{\frac{1}{2}}|u'| \\
&\leqslant |f|^2 + \tfrac{1}{4}|u'|^2 + c|u|\|u\|^2|Au| + \tfrac{1}{4}|u'|^2.
\end{aligned}
$$

So,

$$
|u'|^2 + \nu\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \leqslant 2|f|^2 + C|u|\|u\|^2|Au|.
$$

Thus, there is a constant $C_2 = C_2(C_0, C_1) \geqslant 0$ such that for all $T > 0$

$$
\begin{aligned}
\int_0^T |u'(s)|^2\,\mathrm{d}s &\leqslant \nu\|u_0\|^2 + 2|f|^2 T + c\int_0^T |u|\|u\|^2|Au|\,\mathrm{d}s \\
&\leqslant \nu\|u_0\|^2 + \left\{2|f|^2 + C_2\left(\int_0^T |Au|^2\,\mathrm{d}s\right)^{1/2}\right\}T. \qquad (2.20)
\end{aligned}
$$

Similar to lemma 2.1 of [R], we can have the following useful lemma, the proof of which is omitted here.

Lemma 2.2. *Let $\{u_{0,n}\}$ be a sequence in V, which converges weakly to $u_0 \in V$. Then $S(t)u_{0,n} \rightharpoonup S(t)u_0$, weakly in V, $\forall t \geqslant 0$ and $S(\cdot)u_{0,n} \rightharpoonup S(\cdot)u_0$, weakly in $L^2(0, T; D(A))$, $\forall T \geqslant 0$.*

The next lemma is also important for the proof of our main result.

Lemma 2.3. *Let $\{u_{0,n}\}_n$ be a sequence in H, which converges strongly to $u_0 \in H$. Suppose $u(t) = S(t)u_0$, $u_n(t) = S(t)u_{0,n}$. Then, $\forall T > 0$, $u_n \to u$ in $L^2(0, T; V)$.*

Proof. Using (2.13), it is easy to see that, by (2.8)

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_n - u|^2 + \nu\|u_n - u\|^2 &= b(u_n - u, u_n - u, u) \\
&\leqslant c\|u\||u_n - u|\|u_n - u\| \\
&\leqslant \tfrac{1}{2}\nu\|u_n - u\|^2 + C\|u\|^2|u_n - u|^2.
\end{aligned}
$$

Thus

$$
\frac{\mathrm{d}}{\mathrm{d}t}|u_n - u|^2 + \nu\|u_n - u\|^2 \leqslant c\|u\|^2|u_n - u|^2.
$$

Therefore

$$
\nu\int_0^T \|u_n(s) - u(s)\|^2\,\mathrm{d}s \leqslant |u_{n,0} - u_0|^2 + c\int_0^T \|u(s)\|^2|u_n - u|^2\,\mathrm{d}s.
$$

By Lebesgue's dominant convergence theorem, and noting that

$$
\lim_{n\to\infty}|u_n(t) - u(t)|^2 = \lim_{n\to\infty}|S(t)u_{n,0} - S(t)u_0|^2 = 0 \qquad \forall t \in (0, T)
$$

we have

$$
\lim_{n\to\infty}\int_0^T \|u_n(s) - u(s)\|^2\,\mathrm{d}s = 0. \qquad \square
$$

3. The global attractor in V

First, define $[\![\cdot,\cdot]\!] : D(A) \times D(A) \mapsto R^1$ as

$$[\![u, v]\!] := \nu(Au, Av) - \tfrac{1}{2}\nu\lambda_1((u, v)) \qquad \forall u, v \in D(A).$$

Then

$$[\![u]\!]^2 = [\![u, u]\!] = \nu|Au|^2 - \tfrac{1}{2}\nu\lambda_1\|u\|^2 \geqslant \nu|Au|^2 - \tfrac{1}{2}\nu|Au|^2 = \tfrac{1}{2}\nu|Au|^2.$$

Thus

$$\tfrac{1}{2}\nu|Au|^2 \leqslant [\![u]\!]^2 \leqslant \nu|Au|^2. \tag{3.1}$$

Since

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \tfrac{1}{2}\nu\lambda_1\|u\|^2 + \nu|Au|^2 - \tfrac{1}{2}\nu\lambda_1\|u\|^2 = (f, Au) - b(u, u, Au)$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\lambda_1\|u\|^2 = 2(f, Au) - 2b(u, u, Au) - 2[\![u]\!]^2.$$

By integration, and denoting $S(t)u_0 := u(t)$,

$$\|S(t)u_0\|^2 = \|u_0\|^2 \mathrm{e}^{-\nu\lambda_1 t} + 2\int_0^t \mathrm{e}^{-\nu\lambda_1(t-s)} K(f, S(s)u_0)\,\mathrm{d}s \tag{3.2}$$

where

$$K(f, v) := (f, Av) - b(v, v, Av) - [\![v]\!]^2. \tag{3.3}$$

Now we need to show the asymptotic compactness of the semigroup $\{S(t)\}_{t\geqslant 0}$ in the space V. For the convenience of readers, we give the definition of asymptotic compactness as follows.

Definition 3.1. *The semigroup $\{S(t)\}_{t\geqslant 0}$ is* asymptotically compact *in a given metric space if and only if*

$$\{S(t_n)u_n\} \textit{ is precompact, whenever } \{u_n\} \textit{ is bounded and } t_n \to \infty. \tag{3.4}$$

Let $B \subset V$ be bounded. Consider $\{u_n\}_n \subset B$ and $\{t_n | t_n \geqslant 0, t_n \to \infty \text{ as } n \to \infty\}_n$.

By remark 2.1, there exists $T(B) > 0$ such that

$$S(t)B \subset \mathcal{B} \qquad \forall t \geqslant T(B).$$

By the previous *a priori* estimates, $\mathcal{B}$ can be assumed to be a closed ball, thus a closed and bounded convex set. So, for all $t_n \geqslant T(B)$, $S(t_n)u_n \subset \mathcal{B}$. Since $\{S(t_n)u_n\}_n$ weakly precompact in V, there is $\{u_n\}_{n'}$ such that

$$S(t_{n'} - T)u_{n'} \rightharpoonup w_T \quad \text{weakly in } V \qquad \forall T \in \mathbb{N} \tag{3.5}$$

with $w_T \in \mathcal{B}$.

By section 3 of [R] (see (3.23) of [R]), we have that $S(t_n)u_n \to w$ strongly in H.

By the weak continuity of $S(t)$ in lemma 2.2

$$\begin{aligned} w &= (V_w-)\lim_{n'\to\infty} S(t_{n'})u_{n'} = (V_w-)\lim_{n'\to\infty} S(T)S(t_{n'} - T)u_{n'} \\ &= S(T)(V_w-)\lim_{n'\to\infty} S(t_{n'} - T)u_{n'} = S(T)w_T \end{aligned} \tag{3.6}$$

where V_w–lim is the limit taken in the weak topology of V.

Thus

$$w = S(T)w_T \qquad \forall T \in \mathbb{N} \tag{3.7}$$

and

$$\|w\| \leqslant \liminf_{n'\to\infty} \|S(t_{n'})u_{n'}\|. \tag{3.8}$$

Since V is a Hilbert space, to show the asymptotic compactness, we need only to show that

$$\|w\| \geqslant \liminf_{n'\to\infty} \|S(t_{n'})u_{n'}\|. \tag{3.9}$$

Note that, by (3.2),

$$\begin{aligned}\|S(t_n)u_n\|^2 &= \|S(T)S(t_n - T)u_n\|^2\\ &= \|S(t_n - T)u_n\|^2 e^{-\nu\lambda_1 T} + 2\int_0^T e^{-\nu\lambda_1(T-s)}\{(f, AS(s)S(t_n-T)u_n)\\ &\quad -b(S(s)S(t_n-T)u_n, S(s)S(t_n-T)u_n, AS(s)S(t_n-T)u_n)\\ &\quad -[\![S(s)S(t_n-T)u_n]\!]^2\}\,ds.\end{aligned} \tag{3.10}$$

By (3.5) (and noting that, without loss of generality, we can and will drop the prime for n' thereof), we have

$$S(\cdot)S(t_n - T)u_n \rightharpoonup S(\cdot)w_T \quad \text{weakly in } L^2(0, T; D(A)). \tag{3.11}$$

By (3.11) and noting that $e^{-\nu\lambda_1(T-s)} f \in L^2(0, T; H)$,

$$\lim_{n\to\infty}\int_0^T e^{-\nu\lambda_1(T-s)}(f, AS(s)S(t_n-T)u_n)\,ds = \int_0^T e^{-\nu\lambda_1(T-s)}(f, AS(s)w_T)\,ds. \tag{3.12}$$

Since $[\![\cdot]\!]$ is a norm in $D(A)$, equivalent to $|A\cdot|$ and $0 < e^{-\nu\lambda_1 T} \leqslant e^{-\nu\lambda_1(T-s)} \leqslant 1$, $\forall s \in [0, T]$, we have $(\int_0^T e^{-\nu\lambda_1(T-s)}[\![\cdot]\!]^2\,ds)^{\frac{1}{2}}$ is a norm in $L^2(0, T; D(A))$, equivalent to $|A\cdot|$. Thus

$$\int_0^T e^{-\nu\lambda_1(T-s)}[\![S(s)w_T]\!]^2\,ds \leqslant \liminf_{n\to\infty} e^{-\nu\lambda_1(T-s)}[\![S(s)S(t_n-T)u_n]\!]^2\,ds. \tag{3.13}$$

Now, we show the following:

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T e^{-\nu\lambda_1(T-s)} b(S(s)S(t_n-T)u_n, S(s)S(t_n-T)u_n, AS(s)S(t_n-T)u_n)\,ds\\ &\qquad = \int_0^T e^{-\nu\lambda_1(T-s)} b(S(s)w_T, S(s)w_T, AS(s)w_T)\,ds.\end{aligned} \tag{3.14}$$

Let $u_{0,n} := S(t_n - T)u_n$, $u_0 := w_T$. Then, to show that (3.14) is equivalent, we present the following lemma 3.1.

Lemma 3.1. *Suppose*

$$u_{0,n} \rightharpoonup u_0 \quad \textit{weakly in } V$$
$$u_{0,n} \to u_0 \quad \textit{strongly in } H.$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T e^{-\nu\lambda_1(T-t)} b(S(t)u_{0,n}, S(t)u_{0,n}, AS(t)u_{0,n})\,dt\\ &\qquad = \int_0^T e^{-\nu\lambda_1(T-t)} b(S(t)u_0, S(t)u_0, AS(t)u_0)\,dt \qquad \forall T > 0.\end{aligned}$$

Proof. Let $S(t)u_{0,n} = u_n(t)$, $S(t)u_0 = u(t)$. Then

$$u(s) = S(s)w_T \qquad u_n(s) = S(s)S(t_n - T)u_n.$$

Thus, we need to show that

$$\lim_{n\to\infty} \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} b(u_n(t), u_n(t), Au_n(t))\,\mathrm{d}t = \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} b(u(t), u(t), Au(t))\,\mathrm{d}t. \quad (3.15)$$

We rewrite the difference of the two sides of the above equation and estimate it in terms of the following three parts:

$$\begin{aligned}
&\left| \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} (b(u_n(t), u_n(t), Au_n(t)) - b(u(t), u(t), Au(t)))\,\mathrm{d}t \right| \\
&\qquad \leqslant I_1 + I_2 + I_3 \\
&\qquad := \left| \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} b(u_n - u, u_n, Au_n)\,\mathrm{d}t \right| + \left| \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} b(u, u_n - u, Au_n)\,\mathrm{d}t \right| \\
&\qquad\quad + \left| \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} b(u, u, A(u_n - u))\,\mathrm{d}t \right|. \qquad (3.16)
\end{aligned}$$

Now we want to estimate I_1, I_2 and I_3 one by one.

First,

$$\begin{aligned}
I_1 &\leqslant c \int_0^T |u_n - u|^{\frac{1}{2}} \|u_n - u\|^{\frac{1}{2}} \|u_n\|^{\frac{1}{2}} |Au_n|^{\frac{3}{2}}\,\mathrm{d}t \\
&\leqslant c \left(\int_0^T |u_n - u|^2 \|u_n - u\|^2 \|u_n\|^2\,\mathrm{d}t \right)^{\frac{1}{4}} \left(\int_0^T |Au_n|^2\,\mathrm{d}t \right)^{\frac{3}{4}} \\
&\leqslant c \left(\int_0^T |u_n - u|^2\,\mathrm{d}t \right)^{\frac{1}{4}} \to 0 \qquad \text{as} \quad n \to \infty \qquad (3.17)
\end{aligned}$$

since $u_n \to u$ strongly in H.

Second,

$$\begin{aligned}
I_2 &\leqslant c \int_0^T |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|u_n - u\|^{\frac{1}{2}} |Au_n - Au|^{\frac{1}{2}} |Au_n|\,\mathrm{d}t \\
&\leqslant c \left(\int_0^T |u| \|u\| \|u_n - u\| |Au_n - Au|\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^T |Au_n|^2\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leqslant c \left(\int_0^T |u|^2 \|u\|^2 \|u_n - u\|^2\,\mathrm{d}t \right)^{\frac{1}{4}} \left(\int_0^T |Au_n - Au|^2\,\mathrm{d}t \right)^{\frac{1}{4}} \left(\int_0^T |Au_n|^2\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leqslant c \left(\int_0^T \|u_n - u\|^2\,\mathrm{d}t \right)^{\frac{1}{4}} \to 0 \qquad \text{as} \quad n \to \infty \qquad (3.18)
\end{aligned}$$

where lemma 2.3 is used.

Finally, we estimate I_3. By (2.10),

$$\int_0^T |B(u)|^2\,\mathrm{d}t \leqslant c \int_0^T |u| \|u\|^2 |Au|\,\mathrm{d}t < \infty.$$

Thus, $b(u) \in L^2(0, T; H)$.

While

$$I_3 \leqslant \int_0^T |\langle B(u), A(u_n - u)\rangle| \,\mathrm{d}t.$$

Thus, by lemma 2.2,

$$\lim_{n\to\infty} I_3 \leqslant \lim_{n\to\infty} \int_0^T \mathrm{e}^{-\nu\lambda_1(T-t)} |\langle B(u), A(u_n - u)\rangle| \,\mathrm{d}t = 0. \tag{3.19}$$

Now (3.15) and thus (3.14) are proved by (3.17)–(3.19) and (3.16). □

Now we return to the proof of the asymptotic compactness of $\{S(t)\}_{t\geqslant 0}$ in V.

By (3.10), (3.12)–(3.14), we have

$$\begin{aligned}\limsup_{n\to\infty} \|S(t_n)u_n\|^2 \leqslant (\operatorname{diam}_V(\mathcal{B}))^2 \mathrm{e}^{-\nu\lambda_1 T} &+ 2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)} \langle f, AS(s)w_T\rangle \,\mathrm{d}s \\ &-2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)} [\![S(s)w_T]\!]^2 \,\mathrm{d}s \\ &-2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)} b(S(s)w_T, S(s)w_T, AS(s)w_T) \,\mathrm{d}s. \end{aligned} \tag{3.20}$$

By (3.6) and (3.2)

$$\begin{aligned}\|w\|^2 = \|S(T)w_T\|^2 = \|w_T\|^2 \mathrm{e}^{-\nu\lambda_1 T} + 2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)} \\ \times\left\{(f, AS(s)w_T) - b(S(s)w_T, S(s)w_T, AS(s)w_T) - [\![S(S)w_T]\!]^2\right\} \mathrm{d}s.\end{aligned} \tag{3.21}$$

Thus, from (3.20) and (3.21),

$$\begin{aligned}\limsup_{n\to\infty} \|S(t_n)u_n\|^2 &\leqslant \|w\|^2 + [(\operatorname{diam}_V(\mathcal{B}))^2 - \|w_T\|^2]\mathrm{e}^{-\nu\lambda_1 T} \\ &\leqslant \|w\|^2 + (\operatorname{diam}_V(\mathcal{B}))^2 \mathrm{e}^{-\nu\lambda_1 T}.\end{aligned} \tag{3.22}$$

Let $T \to \infty$ in (3.22). Equation (3.9) is proven. Thus, $S(t_{n'})u_{n'} \to w$, strongly in V. The asymptotic compactness in V is proven and thus $\mathcal{A}$ is compact in V. We now have

Theorem 3.1. *Let Ω be an open set satisfying (2.5) and (2.12). $\nu > 0$, $f \in H$. Then, the semigroup $\{S(t)\}_{t\geqslant 0}$ associated with (2.14) possesses a global attractor $\mathcal{A} \subset V$. More precisely, $\mathcal{A}$ is compact and invariant in the space V, which attracts all bounded sets in H with respect to the metric of V. Moreover, $\mathcal{A}$ is connected in V and maximal.*

Proof. We have already shown the asymptotic compactness of the semigroup $\{S(t)\}_{t\geqslant 0}$ in the space V. Note that remark 2.1 gives $\mathcal{B} \subset V$, an absorbing ball of $\{S(t)\}_{t\geqslant 0}$ in the metric of the space V, while lemma 2.1 shows that $S(t) : V \mapsto V$ is a Lipschitz continuous map (operator) on V for $t \geqslant 0$. Now, by a direct application of the general result of the global attractor theory under the condition of asymptotic compactness (see, for example, [T2] (second edition) section 1.1 or [L2]), to the semigroup $\{S(t)\}_{t\geqslant 0}$ in the space V, we obtain immediately a global attractor $\mathcal{A}$ which is compact in the space V and attracts all the bounded subsets in the space V with respect to the metric of V. Moreover, $\mathcal{A}$ is connected with respect to the metric of V and is also maximal.

However, it is known that (see [L3])

$$t\|S(t)u_0\|^2 \leqslant \exp\{c(1 + |u_0|^4 + t^2\|f\|_{V'}^4)\}. \tag{3.23}$$

This means that the global attractor $\mathcal{A}$ also attracts all the bounded subsets in the space H with respect to the metric of the space V. □

Remark 3.1. By theorem 3.1, we see that the attractor $\mathcal{A}$ obtained above indeed coincides with the attractor obtained in [R] (see remark 3.1 therein), where it is shown that $\mathcal{A}$ is compact in the space H and is bounded in the space V, with only the condition (2.5) imposed for the domain. There, the attractor $\mathcal{A}$ is proved to attract all the bounded subsets of H with respect to the metric of H. The above result shows that, under the additional condition (2.12), $\mathcal{A}$ is not just bounded, but indeed is also compact in the space V. Moreover, it attracts all the bounded subsets of H with respect to the metric of V.

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References

[A1] Abergel F 1989 Attractor for a Navier–Stokes flow in an unbounded domain *Math. Mod. Num. Anal.* **23** 359–70

[A2] Abergel F 1990 Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains *J. Diff. Equ.* **83** 85–108

[Bb] Babin A V 1992 The attractor of a Navier–Stokes system in an unbounded channel-like domain *J. Dyn. Differ. Equ.* **4** 555–84

[BbV] Babin A V and Vishik M I 1990 Attractors of partial differential equations in an unbounded domain *Proc. R. Soc. Edin.* A **116** 221–43

[Bl] Ball J 2000 A Proof of the existence of global attractors for damped semilinear wave equations, to appear (cited in [G])

[CF1] Constantin P and Foias C 1988 *Navier–Stokes Equations* (Chicago, IL: University of Chicago Press)

[CF2] Constantin P and Foias C 1985 Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractor for the 2D Navier–Stokes equations *Commun. Pure Appl. Math.* **38** 1–27

[CFMT] Constantin P, Foias C, Manley O and Temam R 1988 Determining modes and fractal dimension of turbulent flows *J. Fluid Mech.* **150** 427–40

[FLST] Feireisl E, Laurençot P, Simondon F and Touré H 1994 Compact attractors for reaction–diffusion equations in R^N *C. R. Acad. Sci. Paris I* **319** 253–8

[FT] Foias C and Temam R 1979 Some analytic and geometric properties of the solutions of the evolution Navier–Stokes equations *J. Math. Pures Appl.* **58** 334–68

[G] Ghidaglia J M 1994 A note on the strong convergence towards the attractors for damped forced KdV equations *J. Differ. Equ.* **110** 356–9

[Hal] Hale J 1985 *Asymptotic Behaviors and Dynamics in Infinite Dimensions (Research Notes in Mathematics vol 132)* ed J Hale and P Martínnez-Amores (New York: Pitman) pp 1–420

[Har] Haraux A 1985 Two remarks on hyperbolic dissipative problems *Nonlinear Partial Differential Equations and Their Applications (College de France Seminar vol VII)* ed H Brezis and J L Lions (New York: Pitman) pp 161–79

[H] Heywood J G 1980 The Navier–Stokes equations: on the existence, regularity and decay of solutions *Ind. Univ. Math. J.* **29** 639–81

[L1] Ladyzhenskaya O 1972 On the dynamical system generated by the Navier–Stokes equations *Zapiskii of nauchnish seminarovs LOMI* **27** 91–114 (Engl. transl. 1975 *J. Sov. Math.* **3** 458–79)

[L2] Ladyzhenskaya O 1991 *Attractors for Semigroups and Evolution Equations (Lezioni Lincei)* (Cambridge: Cambridge University Press)

[L3] Ladyzhenskaya O 1992 First boundary value problem for the Navier–Stokes equations with nonsmooth boundaries *C. R. Acad. Sci. Paris I* **314** 253–8

[MRW] Moise I, Rosa R and Wang X 1998 Attractors for non-compact semigroups via energy equations *Nonlinearity* **11** 1369–93

[R] Rosa R 1998 The global attractor for the 2D Navier–Stokes flow on some unbounded domains *Nonlinear Anal.* **32** 71–85

[T1] Temam R 1977 *Navier–Stokes Equations Theory and Numerical Analysis* (Amsterdam: North-Holland) (revised edn 1984)

[T2] Temam R 1988 *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (Berlin: Springer) (revised edn 1995)

[W] Wang X 1995 An energy equation for the weakly damped driven nonlinear Schrödinger equations and its applications to their attractors *Physica* D **88** 167–75