Estimates of asymptotic degrees of freedom for solutions to the Navier–Stokes equations

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Abstract. We derive estimates on the upper bounds of the asymptotic degrees of freedom for strong solutions of the two-dimensional Navier–Stokes equations with no-slip and non-homogeneous boundary conditions. The analysis is also extended to an abstract equation modelling some other equations in fluid mechanics. Similar results are obtained.

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1. Introduction

Physical arguments [19, 21] indicate that the asymptotic behaviour of fluid motions can be determined by a finite number of degrees of freedom. This was confirmed mathematically in the case of the two-dimensional (2D) Navier–Stokes equations (NSE) with periodic boundary conditions in [9], where it was shown that the first N Fourier modes determine the asymptotic behaviour of the solution of the NSE completely, for N sufficiently large. An explicit estimate of N , in terms of the Reynolds number, was first given in [8] and later improved in [18]. See also [2] for the estimates for the dimension of the global attractor. Besides Fourier modes, the existence of a large collection of determining sets was asserted in [10]. In particular, the values of the solutions at nodes in the spatial domain [11] and local averages of the solutions over finite volumes [12, 16] were also found to be asymptotically determining. Upper bounds of the asymptotic degrees of freedom were given in [17, 18]. In [6, 7], the discussion of these *determining functionals* was extended to a large collection of *projections* related to finite-element approximations. Similar discussion was made in [15] for the weak solutions to the two- and three-dimensional NSEs.

Definition 1.1. Let $\{\mathcal{L}_i\}_{i=1}^N$ be a set of bounded linear functionals on the space of solutions for an evolutionary partial differential equation (PDE) for $t \in [0, \infty)$. Suppose that u and v are, respectively, the solutions of the PDE with the force term being f and g , respectively. Suppose that if

$$\lim_{t \rightarrow \infty} \mathcal{L}_i(u(t) - v(t)) = 0 \quad \forall 1 \leq i \leq N \quad \text{and} \quad \lim_{t \rightarrow \infty} \|f(t) - g(t)\|_Y = 0$$

then

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_X = 0$$

where X and Y are Banach spaces. Then $\{\mathcal{L}_i\}_{i=1}^N$ is a set of (asymptotically) determining functionals in X , or asymptotically determining in X , when f, g are asymptotically equal in Y , and N gives an upper bound for the asymptotic degrees of freedom in X of the PDE.

Let X_N be a finite-dimensional subspace of X , $\{\phi_i\}_{i=1}^N$ a basis of X_N and $\{l_i\}_{i=1}^N$ a set of bounded linear functionals from X' , the dual space of X , to X . Let

$$P_N v := \sum_{i=1}^N l_i(v) \phi_i.$$

The asymptotic determination of $\{l_i\}_{i=1}^N$ is equivalent to that of P_N . Such projections are frequently encountered in the literature on finite-element methods [1]. For the spatial domain $\Omega \subset \mathbb{R}^d$ and a *regular triangulation* of Ω with the mesh size being h , one has that

$$C_0 |\Omega| h^{-d} \leq N \leq C'_0 |\Omega| h^{-d} \quad (1.1)$$

where h is the maximum of the distance of the diameters of the simplices and C_0, C'_0 are universal constants independent of h, N . In general, one can define a projection operator $R_h : X \mapsto X_N \subset X$, $N := \dim(X_N) < \infty$, such that for $u \in H^1(\Omega) \cap X$,

$$\|u - R_h u\|_{L^2(\Omega)} \leq C_1 h \|u\|_{H^1(\Omega)} \quad (1.2)$$

and for $u \in H^2(\Omega) \cap X$,

$$\|u - R_h u\|_{L^2(\Omega)} + h \|u - R_h u\|_{H^1(\Omega)} \leq C_1 h^2 \|u\|_{H^2(\Omega)} \quad (1.3)$$

where C_1 is a constant independent of h and u . Such projections are used in [6, 7, 15].

For the Navier–Stokes equations, asymptotic degrees of freedom are usually estimated in terms of the *generalized Grashof number*:

$$Gr := \frac{\rho^2 F}{\nu^2} = \frac{F}{\lambda_1 \nu^2} \quad (1.4)$$

where λ_1 is the smallest eigenvalues of the Stokes operator and $\rho = \sqrt{\lambda_1}$ is the related (best) Poincaré constant. Here,

$$F := \limsup_{t \rightarrow \infty} \left(\int_{\Omega} |f(x, t)|^2 dx \right)^{1/2} \quad \text{if } f \in H(\Omega) \text{ for almost all } t \geq 0$$

or

$$F := \limsup_{t \rightarrow \infty} \sqrt{\lambda_1} \|f\|_{H^{-1}(\Omega)} \quad \text{if } f \in V'(\Omega) \text{ for almost all } t \geq 0$$

where H, V and V' are defined in section 2 for the no-slip boundary condition case.

The best known estimate for the determining set size for the 2D NSE with *periodic* boundary conditions and H^2 -regular solutions is of order Gr in V for the upper bound [18]. For the case with *no-slip* boundary conditions, the best known upper bound is of order $(Gr)^2$ in H for H^2 -regular solutions [15, 26]. It seems that this result still holds in V by following the ideas of [7] with a non-trivial analysis provided that for $T \geq (\nu \lambda_1)^{-1}$,

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |Au(\tau)|^2 d\tau \leq C(Gr)^2. \quad (1.5)$$

However, with no-slip boundary conditions, this is not clear.

An interesting question is whether or not the estimate of $(Gr)^2$ which holds in H still holds in V even if (1.5) may not hold. A positive answer is given here with a different analysis

from [7], which estimates the degrees of freedom *directly* in V -space. Instead, we make use of the estimation of [15] in H and prove that the asymptotic determining functionals in H are also determining in V . This idea enables our analysis to extend to the more general non-homogeneous boundary case, which would otherwise be difficult. It also allows us to treat some more complicated systems easily such as the magnetohydrodynamics (MHD) equations (see section 4).

The rest of the paper is organized as follows. In section 2, we recall some notation and preliminary results concerning the NSE with no-slip boundary conditions which will be used in the discussion in section 3. In section 3, we first recall some notation and preliminary results concerning the 2D NSE with non-homogeneous boundary conditions. Then, we give a proof of the estimates of the asymptotic degrees of freedom of the strong solutions. In section 4, we extend the analysis to a general abstract equation modelling some other equations in fluid mechanics, in particular, the magnetohydrodynamics equations.

2. Notation and preliminary results

We recall some preliminary results from [3, 20, 22, 24, 25].

Suppose that $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is open and bounded. Its boundary $\Gamma = \partial\Omega$ is the Lipschitz boundary. Consider the following non-stationary Navier–Stokes equations describing the flow of a viscous incompressible fluid confined in Ω :

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^d u_i \frac{\partial u}{\partial x_i} + \nabla p = f \quad \text{in } \Omega \quad t > 0 \quad (2.1)$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \quad t > 0 \quad (2.2)$$

which are supplemented with the no-slip boundary conditions

$$u|_{\partial\Omega} = 0 \quad (2.3)$$

and the initial condition

$$u|_{t=0} = u_0. \quad (2.4)$$

Here $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ is the velocity, p is the pressure, f represents the density of the body forces and $\nu > 0$ is the kinematic viscosity. When no ambiguity occurs, we use $u(t)$ or simply u to denote the vector function $u(x, t) : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^d$, where $\mathbb{R}_+ = [0, \infty)$. A similar treatment applies to other vector functions and scalar functions.

Define

$$H := \{v | v \in L^2(\Omega)^d, \nabla \cdot v = 0 \text{ in } \Omega, v \cdot \vec{\nu} = 0 \text{ on } \partial\Omega\}$$

$$V := \{v | v \in H_0^1(\Omega)^d, \nabla \cdot v = 0 \text{ in } \Omega\}$$

where $\vec{\nu}$ is the tangential vector of $\partial\Omega$. $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$ are the standard Sobolev vector space. H is a Hilbert space with the induced norm $\|u\| := (u, u)^{1/2}$ and the inner product

$$(u, v) := \int_{\Omega} u(x) \cdot v(x) \, dx = \sum_{i=1}^d \int_{\Omega} u_i(x) v_i(x) \, dx.$$

V is a Hilbert space with the induced norm $\|u\| := ((u, u))^{1/2}$ and the inner product

$$((u, v)) := \sum_{i,j=1}^d \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, dx.$$

Denote H' , V' as the dual spaces of H , V . We can identify H with H' , a subspace of V' .

The well known Stokes operator $A : D(A) = V \cap H^2(\Omega)^d \rightarrow V$ is defined as

$$\langle Au, v \rangle_{V', V} := ((u, v)) \quad \forall u, v \in V.$$

On $D(A)$, $|Au|$ is a norm equivalent to that induced by $H^2(\Omega)^d$.

Define the trilinear form

$$b(u, v, w) := \int_{\Omega} \sum_{i,j=1}^d u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall u, v, w \in V$$

with the associated bilinear operator from $V \times V$ into V' defined as

$$\langle B(u, v), w \rangle_{V', V} := b(u, v, w) \quad \forall u, v, w \in V.$$

With the above notation, problems (2.1)–(2.4) can be reformulated as follows.

Find $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ such that

$$\frac{du}{dt} + \nu Au + B(u, u) = f \quad (2.5)$$

with the initial condition (2.4).

For $d = 2$, we have the following estimates:

$$|b(u, v, w)| \leq c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \quad \forall u \in V \quad v \in D(A) \quad w \in H \quad (2.6)$$

$$|b(u, v, w)| \leq c|u|^{1/2}|Au|^{1/2}\|v\|\|w\| \quad \forall u \in D(A) \quad v \in V \quad w \in H \quad (2.7)$$

$$|b(u, v, w)| \leq c|u|^{1/2}\|u\|^{1/2}\|v\|\|w\|^{1/2}\|w\|^{1/2} \quad \forall u, v, w \in V \quad (2.8)$$

where c is a positive constant independent of u, v and w .

In this paper, we use c , c' and c'' to denote some generic positive constants, which may, on occasion, vary.

3. 2D NSE with non-homogeneous boundary condition

Consider the non-homogeneous Navier–Stokes equations on $\Omega \subset R^2$,

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad (3.1)$$

$$\nabla \cdot u = 0 \quad (3.2)$$

$$u = \varphi \quad \text{on} \quad \partial\Omega. \quad (3.3)$$

Suppose that φ is independent of t and is the trace of ϕ on $\partial\Omega$ such that

$$\phi \in H^2(\Omega)^2 \quad \nabla \cdot \phi = 0 \quad \int_{\partial\Omega} \phi \cdot n = 0 \quad (3.4)$$

where n is the unit outer normal of $\partial\Omega$. Let $u = v + \phi$. Then, it is easy to see that

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + (\phi \cdot \nabla)v + (v \cdot \nabla)\phi + \nabla p = f + \nu \Delta \phi - (\phi \cdot \nabla)\phi \quad (3.5)$$

$$\nabla \cdot v = 0 \quad (3.6)$$

$$v = 0 \quad \text{on} \quad \partial\Omega \quad (3.7)$$

which can be written in functional form as

$$\frac{dv}{dt} + \nu Av + B(v, v) + Ev = \bar{f} \quad (3.8)$$

where

$$\begin{aligned} \bar{f} &:= f + \nu A\phi - B(\phi, \phi) \\ \langle Av, w \rangle &:= ((v, w)) \\ \langle B(u, v), w \rangle &:= ((u \cdot \nabla)v, w) = b(u, v, w) \\ \langle Ev, w \rangle &:= ((v \cdot \nabla)\phi, w) + ((\phi \cdot \nabla)v, w) = b(v, \phi, w) + b(\phi, v, w). \end{aligned}$$

First, we recall some useful results about the function ϕ .

Lemma 3.1 (see [23, 25]). *Suppose that $\Omega \subset \mathbb{R}^2$ is a smooth (at least $\mathcal{C}^3$) bounded domain. Let $\varphi \in \mathcal{C}^3(\partial\Omega)^2$ be given such that $\varphi \cdot n = 0$, on $\partial\Omega$.*

Then for every $\varepsilon > 0$, there exists $\phi = \phi_\varepsilon \in H^2(\Omega)^2$, such that

- (a) $\nabla \cdot \phi = 0$,
- (b) $\phi = \varphi$ on $\partial\Omega$,
- (c) $|b(v, \phi, v)| = |\int_\Omega (v \cdot \nabla)\phi \cdot v \, dx| \leq c'\varepsilon|\varphi|_{L^\infty}|\Omega|^{1/2}\|v\|^2, \forall v \in V$.
- (d) $|\phi| \leq \|\phi\|_1 \leq (c''/\varepsilon^{1/2})|\varphi|_{L^\infty(\partial\Omega)^2}$.

Define

$$U = \max\{|\varphi|_{L^\infty}, |\nabla\varphi|_{L^\infty} \cdot \text{diam}(\Omega)\} \quad L = |\Omega|^{1/2} \quad (3.9)$$

$$Re = \frac{UL}{\nu} \quad (\text{Reynolds number}) \quad (3.10)$$

$$Gr = \frac{LF}{\lambda_1^{1/2}\nu^2} \quad (\text{generalized Grashof number}). \quad (3.11)$$

Using lemma 3.1, one can prove the following existence and regularity results.

Theorem 3.1 (see [25]). *For $\bar{f}$ and v_0 given in H , there exists a unique solution v of (3.8) with $v(x, 0) = v_0(x)$ such that*

$$v \in \mathcal{C}([0, T]; H) \bigcap L^2(0, T; V) \quad \forall T > 0.$$

If $v_0 \in V$, then

$$v \in \mathcal{C}([0, T]; V) \bigcap L^2(0, T; D(A)) \quad \forall T > 0.$$

Moreover, the dynamical system associated with (3.8) possesses an attractor $\mathcal{A}$ which is compact, connected and maximal in H . Furthermore, $\mathcal{A}$ attracts the bounded sets of H and is also maximal among the functional invariants sets bounded in H .

Under the conditions of lemma 3.1, we have the following *a priori* estimates.

Lemma 3.2. *If $v \in L^2(R_+, V)$ is a weak solution of (3.8), then for $T \geq \nu^{-1} > 0$,*

$$\limsup_{t \rightarrow \infty} |v(t)|^2 \leq \frac{c}{\nu^2 \lambda_1} \limsup_{t \rightarrow \infty} |f(t)|_{V'}^2 + \frac{c}{\lambda_1} \|\phi_v\|^2 + \frac{c}{\nu^2 \lambda_1} |\varphi|_{L^\infty}^2 |\phi_v|^2 \quad (3.12)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|v(\tau)\|^2 \, d\tau \leq \frac{c}{\nu^2} \limsup_{t \rightarrow \infty} |f(t)|_{V'}^2 + c' \|\phi_v\|^2 + \frac{c}{\nu^2} |\varphi|_{L^\infty}^2 |\phi_v|^2 \quad (3.13)$$

where ϕ_v is a function to be defined in the proof.

If $u \in L^2(R_+, D(A))$ is a strong solution of (3.8), then for $T \geq v^{-1} > 0$

$$\limsup_{t \rightarrow \infty} \|u(t)\| \leq C(Gr, Re) < \infty \quad (3.14)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |Au(\tau)|^2 \leq C(Gr) < \infty \quad (3.15)$$

where $C(Gr, Re)$ is a function of Gr, Re and v .

Proof. By (3.8), we have that

$$(v', v) + v\|v\|^2 + b(v, v, v) + b(\phi, v, v) + b(v, \phi, v) = (f, v) - v((\phi, v)) - b(\phi, \phi, v).$$

Note that

$$b(v, v, v) = 0 \quad b(\phi, v, v) = 0.$$

Thus,

$$\frac{1}{2} \frac{d}{dt} |v|^2 + v\|v\|^2 + b(v, \phi, v) = (f, v) - v((\phi, v)) - b(\phi, \phi, v). \quad (3.16)$$

Note that, by lemma 3.1,

$$v\langle Av, v \rangle + \langle Ev, v \rangle = v\|v\|^2 + b(v, \phi, v) \geq v\|v\|^2 - c|\psi|_{L^\infty} |\Omega|^{1/2} \varepsilon \|v\|^2.$$

So, for $\varepsilon \leq v/2cUL$,

$$v\langle Av, v \rangle + \langle Ev, v \rangle \geq v\|v\|^2 - cUL\varepsilon\|v\|^2 \geq \frac{1}{2}v\|v\|^2. \quad (3.17)$$

Let ϕ_v be the function ϕ_ε in lemma 3.1 corresponding to $\varepsilon = v/2cUL$. It can be shown that [23, 25],

$$|b(\phi_v, \phi_v, v)| = \left| \int_{\Omega} (\phi_v \cdot \nabla) \phi_v \cdot v \, dx \right| \leq c|\varphi|_{L^\infty} |\phi_v| \|v\|. \quad (3.18)$$

Under the conditions of lemma 3.1, using (3.16)–(3.18), we have

$$\frac{d}{dt} |v|^2 + \frac{1}{2}v\|v\|^2 \leq \frac{c}{v} |f|_{V'} + c'v\|\phi_v\|^2 + \frac{c''}{v} |\varphi|_{L^\infty}^2 |\phi_v|^2. \quad (3.19)$$

Since

$$\frac{d}{dt} |v|^2 + \frac{1}{2}v\lambda_1 |v|^2 \leq \frac{c}{v} |f|_{V'} + c'v\|\phi_v\|^2 + \frac{c''}{v} |\varphi|_{L^\infty}^2 |\phi_v|^2$$

we have

$$|v(t)|^2 \leq e^{-(v\lambda_1/2)t} |v_0|^2 + \frac{2}{v\lambda_1} (1 - e^{-(v\lambda_1/2)t}) \left(\sup_{t \geq 0} \frac{c}{v} |f(t)|_{V'} + c'v\|\phi_v\|^2 + \frac{c''}{v} |\varphi|_{L^\infty}^2 |\phi_v|^2 \right).$$

Thus, $|v(t)|$ is uniformly bounded for $t \in R_+$. Also,

$$\begin{aligned} |v(t)|^2 &\leq |v(s)| e^{-(v\lambda_1/2)(t-s)} \\ &\quad + \frac{2}{v\lambda_1} (1 - e^{-(v\lambda_1/2)(t-s)}) \left(\sup_{s \leq \tau} \frac{c}{v} |f(\tau)|_{V'}^2 + c'v\|\phi_v\|^2 + \frac{c''}{v} |\varphi|_{L^\infty}^2 |\phi_v|^2 \right). \end{aligned}$$

Thus, equation (3.12) holds. Integrating (3.19), we have

$$\frac{1}{T} \int_t^{t+T} \|v(\tau)\|^2 d\tau \leq \frac{2}{vT} |v(t)|^2 + \frac{c}{v^2} \frac{1}{T} \int_t^{t+T} |f(\tau)|_{V'}^2 d\tau + c' \|\phi_v\|^2 + \frac{c}{v^2} |\varphi|_{L^\infty}^2 |\phi_v|^2.$$

Thus, by (3.12), we see (3.13) holds for $T > v^{-1}$.

It can be shown similarly that, if $v \in L^2(R_+, D(A))$ is a strong solution of (3.8), then for $T \geq v^{-1} > 0$, equations (3.14) and (3.15) hold. $\square$

Note that the $C(Gr, Re)$ obtained above will usually not be good for our direct use to estimate the upper bounds of the asymptotic degrees of freedom of the solutions, as larger estimates will be obtained. Note also that for less smooth $\partial\Omega$ and φ , we can obtain similar results on regularity and *a priori* estimates. We omit the details of the discussion here. See [25] for a relevant discussion. Finally, we recall a useful generalized Gronwall lemma, a preliminary version of which was first given in [8].

Lemma 3.3 (see [16]). *Let $T > 0$ be fixed, $\alpha(t)$, $\beta(t)$ be real-valued locally integrable on $(0, \infty)$ satisfying*

$$\liminf_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau = m > 0 \quad \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau) d\tau = M < \infty$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0$$

where $\alpha^- = \max\{-\alpha, 0\}$, $\beta^+ = \max\{\beta, 0\}$. $y(t)$ is non-negative and absolutely continuous on $(0, \infty)$ satisfying $y'(t) + \alpha y(t) \leq \beta(t)$, a.e. on $(0, \infty)$. Then

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

Now we come to the point of estimating the upper bound of the asymptotic degree of freedom of the solutions of (3.1)–(3.3), which is equivalent to that of the solutions of (3.5)–(3.7). We state and prove our main result.

Theorem 3.2. *Suppose $d = 2$. Under the condition lemma 3.2, the upper bounds for the asymptotic degree of freedom for strong solutions to the NSE (3.8) in space V can be estimated as follows:*

$$N \leq C((Gr)^2 + (Re)^2 + (Re)^4).$$

Proof. Suppose that v is the solution of (3.8) and w is the solution of (3.8) with f replaced with g . Then $\forall \psi \in V$,

$$\begin{aligned} \left(\frac{\partial v}{\partial t}, \psi \right) + v((v, \psi)) + b(v, v, \psi) + b(\phi, v, \psi) + b(v, \phi, \psi) \\ = (f, \psi) - v((\phi, \psi)) - b(\phi, \phi, \psi) \end{aligned} \quad (3.20)$$

$$\begin{aligned} \left(\frac{\partial w}{\partial t}, \psi \right) + v((w, \psi)) + b(w, w, \psi) + b(\phi, w, \psi) + b(w, \phi, \psi) \\ = (g, \psi) - v((\phi, \psi)) - b(\phi, \phi, \psi). \end{aligned} \quad (3.21)$$

Let $e = v - w$, and set $\psi = e$. We have, by subtracting (3.21) from (3.20),

$$\left(\frac{\partial e}{\partial t}, e \right) + v\|e\|^2 + b(e, v, e) + b(e, \phi, e) = (f - g, e)$$

which can also be rewritten as

$$(e', e) + v(Ae, e) + (Ee, e) + b(e, v, e) = (f - g, e). \quad (3.22)$$

By (3.17), $\forall \varepsilon \leq v/2cUL$, we can choose $\phi = \phi_\varepsilon$, such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |e|^2 + \frac{v}{2} \|e\|^2 &= (f - g, e) - b(e, v, e) \leq |(f - g, e)| + |b(e, v, e)| \\ &\leq \|f(t) - g(t)\|_{V'} \|e\| + \|v\| |e| \|e\| \\ &\leq \frac{2}{v} \|f(t) - g(t)\|_{V'}^2 + \frac{2}{v} \|v\|^2 |e|^2 + \frac{v}{4} \|e\|^2. \end{aligned}$$

Thus,

$$\frac{d}{dt} |e|^2 + \frac{v}{2} \|e\|^2 \leq \frac{4}{v} \|f(t) - g(t)\|_{V'}^2 + \frac{4}{v} \|v\|^2 |e|^2.$$

By (1.2), we have

$$|e| \leq |R_h e| + |e - R_h e| \leq |R_h e| + Ch \|e\|.$$

So,

$$\frac{d}{dt} |e|^2 + \frac{v}{2} \|e\|^2 \leq \frac{4}{v} \|f(t) - g(t)\|_{V'}^2 + \frac{8}{v} \|v\|^2 |R_h e|_0^2 + \frac{ch^2}{v} \|v\|^2 \|e\|^2.$$

Thus,

$$\frac{d}{dt} |e|^2 + \lambda_1 \left(\frac{v}{2} - \frac{ch^2}{v} \|v\|^2 \right) \|e\|^2 \leq \frac{4}{v} \|f(t) - g(t)\|_{V'}^2 + \frac{8}{v} \|v\|^2 |R_h e|^2. \quad (3.23)$$

Let

$$\begin{aligned} \alpha(t) &:= \lambda_1 \left(\frac{v}{2} - \frac{ch^2}{v} \|v(t)\|^2 \right) \\ \beta(t) &:= \frac{4}{v} \|f(t) - g(t)\|_{V'}^2 + \frac{8}{v} \|v(t)\|^2 |R_h e(t)|^2. \end{aligned}$$

Now we can apply lemma 3.3 to show that

$$\limsup_{t \rightarrow \infty} |e(t)| = 0 \quad (3.24)$$

provided that

$$h^{-2} \geq \frac{c}{v^2} \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|v(\tau)\|^2 d\tau.$$

By the *a priori* estimate (3.13), the above condition holds provided that

$$N \sim h^{-2} \geq \frac{c}{v^4} \limsup_{t \rightarrow \infty} |f(t)|_{V'}^2 + \frac{c}{v^2} \|\phi_v\|^2 + \frac{c}{v^4} |\varphi|_{L^\infty}^2 |\phi_v|^2. \quad (3.25)$$

Thus, one can choose N and h be such that

$$N \sim h^{-2} \geq c((Gr)^2 + (Re)^2 + (Re)^4). \quad (3.26)$$

Now, the only thing left to show is that

$$\limsup_{t \rightarrow \infty} \|e(t)\| = 0. \quad (3.27)$$

Subtracting (3.21) from (3.20), and setting $\psi = Ae$, we have

$$\frac{1}{2} \frac{d}{dt} \|e\|^2 + \nu |Ae|^2 = (f - g, Ae) - b(v, e, Ae) - b(e, w, Ae) - b(\phi, e, Ae) - b(e, \phi, Ae). \quad (3.28)$$

By (2.7),

$$\begin{aligned} |b(u, e, Ae)| &\leq c|u|^{1/2}|Au|^{1/2}\|e\||Ae| \leq c|u|^{1/2}|Au|^{1/2}|e|^{1/2}|Ae|^{3/2} \\ &\leq \frac{\nu}{8}|Ae|^2 + \frac{C}{\nu^3}|u|^2|Au|^2|e|^2. \end{aligned} \quad (3.29)$$

By (2.6),

$$\begin{aligned} |b(e, w, Ae)| &\leq c|e|^{1/2}\|e\|^{1/2}\|w\|^{1/2}|Aw|^{1/2}|Ae| \leq c\|w\|^{1/2}|Aw|^{1/2}|e|^{1/2}|Ae|^{3/2} \\ &\leq \frac{\nu}{8}|Ae|^2 + \frac{C}{\nu^3}|w|^2|Aw|^2|e|^2. \end{aligned} \quad (3.30)$$

It can also be shown that [25],

$$|b(\phi, e, Ae)| \leq c\|\phi\|_{L^\infty}\|e\||Ae| \quad (3.31)$$

$$|b(e, \phi, Ae)| \leq c\|\nabla\phi\|_{L^\infty}|e||Ae|. \quad (3.32)$$

Thus, by (3.31),

$$|b(\phi, e, Ae)| \leq c\|\phi\|_{L^\infty}|e|^{1/2}|Ae|^{3/2} \leq \frac{c}{\nu^3}\|\phi\|_{L^\infty}^4|e|^2 + \frac{\nu}{8}|Ae|^2 \quad (3.33)$$

and by (3.32),

$$|b(e, \phi, Ae)| \leq \frac{c}{\nu}\|\nabla\phi\|_{L^\infty}^2|e|^2 + \frac{1}{8}\nu|Ae|^2. \quad (3.34)$$

By (3.28)–(3.30), (3.33) and (3.34), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e\|^2 + \nu |Ae|^2 &\leq 2\nu|f - g|_0^2 + \frac{3}{4}\nu|Ae|^2 + \frac{C}{\nu^3}(|v|^2 + |w|^2)|Av|^2|e|^2 \\ &\quad + \frac{c}{\nu^3}\|\phi\|_{L^\infty}^4|e|^2 + \frac{c}{\nu}\|\nabla\phi\|_{L^\infty}^2|e|^2. \end{aligned}$$

Thus,

$$\frac{d}{dt} \|e(t)\|^2 + \alpha(t)\|e(t)\|^2 \leq \beta(t) \quad (3.35)$$

where

$$\alpha(t) := \frac{1}{2}\lambda_1\nu$$

$$\beta(t) := 4\nu|f - g|_0^2 + \left(\frac{C}{\nu^3}|v|^2|Av|^2 + \frac{C}{\nu^3}|w|^2|Aw|^2 + \frac{c}{\nu^3}\|\phi\|_{L^\infty}^4 + \frac{c}{\nu}\|\nabla\phi\|_{L^\infty}^2 \right) |e|^2.$$

By (3.35), (3.24), (3.15) and lemma 3.3, equation (3.27) is proved. $\square$

Note that the estimate of N in theorem 3.1 is not directly available by following the analysis of [7], for the reason indicated after the proof of lemma 3.3.

4. An abstract equation in fluid mechanics

In this section, we consider a more general abstract equation in fluid mechanics. In particular, it covers the magnetohydrodynamics equations. We first recall some general results from [25] (see also [13]).

Given two Hilbert spaces V and H , with $V \subset H$, V dense in H and the injection of V into H being *compact*. The scalar product and the norm in V and H are denoted by $(\cdot, \cdot)$, $\|\cdot\|$, $(\cdot, \cdot)$ and $|\cdot|$ as before. We can identify H with H' , a subspace of V' (the dual space of V) and then $V \subset H \subset V'$ where the injections are continuous and each space is dense in the following one.

Consider a symmetric bilinear continuous form on V , $a(u, v)$, which is coercive, i.e. $\exists \alpha > 0$, such that $\forall v \in V$,

$$a(v, v) \geq \alpha \|v\|^2. \quad (4.1)$$

Associate with a the linear operator A that is an isomorphism from V onto V' and can be alternatively considered as a linear unbounded self-adjoint operator in H with domain

$$D(A) = \{v \in V \mid Av \in H \subset V \subset H \subset V'\}$$

where the injections are continuous and each space is dense in the following one.

We are given a linear continuous operator E from V into V' , which maps $D(A)$ into H , and there exists $\theta_1, \theta_2 \in [0, 1)$, and constants $c_1, c_2 > 0$, such that

$$|Ev| \leq c_1 \|v\|^{1-\theta_1} |Av|^{\theta_1} \quad \forall v \in D(A) \quad (4.2)$$

$$|(Ev, v)| \leq c_2 \|v\|^{1+\theta_2} |v|^{1-\theta_2} \quad \forall v \in V. \quad (4.3)$$

Assume that $A + E$ is also coercive on V , i.e. $\exists \alpha' > 0$, such that $\forall v \in V$,

$$a(v, v) + (Ev, v) \geq \alpha' \|v\|^2. \quad (4.4)$$

We are also given a bilinear continuous operator B mapping $V \times V$ into V' and $D(A) \times D(A)$ into H , such that

$$(B(u, v), v) = 0 \quad \forall u, v \in V \quad (4.5)$$

$$|(B(u, v), w)| \leq c_3 |u|^{\theta_3} \|u\|^{1-\theta_3} \|v\| \|w\|^{\theta_3} |w|^{1-\theta_3} \quad \forall u, v, w \in V. \quad (4.6)$$

An alternative of (4.6) is the following:

$$|(B(u, v), w)| \leq c_3 |u|^{\theta_3} \|u\|^{1-\theta_3} \|v\| \|w\|^{\theta_3} |w|^{1-\theta_3} \quad \forall u, v, w \in V. \quad (4.6')$$

Furthermore,

$$|B(u, v)| + |B(v, u)| \leq c_4 \|u\| \|v\|^{1-\theta_4} |Av|^{\theta_4} \quad \forall u \in V \quad \forall v \in D(A) \quad (4.7)$$

$$|B(u, v)| \leq c_5 |u|^{\theta_5} \|u\|^{1-\theta_5} \|v\|^{1-\theta_5} |Av|^{\theta_5} \quad \forall u \in V \quad \forall v \in D(A) \quad (4.8)$$

where constants $c_i > 0$ and $\theta_i \in [0, 1)$, $i = 3, 4, 5$. For simplicity, set $B(u) = B(u, u)$.

For given $f \in H$, consider the following nonlinear evolution equation in H :

$$\frac{du}{dt} + Au + B(u) + Eu = f \quad (4.9)$$

$$u(0) = u_0. \quad (4.10)$$

Under the above assumptions and in particular (4.2), (4.3), (4.5), (4.6) (or (4.5), (4.6')), (4.7) and (4.8), the same existence and regularity results as in theorem 3.1 still hold with $\tilde{f}$ replaced by f .

Following [25], it can be shown that

$$\frac{d}{dt}|u|^2 + \alpha' \|u\|^2 \leq \frac{C}{\alpha'} |f|^2 \quad (4.11)$$

$$\frac{d}{dt}a(u, u) + |Au|^2 \leq C(|f|^2 + \|u\|^2 + \|u\|^4 |u|^{2\theta_5/(1-\theta_5)}). \quad (4.12)$$

Using (4.11) and (4.12), the following *a priori* estimates can be shown following the proof of lemma 3.2.

Lemma 4.1. *If $u \in L^2(R_+, V)$ is a weak solution of (4.9) and (4.10), then for $T \geq \alpha'^{-1} > 0$,*

$$\begin{aligned} \limsup_{t \rightarrow \infty} |u(t)| &\leq \frac{C}{\alpha'^2} \limsup_{t \rightarrow \infty} |f(t)| \\ \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|^2 d\tau &\leq \frac{C}{2\alpha'^2} \limsup_{t \rightarrow \infty} |f|^2. \end{aligned}$$

If $u \in L^2(R_+, D(A))$ is a strong solution of (4.9) and (4.10), then for $T \geq \nu^{-1} > 0$

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|u(t)\| &\leq C(Gr) < \infty \\ \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |Au(\tau)|^2 d\tau &\leq C(Gr) < \infty \end{aligned}$$

where $C(Gr)$ is a function of Gr .

Now we state our main result for this section.

Theorem 4.1. *The upper bound for the asymptotic degree of freedom for strong solutions of equation (4.9) in space V can be estimated as follows:*

$$N \leq C \cdot (Gr)^d$$

where d is the dimension of the physical domain Ω and C is a positive constant independent of Gr .

Proof. Let $g \in H$ and

$$\frac{dv}{dt} + Av + B(v) + Ev = g. \quad (4.13)$$

Let $e = u - v$. By (4.6),

$$|(B(e, u), e)| \leq C|e||e||u|. \quad (4.14)$$

Then (3.24) can still be proved using lemma 4.1.

Subtracting (4.13) from (4.9), and taking the scalar product with Ae , we have

$$(e', Ae) + |Ae|^2 = (f - g, Ae) + (B(e, u), Ae) + (B(v, e), Ae) - (Ee, Ae). \quad (4.15)$$

By (4.8),

$$|(B(e, u), Ae)| \leq C|e|^{\theta_5} \|e\|^{1-\theta_5} \|u\|^{1-\theta_5} |Au|^{\theta_5} |Ae|^{1+\theta_5} \leq C|Au||e||Ae| \leq C|Au||e||Ae|^{3/2}.$$

Thus,

$$|(B(e, u), Ae)| \leq C|Au|^2 |e|^2 + \frac{1}{8} |Ae|^2. \quad (4.16)$$

By (4.7),

$$|(B(v, e), Ae)| \leq C \|v\| \|e\|^{1-\theta_4} |Ae|^{1+\theta_4} \leq C \|v\| |e|^{(1-\theta_4)/2} |Ae|^{(3+\theta_5)/2}.$$

Thus,

$$|(B(v, e), Ae)| \leq C \|v\|^{4/(1-\theta_4)} |e|^2 + \frac{1}{8} |Ae|^2. \quad (4.17)$$

By (4.2),

$$|(Ee, Ae)| \leq C \|e\|^{1-\theta_1} |Ae|^{1+\theta_1} \leq C |e|^{(1-\theta_1)/2} |Ae|^{(3+\theta_1)/2}.$$

Thus,

$$|(Ee, Ae)| \leq C |e|^2 + \frac{1}{8} |Ae|^2. \quad (4.18)$$

By (4.15)–(4.18),

$$\frac{d}{dt} \|e\|^2 + |Ae|^2 \leq c |f - g|^2 + C |Au|^2 |e|^2 + C \|v\|^{4/(1-\theta_4)} |e|^2 + C |e|^2. \quad (4.19)$$

Thus, by (3.24), (4.19) and lemma 3.3, the theorem is proved. $\square$

The following immediate result gives a positive answer to the problem raised in the introduction.

Corollary 4.1. *For $d = 2$, the solutions to the 2D NSE (2.5) are asymptotically determined in V by the projection operator R_h satisfying (1.2) and (1.3), and N , the upper bounds for the asymptotic degree of freedom in space V , can be estimated as*

$$N \leq C \cdot (Gr)^2.$$

Note that results similar to theorem 4.1 hold if (4.6) is replaced by (4.6').

Theorem 4.2. *Suppose that (4.6) is replaced by (4.6'). The upper bounds for the asymptotic degree of freedom for the strong solutions of equation (4.9) in space V can be estimated as follows:*

$$N \leq C \cdot (Gr)^{d/2\theta_3}$$

where d is the dimension of the physical domain Ω .

Proof. It is easy to check that lemma 4.1 still holds. By (4.6'), we have

$$|(B(e, u), e)| \leq C |e|^{2\theta_3} \|e\|^{2(1-\theta_3)} \|u\| \leq \varepsilon \|e\|^2 \|u\| + \frac{C}{\varepsilon^{(1-\theta_3)/\theta_3}} |e|^2 \|u\|.$$

Thus,

$$|(B(e, u), e)| \leq \varepsilon \|e\|^2 \|u\| + \frac{C}{\varepsilon^{(1-\theta_3)/\theta_3}} |e|^2 \|u\| + \frac{C}{\varepsilon^{(1-\theta_3)/\theta_3}} |R_h e|^2 \|u\|.$$

Now follow the proof of theorem 3.1. $\square$

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