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Topological physics illustrated in the laboratory

T. F. Rosenbaum[a] and D. L. Stein
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08544
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Topological ideas can be used to predict a variety of textures and defects in condensed matter systems under certain conditions. Liquid crystals exhibit a rich variety of such textures, and provide an accessible system on which to perform tabletop experiments. One means of creating and observing various liquid crystal textures is described, and results of these observations are presented.

I. INTRODUCTION

In recent years a number of mathematical branches that had previously been applied infrequently to physics have found a wide range of useful physical applications. One of the more extensive and intuitively accessible areas of application has been the use of topology to study defects and textures in both quantum fields and condensed matter systems. The latter case is especially interesting in that a large number of condensed matter systems, and hence a wide diversity of corresponding topological spaces, exists and is readily examined in the laboratory. In this paper we present some of the simpler applications of topology to condensed matter physics and discuss how some of the results obtained can actually be tested in the laboratory.

Any system that demonstrates some kind of ordering has associated with it an order parameter, which for our present purposes we define as a field existing at every point in the system which describes the state of the system at that point (note that this definition differs in certain respects from the thermodynamical order parameter defined by Landau, which is a macroscopic quantity and is nonzero only for a system with long-range order). For example, one can regard a Heisenberg ferromagnet in the continuum approximation as a field of three-dimensional vectors of unit magnitude everywhere. The magnetization, which corresponds to the thermodynamical order parameter, is nonzero only below the Curie point but our microscopic order parameter is nonzero and defined at every point in the system both above and below the transition temperature T_c. It is this microscopic order parameter that is of interest to us here.

Let us look a little more deeply at the Heisenberg magnet order parameter, and ask what is the total set of positions that can be assumed by the order parameter at a point? Clearly the tip of the vector can reside at any point on the surface of the unit sphere (denoted S^2 by the mathematicians—an n-dimensional sphere, which is the surface of a ball in $n+1$ dimensions, is denoted by S^n). Therefore, the total *space* in which the order parameter resides is simply S^2, and is called the order parameter space (some authors use the term "manifold of internal states"[1]) of the system. (Here we should point out to the more sophisticated reader that this is actually something of a cheat. The true order parameter space of a system is the space of transformations of the order parameter; for a three-dimensional vector this is just the coset space SO(3)/SO(2) which is topologically equivalent to the sphere S^2. For simple systems like the Heisenberg ferromagnet the naive intuitive picture presented earlier is acceptable, but it may lead into trouble for more complex order parameters. In either case, one should always keep in mind what we really mean when we use the concept of order parameter. For an extensive discussion of these and related points, see the review article by Mermin.[2])

Using this idea of order parameter space, Toulouse and Kleman[1] and Volovik and Mineev[3] showed how topological methods may be used to completely classify defects of a material given only its order parameter space. This idea has constituted the bulk of topological researches of condensed matter systems, and describes what one may call "sufficient" conditions for appearance of defects. The literature in this area has by now grown quite extensive, and we will not pursue these ideas here. Rather, we wish to look at another side of this problem; namely, given a certain system can one unambiguously predict the appearance of defects? It is this pursuit of "necessary" conditions and their consequences that will interest us here.

Before we begin, though, we should make clearer the topological concept of "defect." Two defects will be said to be topologically equivalent if there exists a continuous transformation that takes the order parameter arrangement characterizing one defect into that of the other. A defect is a discontinuity in the order parameter of lower dimensionality than that in which the system resides, so in three dimensions one may have a zero-dimensional point defect, a one-dimensional line defect, or a two-dimensional wall defect. It should be made clear that this is a rather restricted view of defects and does not always correspond with the older and more conventional view.

Using our example of a Heisenberg ferromagnet, one instance of a point defect is that in which all vectors point outward radially from a single point. This particular example has topological charge one, because if one surrounds the point by a sphere of any nonzero radius and examines the order parameter at every point on this sphere, one finds that every point in order parameter space (also a sphere as earlier mentioned) is represented once. (In general, one requires that every point on the sphere is covered at least once, and at least one point is covered only once.) This "winding number" over the sphere (that is, how many times the sphere is completely covered in a mapping from physical space to order parameter space) is called the topological index, or charge, of the defect. Similarly, one surrounds a line defect in three dimensions by a circle, and computes the winding number of the mapping of the circle into order parameter space to find the topological index of the line defect.

We may now attempt to answer the question posed earlier; that is, given a system can one predict which defects and textures will appear? The rest of the paper is organized as follows: In Sec. II a number of theorems of importance will be presented, followed by a discussion of their appli-

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cations to various selected systems in Sec. III. In Sec. IV experiments which test the predictions of earlier sections will be described, and further work that can be done will be discussed.

II. TOPOLOGY OF VECTOR FIELDS ON SURFACES

In this section we shall prove a useful result which will have several interesting applications in the present context. To do this, we will need several concepts in the mathematical theory of surfaces, and so shall review them here. A good reference on this topic is Massey's book on algebraic topology.[4]

It is fairly straightforward, though tedious, to prove that any orientable closed two-manifold (i.e., surface) is homeomorphic to either a sphere or a connected sum of tori (here we omit the class of surfaces which are homeomorphic to a connected sum of projective planes, which are nonorientable). The genus n of a surface is defined as the number of tori from which the surface is composed, and the sphere is assigned genus 0. We now need the concept of Euler characteristic of a surface, which may be introduced in the following manner: Any surface, orientable or nonorientable, may be triangulated, which involves completely covering the surface with closed subsets homeomorphic to triangles such that any two triangles are either disjoint, have one vertex in common, or have one complete edge in common (see Fig. 1).

Now if v is the number of vertices in a given triangulation of a surface, e the number of edges, and f the number of triangles, then the Euler characteristic of the surface E is defined by

$$E = f - e + v. \tag{2.1}$$

E turns out to be not only invariant for a given surface under any triangulation, but is in fact invariant for *all* surfaces with the same genus (that is, all surfaces homeomorphic to each other) and is given by the formula

$$E = 2(1 - n), \tag{2.2}$$

so a sphere has $E = 2$, a torus $E = 0$, etc. E is thus a topological invariant of great usefulness. Lest the reader be misled into thinking that E can only be defined for two-dimensional manifolds, we point out that a more general definition of E exists for surfaces of arbitrary dimension, and reduces to (2.1) in two dimensions.[5] In this paper, however, we shall only be concerned with the case of two-dimensional surfaces.

We are now in a position to prove the desired result. Our "proof" closely follows that of Ref. 6, which provides an intuitively accessible approach. Suppose we map onto a surface S a unit vector field $\mathbf{v}$ which is everywhere continuous except at a finite number of isolated points and is everywhere tangential to the surface. The line integral around a given triangle T_i of a triangulation of S is

$$\oint_{T_i} \mathbf{v}\cdot d\mathbf{l} = 2\pi(1 - m_i), \tag{2.3}$$

where m_i is the total index of point singularities contained in T_i. The left-hand side of (2.3) may be rewritten as

$$\oint_{T_i} \mathbf{v}\cdot d\mathbf{l} = \sum_{\text{edges}} \Delta\theta + \sum_{\text{vertices}} (\pi - \phi), \tag{2.4}$$

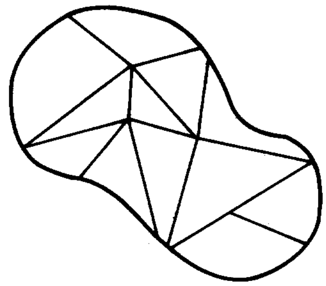

Fig. 1. Triangulation of a surface.

where $\Delta\theta$ is defined over an edge and is given by $\Delta\theta = \cos^{-1} (\mathbf{v}\cdot d\mathbf{l})|_1^2$, 1 and 2 being endpoints of the edge, and ϕ is the interior angle of T_i at a given vertex. If (2.3) and (2.4) are now summed over all triangles T_i, all $\Delta\theta$ terms drop out, since all contours are traversed counterclockwise and every edge is shared between two triangles, leading to cancelling contributions. Since there are twice as many vertices as edges, and since the sum of all interior angles at a vertex is 2π, the sum over all vertices in (2.4) yields $2\pi e - 2\pi v$. Using (2.3), we find that the left-hand side of (2.4) yields $2\pi f - 2\pi m$, where $m = \sum_i m_i$ is the total index (winding number) of all point singularities of $\mathbf{v}$ on S. Hence,

$$2\pi m = 2\pi f - 2\pi v + 2\pi v \tag{2.5}$$

or, using (2.1),

$$m = E. \tag{2.6}$$

We should now pause for a moment and consider the meaning of (2.6). The first thing that comes to mind is that only on a torus, which has $E = 0$, can a vector field be mapped tangentially and everywhere continuously (of course, we can have point singularities on a torus, but the sum of their indices must equal zero). On a sphere, however, it is impossible to do this, and we must have a total index of point singularities equal to two, the Euler characteristic of the sphere.

There exists a complementary theorem to this whose result we will also find useful; for brevity, the proof will be omitted, and the reader is referred to Ref. 7 for details. We shall here merely state the result: Given a normalized vector field $\mathbf{v}$ embedded in a region M of three-dimensional space bounded by a smooth surface ∂M such that $\mathbf{v}$ exists in $\mathbf{M}\cup\partial\mathbf{M}$, is everywhere continous except at a finite number of isolated points in M and on the boundary ∂M points everywhere outward (that is, has no component tangential to ∂M), then

$$\sum_i m_i = (1/2)E(\partial M), \tag{2.7}$$

where now $\sum_i m_i$ is the total index of point singularities of $\mathbf{v}$ in M, and $E(\partial M)$ is the Euler characteristic of the surface ∂M.

Again, we find that only on a torus and its interior can we embed a vector field under the stated conditions such that it is everywhere singularity free. On a sphere and its interior, we recover the rather obvious result that we must have a total index of point singularities equal to one in the interior.

This is all the mathematics we will need from here on. In Sec. III we will see how these theorems can predict which textures will be observed when one places certain condensed matter systems into the appropriate containers.

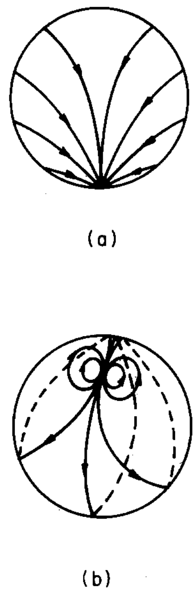

Fig. 2. Spherical boojum in superfluid ^{3}He-A: (a) cross section through sphere showing angular momentum field; and (b) gap parameter vector field on surface.

III. BOOJUMS AND OTHER CURIOSITIES

Suppose one has a sample of superfluid ^{3}He-A in a container of given geometry. The texture of the order parameter of ^{3}He-A in this container may strongly influence the results of experiments on the sample, especially if defects are present. Hence, we wish to know which texture is most likely to be assumed by the sample.

Before this question can be considered, a knowledge of the order parameter of ^{3}He-A is necessary. There will be a vector, denoted by $\mathbf{l}$, which represents the orbital angular momentum of a Cooper pair of ^{3}He atoms. In the plane normal to $\mathbf{l}$ there will also be two perpendicular vectors, $\mathbf{\Delta}^1$ and $\mathbf{\Delta}^2$, called the gap parameter vectors, whose magnitude is related to the degree of superfluid condensation and whose orientation in the plane serves to define a type of superfluid "phase." The three vectors are related by the condition $\mathbf{l} = \mathbf{\Delta}^1 \times \mathbf{\Delta}^2$. This describes the orbital part of the order parameter; however, we need to consider as well the spin degrees of freedom also present. These can be represented by a vector $\mathbf{d}$[8] which in the dipole-locked phase of ^{3}He-A (which is all we consider here) is always parallel or antiparallel to $\mathbf{l}$. Although $\mathbf{d}$ must be included in a complete discussion of possible textures and defects in ^{3}He-A, it is not important for our immediate purposes and we neglect it here. Thus the order parameter is an orthonormal triad of the vectors $\mathbf{l}, \mathbf{\Delta}^1$, and $\mathbf{\Delta}^2$. The group of transformations of this order parameter is then just the group of rotations of a rigid body in three dimensional space and is called SO(3) (for special orthogonal group in three dimensions). For an enlightening treatment of some of the properties of SO(3) [and its universal covering group SU(2)] as they relate to physics, see the discussion in Goldstein.[9]

Owing to the topological properties of the order parameter, there is only one topologically stable line defect in ^{3}He-A (by topological stability, we mean that a defect cannot be *continuously* deformed to the ground or defect-free state—note that this ignores possible energy barriers). This consists of two components of the order parameter rotating by 2π as one circumnavigates the defect. If $\mathbf{l}$ is one of these, the defect is called a disgyration; if $\mathbf{l}$ lies along the defect axis, the defect is called a vortex. Any one of these can be continuously deformed into any other (though again one may have to surmount some energy barrier to do this). A 4π defect, on the other hand, is topologically unstable,[10] which is to say that it can be continuously deformed to the defect-free state.

There is one further fact of interest here. Near a wall the vector will orient itself normal to the surface so that the gap parameter vectors lie in the plane of the surface (assuming that the surface is specularly reflecting).[11] The reason has to do with the fact that if this was not the case, the incident and reflected waves (of the Cooper pair wave function) would interfere destructively, leading to a loss of condensation energy.

Suppose one now places a sample of ^{3}He-A in a sphere of dimensions much larger than a Cooper pair coherence length. A number of different textures are possible, but we will concentrate on one that seems especially likely due to its low bending energy in the order parameter. Recall that the gap parameter vectors must lie tangential to the surface. This implies that we may use our theorem from Sec. II, which states for this case that (if only point singularities appear on the surface) the total vorticity of the gap parameter vectors on the surface must be 4π. (We ignore line singularities on the surface since they tend to have higher energies than point singularities.) How many point singularities actually appear on the surface? Suppose only one is present, which will of necessity have the full 4π vorticity. This will connect to a 4π line defect in the bulk, but we have already seen that such a defect can relax to the defect-free state. Hence we are left with a surface point singularity connected to a singularity-free bulk texture (note this would not be possible for 2π vortices on the surface). The resulting configuration has the lowest bending energy for this geometry, and is displayed in Fig. 2. This is one example of a number of textures that are called "boojums."[12] Among other properties, they may cause a decay in superflow without the necessity of nucleating highly singular vortex line cores.[10]

Let us now turn our attention to liquid crystals. In a nematic liquid crystal, the system consists of long rodlike molecules with an axis of symmetry about the center (rather like "headless" vectors). A cholesteric liquid crystal also consists of these rodlike molecules (called "directors") but in the ground state the directors rotate about a certain direction (normal to their long axis) as one moves through the liquid crystal. The direction of rotation is called the twist, and its period is known as the pitch. One could then take the order parameter of a cholesteric to consist of an orthonormal triad of "headless" vectors: the director $\mathbf{d}$, the twist $\mathbf{t}$, and $\mathbf{t} \times \mathbf{d}$. However, use of this order parameter may cause problems due to its nonlocal nature.[2] In particular, one would normally expect a 4π line defect (called here a disclination) to relax to the defect-free state, as in ^{3}He-A. However, it has been shown[13] that at least certain kinds of 4π defects may not completely relax, so that a singularity may remain in the twist with a core size of the order of the pitch. These problems are not present in simple nematics, which have one class of line defect, also a disclination, in which the director rotates by π as one circumnavigates the disclination. A 2π disclination may relax to the defect-free state.[14]

One may treat the surfaces of certain containers so that directors in nematics or cholesterics line up parallel to them; this may also happen at interfaces between a liquid crystal and a solvent. If we put a nematic or cholesteric in such a container we may again use our theorems of Secs. I and II.

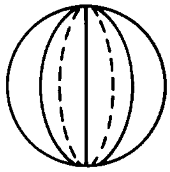

Fig. 3. Onion-skin texture, showing director field on surface. Note singularities at north and south poles.

Consider then a cholesteric in a sphere such that the director molecules are everywhere tangential to the surface of the sphere.[13,15-17] We must again have a total vorticity of 4π on the surface. Here, however, a 4π line defect in bulk will not relax over the volume of the sphere, as previously noted. We will be left with a 4π point surface defect attached to a singularity-free *director* texture in bulk, but connected to a twist texture with a defect of core size of the order of the pitch. Such a configuration will be called a cholesteric boojum.

What about the problem of nematics in a sphere? If the directors line up parallel to the surface, we again have a total vorticity of 4π on the surface. However, in this case we may have *two* 2π point defects on the surface connected to a nonsingular bulk texture[17] (see Fig. 3). Recall that this is due to the fact that in a nematic 2π line defects are topologically unstable.

We may also treat surfaces such that director molecules are normal to the surface (known as "homeotropic" orientation). In this case we may use the second theorem presented in Sec. II to predict the total index of bulk point singularities within the container. For a sphere, or course, there must be a total index of one within.

IV. EXPERIMENTAL DETAILS AND OBSERVATIONS

Although the mathematical concepts involved in a topological discussion of liquid crystals can be subtle, their experimental illustration turns out to be straightforward. It is not quite possible to encase bulk liquid crystal in a sphere, hourglass, or torus, but it is possible to form liquid crystalline "spherulites" in isotropic solvents and to study the rich variety of textures induced by such boundary conditions. Armed only with a polarizing microscope suitably equipped with heating stage and camera mount, the student can thus come to grips with the topological predictions outlined above.

As liquid crystals have varied commercial applications,

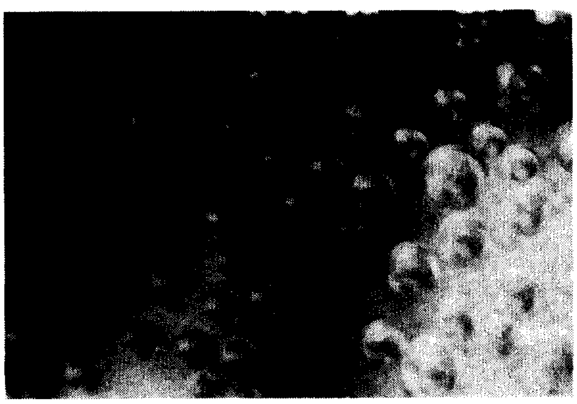

Fig. 4. Nematic spherulites with homeotropic orientation on surface. Maltese cross indicates point defect in bulk. Average diameter of spherulites $\approx 5\ \mu$m.

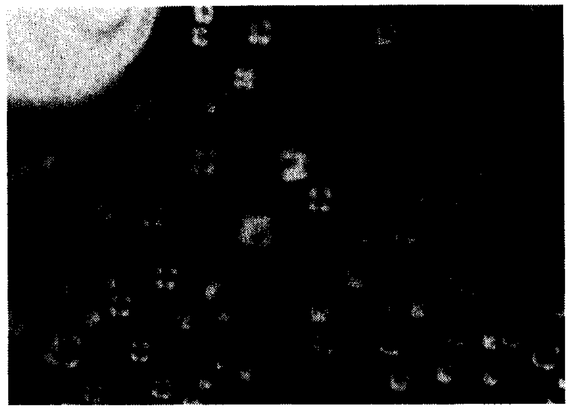

Fig. 5. Cholesteric spherulites with homeotropic orientation on surface. Average diameter $\approx 5\ \mu$m.

such as alphanumeric displays, they are widely available and relatively inexpensive. The experiments described here used the nematic-smectic p-cyanobenzylidene-p′-octyloxy-aniline (CBOOA) for investigations of homeotropic orientation of the director field and 4,4′-Bis(heptyloxy)-azooxybenzene in order to obtain tangential orientations. These two liquid crystals line up differently because CBOOA has a dipole moment parallel to its long axis while the latter liquid crystal has a dipole moment perpendicular to its long axis. In addition, we studied cholesterics using a 0.7% mixture of CBOOA and cholesterol nonanoate (CN). Mixing small amounts of cholesteric liquid with a pure nematic–smectic affords greater flexibility than using the cholesteric alone as the pitch of the resulting mixture is inversely proportional to the concentration of the cholesteric.

The experimental cell consisted simply of a square cover slip attached at two sides to a microscope slide by double-sided mylar tape. A lump of liquid crystal was placed at one of the unattached sides of the cover slip, the solvent (e.g., Canada balsam or microscope immersion oil) at the remaining fourth side, and upon heating the two substances were drawn into the cell by capillary action. Heating the mixture allows the liquid crystal to dissolve in the solvent and upon cooling, or after a few cycles of heating and cooling, spherulites separate out of solution.

It should be noted that a small amount of either the microscope immersion oil or the Canada balsam can interact with the nematic liquid crystal in the same way as does a liquid cholesteric; but as long as the pitch is much greater than the radius of the spherulites the mixture can still be considered a nematic. The isotropic–nematic transition temperature is a sensitive function of the degree of contamination and provides a ready check that this approximation remains valid.

Consistent with the theory, a "maltese cross," representing a point defect in the interior, was observed under cross polarizers for both nematic (Fig. 4) and cholesteric (Fig. 5) homeotropic orientations of CBOOA and CBOOA plus CN, respectively. Observations were made for spherulites with diameters ranging from 2 μ to 30 μ.

The simplicity and symmetry of the director pattern (splaying radially outward from a point) allows the photograph to be easily interpreted: the arms of the cross are just areas of extinction as defined by the axes of the crossed polarizers. More complicated director patterns, however,

Fig. 6. Nematic spherulites with tangential orientation on surface. This texture is believed to correspond to onion skin of Fig. 3. Cross indicates point defect on surface.

Fig. 7. Cholesteric spherulite with tangential orientation on surface. Radial fault is not discernible. Spacing of rings is related to size of the pitch. Diameter $\approx 40\ \mu m$.

are usually the rule. The interpretation of the photographs associated with them involves liquid crystal properties such as uniaxiality and birefringence and requires a well-honed geometric sense. The reader is referred to extensive discussions of at least the pertinent crystal properties in both Hartshorne and Stuart[18] and de Gennes,[19] and to the work of Bouligand[20] for many interpretive examples.

For the case of a tangentially oriented nematic in a sphere, we would expect to observe crosses at both poles of the spherulite, corresponding to two 2π point defects on the surface. This was indeed seen (Fig. 6), although the two-dimensional photograph does not allow portrayal of both poles at once. We checked that the director field was in fact tangential by inserting a quarter wave plate in the path of the light passing through the liquid crystal sample. The sequence of colors across the spherulites caused by the additional twist of the $\lambda/4$ plate establishes either tangential or homeotropic orientation.[21]

Finally, Fig. 7 shows a cholesteric spherulite with a tantentially oriented director field. Both the circumferential pattern on the spherulite and the similarly spaced lines on the bulk liquid crystal in the background are closely related to the twist. We were not able to observe, however, a radial discontinuity which should be characteristic of a cholesteric boojum. Such a picture can be found in Robinson *et al.*[22] whose experiments involve a lyotropic liquid crystal PBLG (as opposed to the thermotropic liquid crystals we used).

The cataloging of textures predicted by topological theory only skims the surface of projects readily pursued. For example:

(1) Defect states of all levels of complexity are usually present. It would be interesting to follow these more complicated textures to see if they decay into the simpler states predicted topologically.

(2) A cholesteric with a very large pitch is effectively a nematic. How small must the pitch become before it behaves like a cholesteric? The interaction of the solvent with the liquid crystal should allow a "twist gradient" to be established across the experimental cell and allow this question to be investigated.

(3) If a very well-controlled temperature stage is available, such as a Mettlar oven, then one can use the liquid crystal alone (without a solvent) and see spherulites form while the system sits at the nematic–isotropic phase transition.

(4) Magnetic field compete with surface boundary effects in the alignment of the director field. Thus the defect patterns observed will be a function of the level of an applied magnetic field. See the work of Candau *et al.*[23] for some striking pictures along these lines.

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[a]Also at Bell Laboratories, Murray Hill, NJ 07974.

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