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Exotic circuit elements from zero-modes in hybrid superconductor–quantum-Hall systems

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In these supplementary materials we characterize the instabilities leading to parafermionic zero-modes in a spin-unpolarized $\nu = 2/3$ quantum Hall system, which for reasons elucidated below is a particularly promising platform for the transport anomalies described in the main text. To begin we summarize the edge theory for this quantum Hall state using K-matrix formalism [S1]. The edge modes of the unpolarized $2/3$ state are described by two bosonic fields $\phi_{\uparrow,\downarrow}$ and an associated K-matrix $\mathbf{K} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and charge vector $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this description, the edge electron density is given by $(1/2\pi)\,\mathbf{q}^\intercal \partial_x \boldsymbol{\phi}$ and the filling fraction reads $\nu = \mathbf{q}^\intercal \mathbf{K}^{-1} \mathbf{q} = 2/3$. A trench etched in the quantum Hall liquid brings two counterpropagating sets of such modes into close proximity with one another. Disregarding tunneling and pairing terms for the moment, the Lagrangian density for the edge modes opposite the trench is

$$\mathcal{L}_0 = \frac{1}{4\pi} \partial_x \phi_I \left(\mathcal{K}_{IJ} \partial_t \phi_J - V_{IJ} \partial_x \phi_J \right), \qquad \text{(S1)}$$

where we have four bosonic fields ϕ_I, two for each side of the trench. The extended K-matrix for this doubled system is $\boldsymbol{\mathcal{K}} = \begin{pmatrix} \mathbf{K} & 0 \\ 0 & -\mathbf{K} \end{pmatrix}$, while the matrix $\mathbf{V}$ captures the density-density interactions both within each edge and across the trench.

For simplicity, we assume the two edges are symmetric and that the interactions are invariant under SU(2) spin rotations. (Note that an equivalent SU(2) symmetry *emerges* due to disorder in the polarized $\nu = 2/3$ state [S2]; here, however, the SU(2) symmetry is manifestly that of spin.) This gives us the generic for m

$$\mathbf{V} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_1 & v_4 & v_3 \\ v_3 & v_4 & v_1 & v_2 \\ v_4 & v_3 & v_2 & v_1 \end{pmatrix}. \qquad \text{(S2)}$$

Of course the SU(2) spin symmetry is not guaranteed microscopically. In particular, this symmetry is broken by a Zeeman field, which takes the form $H_{\text{Zeeman}} = (h/2\pi)\mathbf{n}_t^\intercal \partial_x \boldsymbol{\phi}$, with $\mathbf{n}_t^\intercal = (1, -1, 1, -1)$. This term has scaling dimension 1, but can be absorbed into $\mathcal{L}_0$ via a redefinition $\boldsymbol{\phi} \to \boldsymbol{\phi} - hx\mathbf{V}^{-1}\mathbf{n}_t$. However, this absorbtion will cause other terms involving $\exp\left(i\mathbf{n}_t^\intercal \boldsymbol{\phi}\right)$ to oscillate on the length scale $\pi(v_1 - v_2 + v_3 - v_4)/2h$.

Next we classify perturbations to Eq. (S1) that can generate instabilities. There are six types of gap-opening perturbations that are marginal when $h = 0$ and density–density interactions between the two edges are absent, *i.e.* $v_3 = v_4 = 0$. Charge hopping and pairing terms each come in two varieties due to spin degeneracy. We can divide these into perturbations that form either spin singlets or spin triplets across the trench. In addition, there are marginal perturbations that do not transfer charge but involve singlet or triplet spin correlations across the trench. These perturbations are listed in Table I, along with their scaling dimensions (expressed in terms of parameters u and v defined in the caption).

Process	Operators	Scaling Dimension
singlet hopping	$e^{\pm\frac{1}{2}\mathbf{c}_h^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}} e^{\pm\frac{1}{2}\mathbf{n}_s^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$\frac{1}{2}e^{2u} + \frac{3}{2}e^{2v}$
triplet hopping	$e^{\pm\frac{1}{2}\mathbf{c}_h^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}} e^{\pm\frac{1}{2}\mathbf{n}_t^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$\frac{1}{2}e^{-2u} + \frac{3}{2}e^{2v}$
singlet pairing	$e^{\pm\frac{1}{2}\mathbf{c}_p^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}} e^{\pm\frac{1}{2}\mathbf{n}_s^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$\frac{1}{2}e^{2u} + \frac{3}{2}e^{-2v}$
triplet pairing	$e^{\pm\frac{1}{2}\mathbf{c}_p^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}} e^{\pm\frac{1}{2}\mathbf{n}_t^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$\frac{1}{2}e^{-2u} + \frac{3}{2}e^{-2v}$
neutral singlet	$e^{\pm\mathbf{n}_s^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$2e^{2u}$
neutral triplet	$e^{\pm\mathbf{n}_t^\intercal \boldsymbol{\mathcal{K}}\boldsymbol{\phi}}$	$2e^{-2u}$

TABLE I: The six types of gap-opening perturbations that are marginal when $h = v_3 = v_4 = 0$. There are two representatives of each neutral type and four of each charged type. Here $\mathbf{c}_h^\intercal = (1, 1, -1, -1)$, $\mathbf{c}_p^\intercal = (1, 1, 1, 1)$, $\mathbf{n}_s^\intercal = (1, -1, -1, 1)$, and $\mathbf{n}_t^\intercal = (1, -1, 1, -1)$. The important parameters u and v are defined as $\tanh 2u = -\frac{(v_3 - v_4)}{v_1 - v_2}$ and $\tanh 2v = -\frac{(v_3 + v_4)}{v_1 + v_2}$.

We now focus our attention on a system in which an ordinary s-wave superconductor couples to the trench, and therefore proximity-induces singlet pairing. Likewise, we will assume that electron hopping across the trench acts merely to restore the original quantum Hall state, rather than introducing additional spin flips. We therefore choose parameters such that singlet pairing, singlet hopping, and neutral singlet coupling are the dominant perturbations. Note that these terms are unaffected by the Zeeman field, while the triplet terms will have oscillating coefficients. Using Table I, we can find a region of parameter space for which all three singlet terms are *simultaneously* relevant by simply setting $u < 0$, $v \approx 0$. This is reasonable as a physical regime, given that we must have $|v_1| > |v_2|$ for stability of the individual edges and repulsive interactions will generically give $v_2 > 0$, since v_2 is the density-density interaction between charges on the same edge. Likewise v_3 represents repulsion between charges of the same spin on opposite edges, and v_4 encodes repulsion between charges of opposite spins on opposite edges. It is not unreasonable to expect $v_3 > v_4$, and thus $u < 0$. We shall work in this parameter regime for the remainder of this analysis.

The fact that singlet pairing and direct tunneling across the trench can become simultaneously relevant for the same set of (reasonable) parameters is a great virtue of the unpolarized $\nu = 2/3$ state. That is, the superconductor can induce a gap along the trench *under the same conditions in which the* $2/3$ *state itself would also reform*. This feature is difficult to achieve in other Abelian quantum Hall states. [Consider, *e.g.*, the $\nu = 2/5$ state with the same interaction matrix in Eq. (S2). Here the scaling dimensions for the superconducting and tunneling perturbations respectively read $\Delta_S = \frac{1}{2}e^{2u} + \frac{5}{2}e^{2v}$ and $\Delta_T = \frac{1}{2}e^{2u} + \frac{5}{2}e^{-2v}$. These two terms cannot both be made relevant (*i.e.* $\Delta_{S,T} < 2$) for the same values of u and v.]

Crucially, although both singlet pairing and hopping terms are relevant at $\nu = 2/3$, the gaps favored by these two processes are incompatible with one another. Rather, in any given region of the trench either the pairing *or* the hopping mechanism must win out in order for a gap to open. This is because the (charge-sector) fields that these two types of coupling try to pin are dual to one another. If one of them is pinned, the other must fluctuate. For $\nu = 2/3$, though, there is no such issue in the neutral sector—*i.e.*, all three singlet terms in Table I favor gapping the neutral fields in compatible ways. The neutral sector thus gaps out trivially, independent of which mechanism wins out.

As is the case when there is only one type of (fractional) edge mode, the boundary between a region of the trench gapped by pairing and one gapped by direct hopping supports a parafermionic zero-energy mode [S3–S5]. In this case, the zero mode has a $\mathbb{Z}_3$ character. The main text describes a number of nontrivial consequences for transport when these (and other types of zero modes) are probed with quantum Hall edges. As an aside, we also comment that the unpolarized $2/3$ case may be of particular interest for more traditional topological quantum information applications because of the complete gapping of the neutral sector. This gap partially inoculates the zero modes here against noise. In particular, stray electrons *cannot* directly affect the state of quantum information encoded in these zero modes, because they possess neither the correct charge ($2e/3$, $4e/3$, or $0 \mod 2e$) nor spin (always 0) to do so.

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