Supplementary Material of "Persistence of locality in systems with power-law interactions"

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In this Supplementary Material, we generalize our new bound, Eq. (2) in the main text, to $2D$ and $3D$ infinite cubic lattices (further generalization to arbitrary lattice geometry is also possible). The generalization starts from Eq. (7) in the main text (with $r_{ik} \leq 1$ changed to $r_{ik} < 2$ to include sites with $1 < r_{ik} < 2$ in $2D$ and $3D$),

$$\sum_k J_{ik}J_{kj} \;\leq\; 2(\sum_{r_{ik}<2} J_{ik}J_{kj} + 2^\alpha J_{ij} \sum_{r_{ik}\geq 2} J_{ik}). \tag{S1}$$

Now let x_β be the Cartesian coordinates of the vector $\mathbf{r}_{ik}$, and write $x_\beta = 2y_\beta + w_\beta$, where $w_\beta = 0, 1$ for $y_\beta \geq 1$, $w_\beta = 0, \pm 1$ for $y_\beta = 0$, and $w_\beta = 0, -1$ for $y_\beta \leq -1$. The constraint $r_{ik} \geq 2$ is then equivalent to $\sum_{\beta=1}^{D} y_\beta^2 \neq 0$ for $D = 1, 2, 3$, thus

$$\sum_{r_{ik}\geq 2} r_{ik}^{-\alpha} \;=\; \sum_{\sum_\beta y_\beta^2 \neq 0} \sum_{w_\beta} [\sum_{\beta=1}^{D} (2y_\beta + w_\beta)^2]^{-\alpha/2} \leq \sum_{\sum_\beta y_\beta^2 \neq 0} 3^D [\sum_{\beta=1}^{D} (2y_\beta)^2]^{-\alpha/2} = 3^D 2^{-\alpha}(\lambda - 1). \tag{S2}$$

Combining Eq. (S1) and (S2), and using the inequality that $J_{ij} \leq \sum_{r_{ik}<2} J_{ik}J_{kj}$, we arrive at

$$\sum_k J_{ik}J_{kj} \;\leq\; 2\left[\sum_{r_{ik}<2} J_{ik}J_{kj} + J_{ij}3^D(\lambda - 1)\right] \leq 2\left[3^D\lambda \sum_{r_{ik}<2} J_{ik}J_{kj}\right]. \tag{S3}$$

Applying this result iteratively to Eq. (4) in the main text, we obtain:

$$\mathcal{J}_n(i,j) \leq (3^D 2\lambda)^{n-1} \sum_{r_{ik_1}<2,\ldots,r_{k_{n-2}k_{n-1}}<2} J_{i,k_1} \ldots J_{k_{n-1},j}.$$

which generalizes Eq. (9) of the main text to $D = 2, 3$. The maximum possible value for each summand is still given by $(r - n + 1)^{-\alpha}$. The number of sites with $r_{ik} < 2$ for a D-dimensional cubic lattice is bounded by 3^D, and therefore $\mathcal{J}_n(i,j) \leq (9^D 2\lambda)^{n-1}(r - n + 1)^{-\alpha}$. The generalization of Eq. (10) of the main text follows immediately: the form is identical, but the constants v_2 and c_2 should be replaced with [1]:

$$v_2 = 9^D 4\lambda^2 \quad c_2 = (9^D 2\lambda)^{-1}.$$

The rest of the results in the main text are unchanged.

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[1] We note that, for $D = 1$, the simplified derivation given in the main text actually produces a slightly tighter bound.