CALCULATION OF SMALL DISTANCES*

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Abstract

Tables are presented which facilitate the calculation of small distances between points on the earth's surface whose latitudes and longitudes are given, for mean latitudes from 30° to 40°. The error does not exceed one-tenth of a kilometer in 500 kilometers, and may be reduced by applying specified corrections.

In work relating to earthquakes local to an area under intensive seismological and geological investigation, epicenters are usually placed to within a few kilometers. Times of seismic waves are determined to the tenth of a second. Distances from stations to epicenters assumed for trial are required to the nearest kilometer; and it is desirable that the method used should be capable of furnishing such distances to the tenth of a kilometer, in order to avoid rounding-off errors during the calculations.

The last requirement usually causes trouble. To read distances of the order of 300 kilometers from a map, with an accuracy of 0.1 km., means working on an awkwardly large scale, and checking the measurements with such care that they become very time-consuming. Computation from the coördinates of the stations and epicenter is much more desirable, provided tables and other material are available for carrying through the process with reasonable speed.

The following discussion contains no new contribution, but presents well-known material in the form which has been found most convenient at Pasadena, for use with a computing machine.

Distances up to about 5° (555 km.) are given with an error not exceeding 0.1 km. by a formula of the simple form

$$\Delta^2 = \Delta x^2 + \Delta y^2; \quad \Delta x = A\Delta\lambda, \quad \Delta y = B\Delta\Phi \tag{1}$$

Here Δ is the required distance in kilometers, $\Delta\lambda$ and $\Delta\Phi$ are differences in longitude and latitude between the two given points (epicenter and station), and A and B are conversion factors from circular measure to kilometers. If $\Delta\lambda$ and $\Delta\Phi$ are in minutes of longitude and latitude, then sufficient accuracy is provided by taking A and B as the lengths in kilometers of one minute of arc of the parallel and meridian, respectively, taken at the mean latitude between the two points.

This method is equivalent to that known in navigation as "middle latitude sailing." It is a close approximation to that given by Wiechert (1925). His method, involving tables for finding factors equivalent to A and B, is reasonably rapid in practice, especially after the user is familiar with it. The definition

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here given for the factors A and B leads to the preparation of simple tables for much more rapid computation.

The *Smithsonian Geographical Tables* (Woodward, 1894) contain data from which table 1 is excerpted and derived.

For the accuracy required, five significant figures are usually sufficient. It is necessary then to interpolate the kilometer equivalents given in table 1 to apply to the mean latitude between the two points considered. For the factor B, applying to differences in latitude, this interpolation is simple, since the

TABLE 1

LENGTHS OF TERRESTRIAL ARCS, IN KILOMETERS (CLARKE SPHEROID)

Mean latitude (Φ)	Arc of meridian, B		Arc of parallel, A		$A/\cos \Phi$	
	Degree	Minute	Degree	Minute	Degree	Minute
30°	110.8497	1.847495	96.4893	1.608155	111.4162	1.856937
31	110.8669	1.847781	95.5073	1.591788	111.4220	1.857033
32	110.8844	1.848073	94.4962	1.574937	111.4279	1.857132
33	110.9023	1.848372	93.4563	1.557605	111.4339	1.857231
34	110.9204	1.848673	92.3879	1.539798	111.4399	1.857331
35	110.9388	1.848980	91.2913	1.521522	111.4461	1.857435
36	110.9574	1.849290	90.1668	1.502780	111.4523	1.857538
37	110.9763	1.849605	89.0148	1.483580	111.4586	1.857643
38	110.9953	1.849922	87.8356	1.463927	111.4650	1.857750
39	111.0145	1.850242	86.6296	1.443827	111.4715	1.857858
40	111.0339	1.850562	85.3970	1.423283	111.4779	1.857965

tabular differences in the second column of table 1 are very regular. There results the working table which is given as table 2.

The factor A, applying to differences in longitude, is less simple. The differences in the third and fourth columns of table 1 are not smooth enough for easy interpolation. Each of these figures has accordingly been divided by the cosine of the corresponding latitude, which gives the quotients in the fifth and sixth columns. The differences in these are now regular enough for simple interpolation; this was performed for each 10′ of latitude, the interpolated values multiplied by their corresponding cosines, and a final interpolation performed to give values of A for each minute of latitude. The final working table is given as table 3.

Using tables 2 and 3, with the aid of a computing machine and a good table of squares, routine calculation of distances is very rapid and accurate. With the machine, it is more convenient to form the sums of squares directly instead of referring to the table of squares, which is used only in finding Δ at the last step. If no machine is available, the multiplications by A and B are not particularly onerous; the squarings will then be performed with the aid of the table of squares. The following is an example.

Station, Pasadena, 34° 08′.9 N, 118° 10′.3 W

Trial epicenter, 32° 00′ N, 119° 00′ W

$\Delta\Phi = 128'.9 \qquad \Delta\lambda = 49'.7$

Mean latitude $\Phi = 33° 04'.45$

From table 2, $B = 1.8484$

From table 3, $A = 1.5562$

Hence $\Delta x = 77.3$ km., $\Delta y = 238.3$ km.

$\Delta = \sqrt{77.3^2 + 238.3^2} = 250.5$ km.

TABLE 2

COEFFICIENTS B

(Lengths in kilometers of 1′ of the meridian)

Mean latitude Φ	B	Mean latitude Φ	B
29° 51′ to 30° 11′	1.8475	35° 14′ to 35° 33′	1.8491
30 12 to 32	76	34 to 52	92
33 to 53	77	35 53 to 36 11	93
54 to 31 14	78	36 12 to 30	94
31 15 to 34	79	31 to 49	95
35 to 54	80	50 to 37 08	96
55 to 32 14	81	37 09 to 27	97
32 15 to 35	82	28 to 46	98
36 to 55	83	47 to 38 06	1.8499
56 to 33 15	84	38 07 to 24	1.8500
33 16 to 35	85	25 to 43	01
36 to 55	86	44 to 39 01	02
56 to 34 15	87	39 02 to 20	03
34 16 to 34	88	21 to 39	04
35 to 54	89	40 to 57	05
55 to 35 13	1.8490	58 to 40 16	1.8506

It remains to discuss the errors of the method. These fall into three classes: (1) errors due to the curvature of the earth considered as a sphere, (2) errors due to the departure of the Clarke spheroid (on which the geographical tables used are based) from a sphere, and (3) errors due to departure of the Clarke spheroid from the true figure of the earth, or from presumably better approximations such as the International Ellipsoid.

Since errors of the first class vanish when the two points are on the same meridian ($\Delta\lambda = 0$), while those of the other classes are then at maximum, a direct comparison is of interest. The following figures are typical:

Length of arc of the meridian between latitudes 31° 30′ and 36° 30′:

1) multiplying by 300 the length given for 1′ of the meridian at 34° in table 2, 554.61 km.

TABLE 3

Coefficients A for Given Mean Latitude Φ (Lengths in kilometers of 1′ of the parallel)

	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°
00′	1.6082	1.5918	1.5749	1.5576	1.5398	1.5215	1.5028	1.4836	1.4639	1.4438
01	79	15	47	73	95	12	25	33	36	35
02	76	12	44	70	92	09	21	29	33	32
03	74	10	41	67	89	06	18	26	29	28
04	71	07	38	64	86	03	15	23	26	25
05	1.6068	1.5904	1.5735	1.5561	1.5383	1.5200	1.5012	1.4820	1.4623	1.4421
06	65	1.5901	32	58	80	1.5197	09	16	19	18
07	63	1.5898	29	56	77	94	06	13	16	15
08	60	96	27	53	74	90	1.5002	10	13	11
09	57	93	24	50	71	87	1.4999	07	09	08
10	1.6055	1.5890	1.5721	1.5547	1.5368	1.5184	1.4996	1.4803	1.4606	1.4404
11	52	87	18	44	65	81	93	1.4800	1.4603	1.4401
12	49	85	15	41	62	78	90	1.4797	1.4599	1.4398
13	46	82	12	38	59	75	87	94	96	94
14	44	79	09	35	56	72	83	90	93	91
15	1.6041	1.5876	1.5706	1.5532	1.5353	1.5169	1.4980	1.4787	1.4589	1.4387
16	38	73	04	29	50	66	77	84	86	84
17	36	71	1.5701	26	47	63	74	81	83	81
18	33	68	1.5698	23	44	60	71	78	79	77
19	30	65	95	20	41	56	67	74	76	74
20	1.6028	1.5862	1.5692	1.5517	1.5338	1.5153	1.4964	1.4771	1.4573	1.4370
21	25	59	89	14	35	50	61	68	69	67
22	22	57	86	11	32	47	58	64	66	63
23	19	54	83	08	28	44	55	61	63	60
24	17	51	81	05	25	41	52	58	59	56
25	1.6014	1.5848	1.5678	1.5502	1.5322	1.5138	1.4948	1.4754	1.4556	1.4353
26	11	45	75	1.5499	19	35	45	51	53	50
27	08	43	72	96	16	31	42	48	49	46
28	06	40	69	93	13	28	39	45	46	43
29	03	37	66	91	10	25	36	41	43	40
30	1.6000	1.5834	1.5663	1.5488	1.5307	1.5122	1.4932	1.4738	1.4539	1.4336
31	1.5998	31	60	85	04	19	29	35	36	33
32	95	29	58	82	1.5301	16	26	32	33	29
33	92	26	55	79	1.5298	13	23	28	29	26
34	89	23	52	76	95	10	20	25	26	22
35	1.5987	1.5820	1.5649	1.5473	1.5292	1.5106	1.4916	1.4722	1.4523	1.4319
36	84	17	46	70	89	03	13	18	19	16
37	81	15	43	67	86	1.5100	10	15	16	12
38	78	11	40	64	83	1.5097	07	12	13	09
39	76	09	37	61	80	94	04	09	09	05
40	1.5973	1.5806	1.5634	1.5458	1.5277	1.5091	1.4900	1.4705	1.4506	1.4302
41	70	03	31	55	74	88	1.4897	1.4702	1.4502	1.4298
42	68	1.5800	29	52	71	85	94	1.4699	1.4499	95
43	65	1.5798	26	49	67	81	91	95	96	92
44	62	95	23	46	64	78	87	92	92	88
45	1.5959	1.5792	1.5620	1.5443	1.5261	1.5075	1.4884	1.4689	1.4489	1.4285
46	57	89	17	40	58	72	81	86	86	81
47	54	86	14	37	55	69	78	82	82	78
48	51	83	11	34	52	66	75	79	79	74
49	48	81	08	31	49	63	71	76	75	71
50	1.5946	1.5778	1.5605	1.5428	1.5246	1.5059	1.4868	1.4672	1.4472	1.4267
51	43	75	1.5602	24	43	56	65	69	69	64
52	40	72	1.5599	22	40	53	62	66	65	60
53	37	69	97	19	37	50	58	62	62	57
54	34	66	94	16	34	47	55	59	59	54
55	1.5932	1.5764	1.5591	1.5413	1.5231	1.5044	1.4852	1.4656	1.4455	1.4250
56	29	61	88	10	28	40	49	52	52	47
57	26	58	85	07	24	37	45	49	48	43
58	23	55	82	04	21	34	42	46	45	40
59	20	52	79	1.5401	18	31	39	43	42	36
60	1.5918	1.5749	1.5576	1.5398	1.5215	1.5028	1.4836	1.4639	1.4438	1.4233

2) multiplying by 5 the length given for 1° of the meridian at 34° in table 1, 554.6020 km.

3) by summing the lengths given at 32°, 33°, 34°, 35°, and 36° in table 1, 554.6033 km.

4) from tables based on the International Ellipsoid (Lambert and Swick, 1935), 554.6267 km.

The third result is the correct length for the Clarke Spheroid. The first and second computations differ from this by the deviation of the spheroid from a sphere, plus rounding-off errors. This deviation is seen to be actually less than that between lengths computed on the two spheroids, and amounts to less than 0.01 km., or less than ten meters.

The deviation between distances computed from formula (1) and distances over the sphere is increased when the end points also differ in longitude, while the effect of the ellipticity of the earth becomes less. Hence, in estimating errors of the method it will be sufficient to determine the errors when applied to points at the same latitude and longitude on a true sphere of nearly the size of the earth—conveniently, a sphere of 40,000 km. circumference. The theoretical foundation is as follows.

Exact angular distances over the sphere are given by

$$\cos \Delta = \cos \Phi_1 \cos \Phi_2 \cos \Delta\lambda + \sin \Phi_1 \sin \Phi_2 \tag{2}$$

where Φ_1 and Φ_2 are the latitudes of the two given points, the longitudes of which differ by $\Delta\lambda$. If we put $\Phi_1 - \Phi_2 = \Delta\Phi$ and $\Phi_1 + \Phi_2 = 2\Phi$, and suppose $\Delta\Phi$ and $\Delta\lambda$ to be small compared to one radian, then we find

$$\cos \Delta = 1 - \frac{\Delta\Phi^2}{2} - \cos^2 \Phi \frac{\Delta\lambda^2}{2} + \frac{\Delta\Phi^4}{24} + \frac{\Delta\Phi^2 \Delta\lambda^2}{8} + \cos^2 \Phi \frac{\Delta\Phi^4}{24} + \cdots \tag{3}$$

By applying the expansion of $\cos \Delta$ and using the method of undetermined coefficients it is easy to derive

$$\Delta^2 = \Delta\Phi^2 + \cos^2 \Phi \Delta\lambda^2 + \left(-\frac{1}{4} + \frac{\cos^2 \Phi}{6}\right) \Delta\Phi^2 \Delta\lambda^2 - \frac{\sin^2 \Phi \cos^2 \Phi}{12} \Delta\lambda^2 + \cdots \tag{4}$$

Writing $\Delta_0^2 = \Delta\Phi^2 + \cos^2 \Phi \Delta\lambda^2$, there results

$$\Delta = \Delta_0 - \frac{1}{24} \sin^2 \Phi \Delta_0 \Delta\lambda^2 - \frac{1}{24} (1 + \sin^2 \Phi) \frac{\Delta\Phi^2 \Delta\lambda^2}{\Delta_0} + \cdots \tag{5}$$

Δ_0 is the value of Δ given by the method of this paper. The following terms give the correction, including small quantities of the third order. If $\Delta\Phi$, $\Delta\lambda$, and Δ_0

are near 6° they are of the order of one-tenth of a radian, and the correction terms given above are of the order of one-thousandth of a radian divided by 24, or about one-fourth of a kilometer.

In computing tables showing these errors it has been desirable to have $\Delta\Phi$ and $\Delta\lambda$ expressed in integral numbers of degrees, and Δ, with the corrections, in kilometers. If $\Delta\Phi = m$ degrees and $\Delta\lambda = n$ degrees, then

$$\Delta = \Delta_0 - 0.00140 \frac{n^2}{\sqrt{m^2 + \cos^2 \Phi}} \left[(1 + 2 \sin^2 \Phi) m^2 + \sin^2 \Phi \cos^2 \Phi m^2 \right] \quad (6)$$

for a sphere of 40,000 km. circumference. The exact value of the numerical coefficient in the correction term is $\pi^2/6998.4$.

TABLE 4

ERRORS OF FORMULA (1), IN KILOMETERS

a. Mean latitude $\Phi = 30°$

$\Delta\lambda$

$\Delta\Phi$	1°	2°	3°	4°	5°	10°
0°	.0003	.002	.008	.02	.04	.31
1	.002	.006	.01	.03	.05	.33
2	.004	.01	.03	.05	.08	.39
3	.006	.02	.05	.08	.12	.50
4	.008	.03	.07	.12	.17	.63
5	.01	.04	.09	.15	.22	.79
10	.02	.08	.19	.32	.50	1.80

b. Mean latitude $\Phi = 45°$

$\Delta\lambda$

$\Delta\Phi$	1°	2°	3°	4°	5°	10°
0°	.0005	.0004	.01	.03	.06	.50
1	.003	.01	.02	.05	.08	.53
2	.005	.02	.05	.08	.12	.63
3	.008	.03	.07	.12	.18	.79
4	.01	.04	.10	.17	.25	.99
5	.01	.06	.12	.21	.32	1.22
10	.03	.11	.25	.44	.69	2.59

Tables 4*a* and 4*b* have been computed from formula (6). Table 4*a* gives the value of the third-order terms for mean latitude $\Phi = 30°$; table 4*b*, for $\Phi = 45°$. Interpolation can be made very exact, if desired, by noting that when both $\Delta\Phi$ and $\Delta\lambda$ are multiplied by the same constant k, the errors as tabulated are

multiplied by k^3. These positive numbers are errors of formula (1); to be applied as corrections they must be subtracted, in accordance with the negative sign in (6).

The errors near 45° are rougly half again as large as those near 30°, and errors increase further at higher latitudes. The following are the errors in meters for $\Delta\Phi = \Delta\lambda = 1°$, at the indicated mean latitudes:

Φ	30°	35°	40°	45°	50°	55°	60°
Error	1.80	2.05	2.32	2.59	2.87	3.14	3.39 meters

To compare with table 4, divide by 1000.

While these corrections are computed for a sphere of 40,000 km. circumference, so that 1° of a great circle has the length 111.1111 km., they will apply with their tabulated accuracy to the true earth and to the results from formula (1) and tables 2 and 3.

The method of construction of tables 2 and 3 eliminates that part of the effect of the ellipticity of the earth which expresses itself in the difference between the ordinary, geodetic or geographical latitude (which is that used in the tables and on all ordinary maps), and the geocentric latitude. This effect must be considered in any attempt to make direct use of equation (2), since neglecting it will introduce comparatively large errors. As an example, suppose formula (1) and the tables used to find a distance between two points at latitudes 32° and 36°, differing 4° in longitude. The mean latitude Φ is then 34°. Tables 2 and 3 give $\Delta x = 369.552$ and $\Delta y = 443.688$, whence $\Delta = 577.432$ kilometers. On the other hand, equation (2) with $\Phi_1 = 32°$, $\Phi_2 = 36°$, and $\Delta\lambda = 4°$, gives $\cos \Delta = 0.9958928$, whence $\Delta = 5° \, 11' \, 40''.9 = 5.19469°$. Now the length of 1° of a great circle at the mean latitude is that given in table 1 for 1° of the meridian centering at 34°, namely, 110.9204 km. Multiplying this by the last value for Δ expressed in degrees, we find $\Delta = 576.197$. There is a discrepancy of 1.235 km. between the two results.

The error originates in the use of geodetic latitudes in equation (2). The computed value of $\cos \Delta$, and of Δ derived from it, should refer to points at geocentric latitudes 32° and 36°. The corresponding geodetic latitudes are $32° \, 10' \, 30''$ and $36° \, 11' \, 07''$. The arc of the meridian between these latitudes is found by interpolation in table 1 to be 444.814 km., so that 1° of the great circle is 111.203 km. Multiplying this by 5.19469 gives $\Delta = 577.665$ km., which is probably the best result obtainable by simple means. The values of Δx and Δy used in formula (1) also need correction, since the geodetic mean latitude Φ should now be taken as $34° \, 10'.8$, while the difference in latitude is not $240'$ but $240'.61$. Tables 2 and 3, then, give $\Delta x = 368.760$ km., $\Delta y = 444.816$ km., whence $\Delta = 577.793$ km. A more exact value of Δx, derived from table 1, is 368.770 km., while more exactly $\Delta y = 444.814$ km. as found earlier. These give $\Delta = 577.798$ km. as the best result using formula (1). The correction to this derived from formula (6) is -0.0127 km., giving a corrected result of $\Delta = 577.671$ km. This differs from the result of (2) by 0.006 km., which is due to slight rounding-off errors.

The quantities Δx and Δy are independently useful in least-square determinations of epicenter and origin time. Their quotient gives the tangent of the angle which measures the azimuth of the station at the epicenter.

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