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A FAST, CONVENIENT PROGRAM FOR COMPUTATION OF SURFACE-WAVE DISPERSION CURVES IN MULTILAYERED MEDIA

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ABSTRACT

Surface wave analysis has become an important tool for exploration of crustal and mantle structure. The need exists for fast, convenient digital computer programs for computing theoretical dispersion curves and displacements for Rayleigh waves and Love waves. One such program for an IBM 7090 computer is described and made available to the scientific community. Among the conveniences are mail-order service, high speed, and choice of many options.

Introduction

In recent years, surface waves have been used extensively to determine crustal and upper mantle structure. More recently, surface waves have been used to infer some properties of the source. The introduction of electronic digital computers in determining dispersion curves in multilayered media is responsible for the renewed effort in this field (Stoneley, 1955; Dorman, Ewing, and Oliver, 1960). Whereas previously, tedious hand calculations for oversimplified models were necessary to interpret dispersion data, the use of computers permits rapid calculation of dispersion curves for realistic models. Dorman *et al* (1960) demonstrated that the accuracy of the multilayered approximation for continually varying elastic-depth functions is adequate when the number of homogeneous layers used to represent the function is sufficiently large (say, 20 layers to represent the outermost 300 km of the earth).

With the wider distribution of long period seismographs now taking place, the importance of surface wave dispersion analysis as a tool for geophysical exploration will increase further. Fast, convenient programs for computation of the dispersion curves will be needed. In this paper we make available our results for such a program based on our experience in coding the dispersion problem for three different computers.

We first programmed the dispersion computation for our small computer, the Bendix G-15D. Although this program gave the required precision, the speed was not high enough for large scale use (it required an overnight, unattended run for a typical 20 layer Rayleigh wave dispersion calculation). Much was learned from this initial attempt and a subsequent one written for an IBM 704 computer. Our final program is for the IBM 7090 computer.

Among the conveniences of the program are the following: speed, ability to specify one or more Love or Rayleigh modes, automatic layer reduction, optional determination of displacements at depth, optional insertion of approximate correction due to curvature of earth, root prediction for increasing speed, alternate group velocity computation methods, and ability to handle laminated plates and combinations of solid and liquid layers.

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METHOD

As for Dorman *et al* (1960), the theoretical basis for this program is the Thomson-Haskell matrix formulation (Haskell, 1953) for the dispersion of Rayleigh and Love type surface waves on a plane layered elastic media. In the Thomson-Haskell formulation the fundamental relationship is the motion stress vector equation relating the vector at one side of an elastic solid layer to the vector at the other side of the same layer, or

$$U_m = a_m{}^R U^*_{m-1} ; \qquad V_m = a_m{}^L V^*_{m-1} \tag{1}$$

where

$$U_m = (\dot{u}_m/c,\ \dot{w}_m/c,\ \sigma_m ,\ \tau_m); \qquad V_m = (\dot{v}_m/c,\ Y_{zm}).$$

The quantities $\dot{u}_m$ and $\dot{w}_m$ are the horizontal and vertical particle velocities respectively, which are associated with Rayleigh type motion, and $\dot{v}_m$ is the transverse horizontal velocity which is associated with Love type motion. The symbols σ_m, τ_m, and Y_{zm} represent the normal stress, tangential stress in the direction of wave propagation, and the transverse tangential stress, respectively. The asterisk on the U^*_{m-1} and V^*_{m-1} vectors indicates that the vector refers to the side of the m^{th} layer next to the $m - 1^{\text{th}}$ layer. The $a_m{}^R$ and $a_m{}^L$ are the solid layer matrices for Rayleigh and Love type motion respectively. For the former this is a four by four matrix and for the latter a two by two. Each matrix is a function of phase velocity, period, layer thickness, and elastic constants of the layer.

Now at a welded contact, the motion stress vectors are continuous across the interfaces from one layer to the next, thus by repetitive substitution of eq. (1) and matrix multiplication, we have a relationship between two motion stress vectors separated by any number of welded elastic solid layers. With this relationship and the boundary conditions of the particular problem being considered, one obtains transcendental relationships which are functions only of the matrix elements: phase velocity, period, and the elastic constants of the media. The usual boundary conditions used are the vanishing of stress at the free surfaces and the vanishing of motion at infinite depth in the underlying half space.

Whenever liquid layers are involved, the matrix multiplication is performed for the solid layers above and below the liquid layer. Then the transcendental equation which relates the vertical stress to vertical particle velocity ratio at one side of the liquid layer to the other side is used as the basic phase velocity-period equation, since these quantities are continuous across liquid-solid interfaces.

A by-product of solving the period equation for Rayleigh waves is the horizontal to vertical particle velocity amplitude ratio at the free surface. Therefore, with this value and by iterative use of eq. (1) the corresponding vertical and horizontal particle velocity amplitude ratios with respect to the vertical free surface amplitude can be calculated at each interface. For Love waves the iterative use of eq. (1) gives the horizontal to transverse velocity amplitude ratio with respect to the free surface amplitude at each interface.

The program is written in FAP for the IBM 7090 and can be used to find the dis-

persion (phase velocity versus period) and displacement or velocity ratios at each interface for Rayleigh or Love type motion in the following problems: elastic layers over an elastic half space, where up to two liquid layers are possible; laminated plates, or elastic layers as above, bounded on both sides by free surfaces; and finally, a curvature approximation for the above arrays. If two liquids are used, one must be the surface layer. The half space can be either solid or liquid, but if it is a liquid, it must be counted as one of the two possible liquids.

The general computational procedure is to find the zeros of a function F (the characteristic or frequency function), which is a function of phase velocity, wave number, and the elastic constants of the layers. This is done by initially specifying the phase velocity and a trial value of the wave number k. The elements of the matrix in eq. (1) are formed for each layer and then multiplied by the matrix for the layer below it (increasing m index), starting with the solid layer nearest the free surface. The numerical value of the F function is then calculated from elements of the final product matrix and stored. The F function is formed so that its value is positive for k's less than the root in the gravest mode. Thus new trial values of k (of increasing or decreasing size depending on whether the initial F is positive or negative respectively) are used until a change in sign of F is detected. Linear interpolation and extrapolation are then repeatedly used to find smaller F values until k's corresponding to F's of different sign are within the precision interval desired. The root is then designated as the resulting interpolated value of k.

After each root has been found for Rayleigh waves, the elements used to calculate F are recomputed using the root k. These new elements are then combined to form the horizontal to vertical particle velocity amplitude ratio at the free surface. Then the group velocity is computed by numerical differentiation of the phase velocity values. This is accomplished by perturbing the phase velocity c by a small amount and then finding a new k root to be used in the difference relation. If the displacement option has been selected, the program will now calculate the particle velocity ratios at each interface by iterative use of eq. (1).

The phase velocity is decremented by a specified amount, and then a new k and group velocity are calculated by repeating the above procedure. At first the initial trial k is the root of the previous phase velocity, but after three roots have been found, the trial k for the next phase velocity is determined from the three previous roots by Newton's formula for backward extrapolation. The program selects the normal mode order by counting the number of zero crossings of the F function for a given phase velocity as k increases from zero. The fundamental mode being the first zero crossing. In this way any particular mode can be specified.

Some Programming Problems

In most problems phase velocity decreases as k increases, and since numbers of the order of $\exp\left(k\sum_{i=1}^{n-1} d_i\right)$, where d_i is the layer thickness are involved in calculating F, the larger root values of k will lead to machine overflow (even though floating point arithmetic is used) if the total number of layers remains constant. Now Haskell showed in the case of solid layers overlying a solid half space, that whenever the phase velocity became less than the shear velocity of a layer, say the i^{th} layer, and also less than the shear velocities of the layers between that layer

and the half space, that the F function in the form he used can be factored at large k. Only one of the factors is a function of k, the rest of the factors represent the Stoneley wave velocity equations for the solid-solid interfaces between the layer and the half space when individually equated to zero. The k dependent factor, when equated to zero, is equivalent to an F function where the half space and the layers between it and the i^{th} layer have been replaced by a half space with the same elastic constants as the i^{th} layer. This also holds for problems where liquid layers are included provided the F function is arranged properly. Physically this means that the larger the k value the shorter the wave length and thus the shallower the depth of penetration for surface waves and the smaller the influence of the deeper layers. We use this feature of surface waves in our program to prevent overflow and reduce computing time. Thus when the F factor becomes larger than a specified amount, the program will reduce the number of layers, starting with the half space and will continue to eliminate layers with shear velocity greater than the phase velocity until the F function is small again. Then before proceeding to calculate the new root, the program will recalculate the root for the preceding phase velocity in order to verify that no loss in the precision of k was caused by the layer reduction.

Precision problems occur in another way. Most operations are executed in the single precision mode. There are two exceptions to this which rarely occur but whenever they arise the machine automatically shifts to double precision calculations. The first case is when the computed value of F (including its sign) has lost significance in single precision for values of k which are separated by more than the required root precision. The second case occurs whenever the k root for the perturbed phase velocity is identical in single precision with the unperturbed phase velocity root in calculating the group velocity. If the perturbation of c is too small, especially when k is a slowly varying function of c, the shift to double precision is sometimes not enough to give a significant difference in k. Therefore the group velocity will be in error. As a check against this, group velocities are also calculated from successive roots, using the c's for which the k roots have been calculated. These group velocities are printed out at the end of each problem.

As mentioned earlier, it is also possible to reduce the number of layers used in computation as k increases for the liquid layer cases, but care must be taken that the terms in the F function are properly arranged so that mode jumping does not occur at high frequencies. This is because for values of c less than the compressional velocity of the liquid and for kd_l large ($d_l \equiv$ liquid thickness), the F function can be factored into two terms, both of which are functions of k. One term when equated to zero is equivalent to the period relationship or F function for solid layers between a solid half space on one side and a liquid half space on the other. The other factor when equated to zero corresponds to the case for solid layers between a liquid half space and a free surface. At still larger k the latter term factors asymptotically into two more terms. The one which is not a function of k when equated to zero is equivalent to the Stoneley velocity equation for the liquid-solid interface at the liquid half space. The other is the F function where the liquid half space has been replaced by a solid half space with the same elastic properties as the solid layer above the liquid. Thus since the liquid case factors into two terms which are both functions of k, it is possible to make the unfortunate arrangement

of terms of the F function so that one of these factors appears in the denominator of the expression. Then at large k it will be possible to find only the root which corresponds to the other factor. This was accidentally done in the first liquid case programs and the factor which represented the solid layers above the liquid layer near the free surface was lost at large k. Thus instead of approaching the curve for solid layers without a liquid layer at short wavelengths, the dispersion curve asymptotically approached the Stoneley velocity for the solid-liquid interface below the liquid layer.

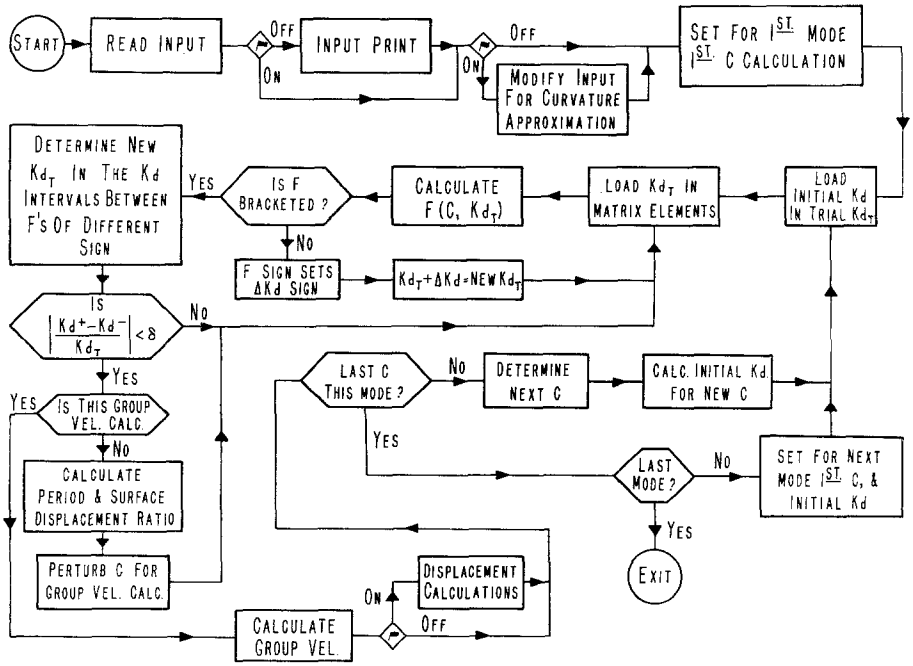

FIG. 1. Simplified flow diagram of dispersion computation.

RESULTS

Our program was designed for "mail order" service on an IBM 7090 computer. The user fills out an order sheet specifying the layer constants (up to 500 layers), the particular modes desired, the initial phase velocity, the final phase velocity, and the number of roots. An initial guess of wave number is desirable but unnecessary. Any of the following options may also be specified:

1) Love or Rayleigh wave dispersion or both in layered medium with or without underlying solid or liquid half space.
2) One or two liquid layers for Rayleigh wave dispersion.
3) Displacements at each interface and at selected depths in half space normalized to surface displacement.
4) Approximate allowance for curvature[1] valid for periods up to 200–300 sec.

[1] Alterman, Jarosch and Pekeris (1961) showed that the curvature of the Earth affects phase velocities for periods greater than 50 sec. and group velocity for periods greater than 250 sec. They further showed that a linear modification of layer velocities was sufficient to yield phase velocities correct to 1% for periods up to 300 sec. The linear increase is given by the factor $(1 + z/a)$ where z is depth from the surface and a is the earth radius.

RAYLEIGH WAVE DISPERSION CASE 104 (B2) PAGE 1

N= 18 DELT=.100 -04 DKD1=.05000 KDR= .5000 SUM DI= 1100.00
L=0, 0 CPRT=1.0010 DKD2=.10000 KDN= .0010 DRED=.100 +21..2500

D	ALPHA	BETA	RHO
22.00	6.0300	3.5300	2.7800
15.00	6.7000	3.8000	3.0000
13.00	7.9600	4.6000	3.3700
25.00	7.8500	4.5000	3.3900
25.00	7.8500	4.4100	3.4200
5.00	5.0000	2.0000	3.2400
20.00	7.8500	4.4100	3.4200
75.00	8.0000	4.4100	3.4500
50.00	8.2000	4.5000	3.4700
100.00	8.4000	4.6000	3.5000
100.00	9.0000	4.9500	3.6300
100.00	9.6300	5.3100	3.8900
100.00	10.1700	5.6300	4.1300
100.00	10.5850	5.9150	4.3300
100.00	10.9550	6.1400	4.4900
100.00	11.2750	6.2850	4.6000
150.00	11.4600	6.3840	4.6900
S-INF S	11.7550	6.5000	4.8000

MODE M 1, 1

KD	T	C	U	W*	N	SUM DI
.55520	65.347	3.8100	3.6900 *	-.82932	16	825.00
.56552	64.189	3.8080	3.6972 *	-.82992		
.57712	62.932	3.8060	3.7050 *	-.83044		
.59045	61.543	3.8040	3.7136 *	-.83084		
.60631	59.965	3.8020	3.7234 *	-.83107		
.62631	58.081	3.8000	3.7346 *	-.83097		
.65485	55.578	3.7980	3.7486 *	-.83013		
.65485	55.578	3.7980	3.7486 *	-.83013	14	618.75
1.01236	35.970	3.7960	3.7898 *	-.77585		
1.01235	35.970	3.7960	3.7898 *	-.77585	13	464.06
1.01236	35.970	3.7960	3.7898 *	-.77585	11	348.05
1.05076	34.674	3.7940	3.7610 *	-.76796		
1.07915	33.780	3.7920	3.7298 *	-.76215		
1.10255	33.080	3.7900	3.7076 *	-.75741		

GROUP VELOCITIES CALCULATED FROM SUCCESSIVE ROOTS

KD BAR	T	C BAR	U
.56036	64.763	3.8090	3.7004
.57132	63.554	3.8070	3.7085
.58378	62.230	3.8050	3.7174
.59838	60.744	3.8030	3.7275
.61631	59.007	3.8010	3.7394
.64058	56.802	3.7990	3.7541
.83360	43.672	3.7970	3.7923
1.03156	35.310	3.7950	3.7413
1.06495	34.221	3.7930	3.7180
1.09085	33.426	3.7910	3.6978

FIG. 2. Output format for Rayleigh wave dispersion calculation.

5) Printout of details of "F" value calculation to determine root regions in the c, k plane.

The computation speed for Rayleigh waves is such that a 20 layer dispersion curve with 20 values of phase and group velocity can be obtained in 2 minutes. Love wave calculations are approximately eight times faster.

A simplified flow diagram of the program appears in fig. 1. The sequence of computation and the flags for the various options are shown. Sample results for typical calculations appear in figs. 2 and 3.

RAYLEIGH WAVE DISPERSION CASE 16 (C1) PAGE 1

N= 24 DELT=.100 -04 DKD1=.01000 KDR= .0400 SUM DI= 2500.00
L=0, 7 CPRT=1.0010 DKD2=.10000 KDN= .0010 DRED=.100 +21,.2500

D	ALPHA	BETA	RHO
22.00	6.0300	3.5300	2.7800
15.00	6.7000	3.8000	3.0000
13.00	7.9600	4.6000	3.3700
25.00	7.8500	4.5000	3.3900
50.00	7.8500	4.4100	3.4200
75.00	8.0000	4.4100	3.4500
1.00	4.5900	.0000	3.2400
49.00	8.2000	4.5000	3.4700
100.00	8.4000	4.6000	3.5000
100.00	9.0000	4.9500	3.6300
100.00	9.6300	5.3100	3.8900
100.00	10.1700	5.6300	4.1300
100.00	10.5850	5.9150	4.3300
100.00	10.9550	6.1400	4.4900
100.00	11.2750	6.2850	4.6000
150.00	11.4600	6.3840	4.6900
200.00	11.7550	6.5000	4.8000
200.00	12.0200	6.6100	4.9100
200.00	12.2800	6.7400	5.0300
200.00	12.5400	6.8500	5.1300
200.00	12.8000	6.9600	5.2400
200.00	13.0200	7.0000	5.3400
200.00	13.2400	7.1000	5.4400
S-INF S	13.4800	7.2000	5.5400

MODE M 1, 1

KD	T	C	U	W*	N	SUM DI
.03222	715.031	6.0000	5.1579	-.45504 -01	24	2500.00

DISPLACEMENT CALCULATION

M	UN/WI	WN/WI
0	-.45503630 -01	.10000000 +01
1	-.13283793 -01	.99978399 +00
2	.86607442 -02	.99908266 +00
3	.27661902 -01	.99836610 +00
4	.64136379 -01	.99601465 +00
5	.13667874 +00	.98745370 +00
6	.24392137 +00	.96532759 +00
6*	.29679528 +00	.96532759 +00
7	.29519280 +00	.96501878 +00
7*	-.74332377 +00	.96501878 +00
8	-.67618555 +00	.97324877 +00
9	-.55276150 +00	.97850528 +00
10	-.44490407 +00	.97347850 +00
11	-.34954453 +00	.96247401 +00
12	-.26515488 +00	.94648387 +00
13	-.19036464 +00	.92574350 +00
14	-.12455331 +00	.90137411 +00
15	-.68114199 -01	.87407532 +00
16	-.12265012 -02	.82732473 +00

FIG. 3. Output format of displacements depth calculation.

The notation is as follows:

D	Layer thickness
ALPHA	Compressional wave velocity in layer
BETA	Shear wave velocity in layer
RHO	Density in layer
KD	Dimensionless wave number with respect to top layer
T	Period in sec.
C, U	Phase and group velocity in km/sec. Asterisk on U indicates automatic use of double precision

W	Ratio of vertical to horizontal displacement at free surface
N	Number of layers used in computation. Change in N indicates automatic layer reduction is operative
M	Interface at which displacement is calculated starting at free surface; repeated number with asterisk indicates liquid-solid interface
UN/WI, WN/WI	Horizontal and vertical displacement at M^{th} interface with respect to vertical displacement at free surface (floating point notation)

Fig. 2 shows output format (printed off-line) for an 18 layer calculation of Rayleigh wave dispersion between periods of 65 to 33 sec. The layer constants are printed in the upper part and the results for the Rayleigh mode (M 1, 1) appear in the lower part. Group velocity was computed by both methods described earlier. Automatic layer reduction and switching to double precision was used. Fig. 3 illustrates results for displacements at depth for Rayleigh waves in a 24 layer structure with an intermediate liquid layer. Note continuity of vertical displacement and discontinuity in horizontal displacement at liquid-solid interfaces.

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References

Alterman, Z., H. Jarosch, and C. L. Pekeris
1961. "Propagation of Rayleigh Waves in the Earth," *Technical Report* of the Weizmann Institute, Rehovot, Israel.

Dorman, J., M. Ewing, and J. Oliver
1960. "Study of Shear-Velocity Distribution in the Upper Mantle by Mantle Rayleigh Waves," *Bull. Seism. Soc. Am.*, 50: 87–115.

Haskell, N. A.
1953. "Dispersion of Surface Waves on Multilayered Media," *Bull. Seism. Soc. Am.*, 43: 17–34.

Stoneley, R.
1955. "Rayleigh Waves in a Medium with Two Surface Layers (second paper)," *Mon. Not. Roy. Astron. Soc.*, 7: 71–75.

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