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THE FIELD FROM AN *SH*-POINT SOURCE IN A CONTINUOUSLY LAYERED INHOMOGENEOUS MEDIUM:

I. THE FIELD IN A LAYER OF FINITE DEPTH

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ABSTRACT

Expressions are derived for the field from an *SH* point source in a stratified heterogeneous layer of finite depth. It is found, that for a periodic disturbance, the contribution to the far field is mainly due to at most a finite number of unattenuated normal Love modes. The transient response of the medium is obtained by a Fourier synthesis. The final expressions are of a simple form, involving the eigenfunctions of a Sturm-Liouville boundary value problem. The excitation of a certain mode as a function of frequency and source depth is formulated in a concise form.

1. Introduction

In recent times it has become possible, by means of electronic computers, to obtain dispersive properties and the amplitude versus depth dependence of surface waves, as a function of period, for any parallel layered heterogeneous halfspace in which propagating surface waves exist. One of the main aims in recent investigations concerned the influence of the presence of a low velocity layer (Gutenberg channel) on the dispersive properties and amplitudes of surface waves. This problem has been approached from two different sides.

One approach is by assuming the medium to consist of plane-parallel homogeneous layers of different elastic properties and densities, the other is by assuming that the halfspace has piecewise continuous properties, which vary with depth. By the first method, solutions in each homogeneous layer are found involving an exponential or sinusoidal behavior with depth. These solutions have to match boundary conditions at each interface separating two layers, (the Thomson-Haskell method). By the latter method, the problem is cast in a shorter form by the requirement to solve numerically one second order differential equation in the case of Love waves. This method is convenient for the study of the propagation of free surface waves.

The analysis of forced surface waves emitted from sources within the medium has been done only in the case of a multilayered elastic medium, (Harkrider, 1964). The analysis is rather complicated as the analytical solution in each layer is involved. As far as the present author is aware, no attempt has been made to give a straightforward analysis for the case that sources are imbedded in a medium in which the inhomogeneity is both vertical and arbitrary.

In this paper, a beginning has been made by considering the relatively simple case of the field from an *SH*-point source in a heterogeneous layer of finite depth, with its lower boundary rigidly fixed. The corresponding case of free Love wave propagation has been studied extensively by Hudson (1962).

No assumptions have to be made regarding the way in which the properties of the medium vary with depth and no analytical expressions for the depth dependence appear. The method reduces to finding an expression for the Green's function of a second order Sturm-Liouville boundary problem.

The final expression for the field is cast in a very simple form. Once a numerical solution of a normal mode as a function of depth and frequency is known, the measure of excitation of this mode as a function of source depth and frequency can be found by simple means.

2. The Formal Solution

2.1 We shall consider a medium that consists of a layer of finite depth ($0 < z < H$), bounded by the planes $z = 0$ and $z = H$. The layer is considered to be an isotropic, elastic solid, where the rigidity and the density are arbitrary continuous functions of the z-coordinate only. The halfspace $z < 0$ is assumed to be vacuum which gives rise to the boundary condition of vanishing shear stress at $z = 0$. The halfspace $z > H$ is assumed to be a perfectly rigid solid giving rise to the displacement at $z = H$.

At a depth $z = z_0$ ($0 < z_0 < H$), a point source emits a transient disturbance of the SH type. Thus, the problem can be considered to have axial symmetry with respect to a z-axis passing through the point source and so the use of cylinder coordinates r, ϕ, z is justified. In the case of an SH point source the displacement vector $\bar{s}$ has only its azimuthal component different from zero:

$$\bar{s} = (0, v_0, 0)$$

which component, by symmetry properties is independent of the azimuthal variable ϕ. From the equations of elastodynamics it follows that $v_0(r, z, t)$ satisfies a scalar wave equation given by:

$$1/r\partial/\partial r(r\partial v_0/\partial r) + 1/\mu\partial/\partial z(\mu\partial v_0/\partial z) - \rho/\mu\partial^2 v_0/\partial t^2 = -2\delta(r)/r\cdot\delta(z - z_0)F(t), \tag{2.1}$$

where $\mu(z)$ and $\rho(z)$ are the shear modulus and the density respectively.

The right hand member of (2.1) containing delta functions represents the field singularity due to the presence of the source at $r = 0$, $z = z_0$, and gives the action in the source as a function of the time t by the function $F(t)$, which is assumed to vanish for $t \leqq 0$ and to have a Fourier transform $g(\omega)$.

The plane $z = 0$ is stress-free and so the component of the stress tensor

$$\tau_{z\phi} = \mu\partial v_0/\partial z$$

will vanish for $z = 0$. This leads to the boundary condition

$$\partial v_0/\partial z = 0 \quad \text{at} \quad z = 0. \tag{2.2a}$$

At $z = H$, the boundary condition of vanishing displacement must be satisfied, i.e.

$$v_0 = 0 \quad \text{at} \quad z = H. \tag{2.2b}$$

2.2 The transient disturbance can be obtained by Fourier synthesis of the time transformed disturbance. We thus depart from a time harmonic field by putting

$$v_0 = ve^{i\omega t} \qquad (\omega > 0)$$

where v satisfies

$$1/r\partial/\partial r(r\partial v/\partial r) + 1/\mu\partial/\partial z(\mu\partial v/\partial z) + (\omega^2\rho/\mu)v = -2\delta(r)/r\cdot\delta(z - z_0) \tag{2.3}$$

and the boundary conditions (2.2a) and (2.2b) with v_0 replaced by v.

A formal solution of (2.3) now is easily verified to be given by

$$v(r, z) = 2\int_0^\infty \mathfrak{J}_0(kr)Z(z, z_0, k)k\,dk \tag{2.4}$$

where Z has to be determined such that v satisfies source, boundary, and radiation conditions. From (2.4) it is clear that v and $2Z$ form a pair of Fourier-Bessel transforms, and so we have

$$2Z(z, z_0, k) = \int_0^\infty \mathfrak{J}_0(kr)v(r, z)r\,dr.$$

By multiplying (2,3) with $\mathfrak{J}_0(kr)r\,dr$ and integrating both members of the equation between the limits $r = 0$ and $r = \infty$, it is easily demonstrated that Z satisfies the inhomogeneous ordinary differential equation

$$d/dz(\mu\,dZ/dz) + (\rho\omega^2 - \mu k^2)Z = -\mu(z_0)\cdot\delta(z - z_0) \tag{2.5}$$

the solution of which is seen to be an even function of the parameter k.

From (2.2a, b) and (2.4) follow the boundary conditions

$$dZ/dz = 0 \quad \text{at} \quad z = 0 \tag{2.6a}$$

$$Z = 0 \quad \text{at} \quad z = H. \tag{2.6b}$$

It is no restriction to let the path of integration in (2.4) run slightly above the positive real axis from $k = 0$ to $k = \infty$ in the first quadrant of the complex k-plane.

By employing the identity

$$2\mathfrak{J}_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$$

and the following symmetry property of Hankel functions

$$H_0^{(1)}(kr) = -H_0^{(2)}(-kr)$$

and by using the fact that Z is even with respect to k, (2.4) transforms into

$$v(r, z) = \int_L H_0^{(2)}(kr)Z(z, z_0, k)k\, dk \tag{2.7}$$

where the path of integration L runs from $k = -\infty$ to $k = +\infty$, slightly below the negative real axis in the third, and above the positive real axis in the first quadrant.

As $H_0^{(2)}(kr)$ behaves exponentially damped with increasing $\mathfrak{Im}k$, the value of the integral (2.7) will depend on singular points of the integrand, and thus of Z, which are situated below L in the lower half of the k-plane, at least if Z behaves properly at infinity.

2.3 The remaining task is to determine the character and location of the singular points of Z in the k-plane, that is, to find a solution of (2.5) subject to the boundary conditions imposed.

This reduces to the theory of ordinary second order differential equations on a finite interval, much of which can be found in standard texts such as Ince (1927).

In the present paper the theory is extended and modified, if necessary, to make it relevant to the subject under consideration.

The theory involved for this simple case is the classical Sturm-Liouville and associated Green's function theory.

In the present paper $Z(z, z_0, k)$ will prove to be a meromorphic function of k, i.e., an analytic function having an infinity of discrete poles as the only singularities.

3. Eigenvalues and Eigenfunctions

3.1 We will investigate equation (2.5) on the finite closed interval $0 \leqq z \leqq H$. The solution $Z(z, z_0, k)$ is to satisfy the following boundary conditions

$$dZ/dz = 0 \quad \text{at} \quad z = 0 \tag{3.1a}$$

$$Z = 0 \quad \text{at} \quad z = H. \tag{3.1b}$$

It should be remarked here that these boundary conditions could be replaced by any linear, real homogeneous boundary condition involving Z and its derivative at $z = 0$ and $z = H$.

The homogeneous equation corresponding to (2.5) is given by:

$$d/dz(\mu\, d\phi/dz) + (\rho\omega^2 - \mu k^2)\phi = 0 \tag{3.2}$$

Combined with (3.1a, b) with Z replaced by ϕ, equation (3.2) constitutes a self adjoint problem of the Sturm-Liouville type, for real k^2.

3.2 The following statements and derivations are largely based on well-known results of the classical Sturm-Liouville theory.

(1) As $\mu(z)$ is positive and assumed to be continuous (any discontinuity in μ can be approximated as closely as desired by a continuous μ) on $0 \leqq z \leqq H$, (3.2) possesses two linear independent regular solutions. Any solution can be obtained as a linear combination of the two.

(2) A real solution ϕ can be found which fits the boundary condition (3.1a) at $z = 0$. This solution is an integral function of k^2, i.e. is analytical for all complex finite values of k^2. *A forteriori* ϕ is an integral function of k as well.

(3) If ϕ is to satisfy also condition (3.1b) it is called an eigenfunction. This is possible only for an infinity of discrete real values of k^2: $k_0^2, k_1^2, k_2^2, \cdots$, which can be arranged in decreasing order of magnitude

$$k_0^2 > k_1^2 > k_2^2 > \cdots$$

having an upper bound for any finite value of $\rho\omega^2$ and the sole limit point $k^2 = -\infty$.

The value of k^2 corresponding to the i-th eigenfunction is called the i-th eigenvalue k_i^2. The corresponding eigenfunction is

$$\phi(z, k_i^2) = \phi_i(z).$$

From the self adjointness of the problem follows the fact that the eigenvalues are real and also the following orthogonality property of the eigenfunctions:

$$\int_0^H \mu\phi_i \cdot \phi_j \, dz > 0 \qquad \text{if } i = j$$

$$= 0 \qquad \text{if } i \neq j.$$

The orthogonal functions ϕ_i can be normalized by introducing the orthonormal functions ψ_i :

$$\psi_i = \phi_i \Big/ \left\{ \int_0^H \mu\phi_i^2 \, dz \right\}^{1/2}$$

for which holds:

$$\int_0^H \mu\psi_i \psi_j \, dz = \delta_{ij} \qquad (\delta_{ij} = 1 \text{ if } i = j, = 0 \text{ if } i \neq j). \quad (3.3)$$

(4) Let ϕ be a solution of (3.2) and ϕ_+ a solution of the same differential equation with k^2 replaced by k_+^2. From the pair of corresponding differential equations Green's formula can be derived:

$$[\mu(\phi_+\phi' - \phi\phi_+')]_0^z = (k^2 - k_+^2) \int_0^z \mu\phi\phi_+ \, dz. \quad (3.4)$$

The quantity in brackets we denote by $D(\phi_+, \phi)$. If $k_+^2 = k^2$, thus if ϕ and ϕ_+

are two (independent or dependent) solutions of the same equation (3.2), the first member of (3.4) will vanish and so $D(\phi_+, \phi)$ is independent of the variable z and an integral function of k only.

(5) Let ϕ_i be the eigenfunction corresponding to the eigenvalue k_i^2 and let $\phi = \phi(z, k^2)$ be a solution of (3.2) which only satisfies the condition (3.1a) at $z = 0$. Now put in (3.4) $\phi = \phi$ and $\phi_+ = \phi_i$. The quantity $D(\phi_i, \phi)$ vanishes at $z = 0$ as a consequence of ϕ and ϕ_i satisfying the same initial conditions at $z = 0$. We then arrive at

$$\mu(H)\phi_i'(H, k^2)\phi(H, k^2) = (k_i^2 - k^2) \int_0^H \mu\phi_i\phi \, dz$$

from which it follows that $D(\phi_i, \phi)$ has a simple zero at $k^2 = k_i^2$ as the integral

$$\int_0^H \mu\phi_i\phi \, dz$$

tends to a positive limit as $k^2 \to k_i^2$. Hence the eigenvalues k_i^2 are simple roots of the equation

$$D(\phi_i, \phi) = 0.$$

If we choose ϕ_i to be the normalized eigenfunction ψ_i and ϕ to be a solution ψ, which for $k^2 \to k_i^2$ tends to the eigenfunction ψ_i, we see that

$$\{D(\psi_i, \psi)\}^{-1}$$

is a function having simple poles in the k-plane at $k^2 = k_i^2$ with a residue equal to unity in virtue of the property (3.3).

4. Green's Function

4.1 Apart from the multiplicative factor $\mu(z_0)$, the solutions of (2.5) subject to the boundary conditions (3.1a, b) is the Green's function of the boundary value problem concerned.

As the variable of integration k in (2.7) is in the first and third quadrant of the k-plane, it follows that Im k^2 is positive. So k^2 will not coincide with a real eigenvalue of the corresponding homogeneous problem. (We can choose ω such that $k^2 = 0$ is not an eigenvalue.)

Let $\phi(z, k^2)$ and $\eta(z, k^2)$ be solutions of the homogeneous equation, ϕ satisfying the boundary condition imposed on Z at $z = 0$, η that at $z = H$:

$$d\phi(0, k^2)/dz = 0, \qquad \eta(H, k^2) = 0.$$

We now put

$$\begin{aligned} Z(z, z_0, k) &= \mu(z_0)\phi(z, k^2)\eta(z_0, k^2)/D(\eta, \phi) \qquad z < z_0 \\ &= \mu(z_0)\eta(z, k^2)\phi(z_0, k^2)/D(\eta, \phi) \qquad z > z_0 \end{aligned} \tag{4.1}$$

where k^2 is the complex parameter in (2.5). The quantity $D(\eta, \phi)$ being independent of z, may be evaluated at $z = z_0$ or at $z = H$.

It is clear that for $z \neq z_0$, Z satisfies the homogeneous equation (3.2) and the boundary conditions (3.1a, b).

At $z = z_0$ the derivative of Z makes a jump of magnitude -1 and so Z satisfies also the imhomogeneous equation (2.5) with the delta function in the right-hand member. Z is thus the required solution of (2.5).

4.2 Let ϕ and η be such that they both tend to the normalized eigenfunction ψ_i if $k^2 \to k_i^2$.

In virtue of D^{-1} having simple poles at $k^2 = k_i^2$ with residue equal to one, $Z(z, z_0, k)$ also has simple poles at $k^2 = k_i^2$. In the neighborhood of k_i^2 we then have

$$Z \sim \mu(z_0)\psi_i(z)\psi_i(z_0)/(k^2 - k_i^2).$$

It then follows that the residue of Z at the simple pole $k = k_i$ in the k-plane is given by

$$\text{Res}\, Z(z, z_0, k_i) = \mu(z_0)\psi_i(z)\psi_i(z_0)/2k_i. \tag{4.2}$$

5. The Time Harmonic Field

5.1 In order to shift the path of integration L in (2.7) into the lower half of the k-plane, poles of Z must be taken into account and Z must behave properly at infinity.

As a consequence of Z being regular on $0 \leqq z \leqq H$ if $k^2 \neq k_i^2$, Z is bounded except for a neighborhood of its poles. (We must require that on the shifted path $| k - k_i | > \delta$, where δ is an arbitrarily small positive quantity.) This property then holds uniformly in the k-plane.

In virtue of the asymptotic behavior of the Hankel Function for large values of its argument

$$H_0^{(2)}(kr) \sim (2/\pi kr)^{1/2} \exp\{i(\pi/4 - kr)\} \qquad | kr | \gg 1, \tag{5.1}$$

it is justified to replace the integral in (2.7) by the residue contributions from the poles which are situated on the positive real axis and in the lower half plane, at least if ω^2 is such that a pole k_i is not at $k = 0$.

5.2 All eigenvalues k_i^2 are real and distinct. As there is an upper bound for k_i^2, there is at most a finite number of positive real eigenvalues and in any case an infinity of negative k_i^2. If $k_i^2 > 0$, the corresponding poles are at $k = \pm k_i$ on the real axis. If $k_i^2 < 0$ the poles are on the imaginary axis at $k = \pm k_i$, k_i being a purely imaginary quantity in this case. ($k_i^2 = 0$ has been excluded.)

In virtue of Cauchy's theorem we can put

$$\psi = \int_L H_0^{(2)}(kr)Z(z, z_0, k)k\, dk = -\pi i \sum_i H_0^{(2)}(k_i r)\mu(z_0)\psi_i(z)\psi_i(z_0) \tag{5.2}$$

where the summation extends over the poles situated on the positive real and the negative imaginary axis.

For r large (and $k_i \neq 0$) each of the terms in (5.2) behaves like

$$\mu(z_0)(2\pi/k_i r)^{1/2} \exp\{i(-\pi/4 - k_i r)\}\psi_i(z)\psi_i(z_0).$$

This form shows that for negative imaginary k_i the contribution from the corresponding term is exponentially damped with increasing distance r from the source and does not represent a progressing wave. The contribution to the far field thus comes mainly from at most a finite number of real positive k_i. Each of the corresponding terms in (5.2) represents a cylindrically damped $(O(r^{-1/2}))$ sinusoidally varying outgoing wave, propagating with phase velocity $c_i = \omega/k_i$. The term corresponding to real k_i is the i-th normal Love mode of seismology

$$\bar{N}_i(r, z, \omega) = -\pi i e^{i\omega t} H_0^{(2)}(k_i r)\mu(z_0)\psi_i(z)\psi_i(z_0). \tag{5.3}$$

5.3 In the foregoing we assumed that no eigenvalue k_i coincides with $k = 0$. This is accomplished by omitting certain values of the circular frequency ω. If ω is such that an eigenvalue is zero, the analysis becomes troublesome in virtue of the corresponding pole being situated at the path of integration and by the fact that at $k_i = 0$ the factor $H_0^{(2)}(k_i r)$ has a logarithmic singularity.

This gives rise to resonance whereby the i-th mode is excited beyond all bounds. The way of overcoming this difficulty is by giving ω a small negative imaginary part, which, as the field has to satisfy real boundary conditions, will shift the poles on the positive real and the negative imaginary axis into the fourth quadrant. Now the analysis can proceed without difficulty.

5.4 One of the interests of seismology concerns the relation between a layered heterogeneous medium and the dispersive properties of surface waves guided parallel to the layering. Dispersion in every distinct mode is caused by the horizontal phase velocity $c_i(\omega) = \omega/k_i(\omega)$ being a function of the circular frequency ω. The relationship between c_i and ω is given by the fact that the zeros k_i of the denominator in (4.1) are dependent on ω.

Several authors have discussed the dispersive properties of free Love waves in an arbitrarily varying layered inhomogeneous halfspace.

A study of Love waves in a heterogeneous layer of finite depth (as considered in the present paper) was made by Hudson (1962). Several of his conclusions will be restated here:

(1) The i-th mode is propagated only if $c_i > \beta_0$, where β_0 is the minimum value of the shear wave velocity $(\mu/\rho)^{1/2}$ within the layer.

(2) The i-th mode has a finite cutoff frequency ω_{ci}. If $\omega > \omega_{ci}$ this mode is not propagated. In the limit if $\omega \to \omega_{ci}$ the phase velocity c_i tends to infinity, and the group velocity $U_i(\omega) = d\omega/dk_i$ tends to zero.

(3) In the high frequency limit $\omega \to \infty$ both U_i and c_i go to the same asymptotic value β_0.

(4) The phase velocity is a monotonically decreasing function of ω, $\beta_0 < c_i < \infty$ if $\infty > \omega > \omega_{ci}$. The group velocity has an upper bound β_1^2/β_0 where β_1 is the maximum shear wave velocity in the layer. We always have $U_i < c_i$.

6. The Normal Modes in the Time Domain

The response due to a transient source action can be obtained by Fourier synthesizing the time harmonic solution (5.3).

The transient action is represented by

$$F(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\, d\omega = 2 \operatorname{Re} \int_0^{\infty} g(\omega)e^{i\omega t}\, d\omega$$

where $F(t) = 0$ for $t \leqq 0$.

The corresponding response for the i-th mode then is given by

$$N_i(r, z, t) = 2\mu(z_0) \operatorname{Re}\left\{ -\pi i \int_0^{\infty} g(\omega) H_0^{(2)}(k_i r)\psi_i(z, k_i^2)\psi_i(z_0\ k_i^2) d\omega \right\}. \quad (6.1)$$

As the i-th mode is propagated only for $\omega > \omega_{ci}$, (ω_{ci} is the cutoff frequency), the lower limit of integration can be replaced by ω_{ci}. At $\omega = \omega_{ci}$ we have $k_i = 0$ and so the integrand will have a logarithmic singularity at the lower integration limit due to the presence of the Hankel function. This singularity, however, being integrable, will not be felt. For r large we then can replace (6.1) by

$$N_i(r, z, t) \sim 2\mu(z_0)(2\pi/k_i r)^{1/2} \operatorname{Re} \int_{\omega_{ci}}^{\infty} g(\omega) e^{i(\omega t - k_i r - \pi/4)} \psi_i(z, k_i^2)\psi_i(z_0, k_i^2)\ d\omega. \quad (6.2)$$

This expression can be approximated by standard asymptotic methods.

If knowledge is assumed regarding the spectral function $g(\omega)$, the dependence of the spectral response as a function of source depth and frequency is given by the asymptotic approximations to (6.2). This dependence is governed completely by the normalized eigenfunction $\psi_i(z)$ which can be evaluated numerically as a function of z and ω. The source depth influence is then simply given as the amplitude of $\psi_i(z)$ at $z = z_0$ as a function of frequency.

7. Conclusion

The analysis given in this paper leads to a very simple representation of the field for the radiation problem concerned. The influence of source depth on the normal mode contribution to the field in the frequency domain can be found easily, once numerical knowledge of free normal modes as a function of frequency is obtained. The method followed seems, therefore, preferable to the approach in which the earth is represented by a layered half space consisting of a large number of homogeneous layers. In the latter approach the analytical expressions of the field in each of the layers is involved in the final representations.

Though the present paper concerns a relatively simple problem (*SH* source and layer of finite depth), it seems desirable to extend the theory to a half space and more complicated sources involving also the propagation of Rayleigh waves.

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