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THE FIELD FROM AN *SH* POINT SOURCE IN A CONTINUOUSLY LAYERED INHOMOGENEOUS HALF-SPACE II. THE FIELD IN A HALF-SPACE

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ABSTRACT

The propagation of waves due to the presence of an *SH* point source in the interior of a piecewise continuously stratified half-space is studied. The physical parameters governing the wave propagation, i.e. the rigidity and the density, are assumed to be arbitrary piecewise continuous functions of depth with constant finite limiting values as depth goes to infinity. The analysis is based on spectral theory of boundary value problems associated with ordinary linear second order differential equations. It is found for the time harmonic case that the final field representations are given in the form of a finite residue series, plus a branch line integral, the first representing the normal mode contribution to the field. The field expression appears to have a symmetrical form with respect to field point depth and source depth, involving solutions connected with free wave propagation. This enables one to draw immediately conclusions regarding the influence of the source depth and the frequency on the spectral excitation of the normal modes if numerical knowledge of free Love waves is assumed to be known.

1. Introduction

In a foregoing paper (Vlaar, 1966, referred to as Part I in the following) expressions were derived for the field due to the presence of an *SH* point source in the interior of a stratified inhomogeneous layer of finite depth. The upper surface was assumed to be stress free, the lower to be rigid.

In the present paper the theory is extended to the case of a stratified inhomogeneous layer of finite (but arbitrary large) depth on top of a homogeneous layer which extends to infinity. There also the point source is of *SH* type. The physical parameters (shear modulus and density) are assumed to be piecewise continuous functions of the depth coordinate only, thus allowing for a finite number of discontinuities to exist in the medium. This model of a half-space comes close to the situation we find in a real earth (if sphericity has not to be taken into account).

The usual model used for the study of wave propagation in a stratified half-space is the multilayered model in which the half-space is assumed to consist of a large number of plane parallel homogeneous layers. The depth-dependence of the field in each of the layers then is sinusoidal or exponential and the solutions in each of the layers has to match boundary conditions at each of the interfaces separating two layers. The solution thus involves the analytical field expressions in each of the layers from the onset of the formulation of the problem, which makes things rather complicated.

The analysis for the case of forced surface waves due to the presence of point

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sources in a multi-layered half-space has been studied extensively by Harkrider (1964).

The propagation of free surface waves on an arbitrary continuously layered half-space has been the subject of a limited number of papers. Sato (1959) gave a presentation of surface wave propagation on a heterogeneous half-space and compared results for the case of linear velocity dependence with depth obtained by numerical and by analytical means. This half-space model was limited by the requirement of continuity. Keilis-Borok *et al* (1965) formulated the problem in terms of ordinary differential equations and indicated how these could be used for numerical purposes. Both of the publications dealt with Love as well as Rayleigh waves.

As far as the present author is aware, up to now no theory of the excitation of the field from a point source in a continuously stratified half-space has been given. In this paper, this has been done for the case of *SH* waves. Apart from its intrinsic theoretical interest, the theory proves to be useful also for seismological purposes. The final field expressions are cast in a concise form from which immediately the influence of source depth and frequency on the excitation of each Love mode is given, if once numerical knowledge regarding free Love waves is assumed to be known. The numerical dependence of the physical parameters with depth need only to be known a posteriori, when the problem has reached its final formulation.

Also included in this paper, is the derivation of criteria for the existence of Love waves on a heterogeneous half-space.

The theory involved is largely based on the spectral theory of singular boundary value problems associated with ordinary linear second order differential equations.

2. The Formal Solution

2.1 We shall consider the problem of the propagation of waves radiated from an *SH* point source located in the interior of an isotropic, stratified inhomogeneous halfspace, in which the physical properties change with the depth coordinate only.

The plane $z = 0$ separates the elastic half-space $z > 0$ from vacuum. This gives rise to the boundary condition of vanishing shear stress at $z = 0$. The shear modulus $\mu(z)$ and the density $\rho(z)$ are assumed to be piece-wise continuous positive functions of the depth z, thus allowing for finite jump discontinuities to exist in our model, as may actually be the case in the interior of the earth. The source is assumed to be located at $z = z_0$.

In Part I it was shown that the time transformed field from an *SH* point source at $z = z_0$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{\mu}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) + (\omega^2\rho/\mu)v = -\frac{2\delta(r)}{r}\,\delta(r - r_0); \tag{2.1}$$

and the boundary conditions of vanishing shear stress at the free surface

$$\mu\partial v/\partial z = 0 \quad \text{at} \quad z = 0$$

where v is the only (azimuthal) component of the displacement vector. The circular frequency ω is assumed to be positive.

It was also shown that v can formally be represented by

$$v(r, z) = \int_L H_0^{(2)}(kr)Z(z, z_0, k)k\, dk \tag{2.2}$$

where the path of integration L in the k-plane runs from $k = -\infty$ to $k = +\infty$ slightly below the real axis in the third, and above the real axis in the first quadrant. This amounts to the condition $\mathrm{Im}k^2 > 0$ on L, except at $k = 0$.

The function $Z(z, z_0, k)$ has to satisfy the inhomogeneous differential equation

$$\frac{d}{dz}\left(\mu \frac{dZ}{dz}\right) + (\rho\omega^2 - \mu k^2)Z = -\mu(z_0)\delta(z - z_0) \tag{2.3}$$

and the boundary condition

$$\mu dZ/dz = 0 \quad \text{at} \quad z = 0. \tag{2.4}$$

If Z is sufficiently well-behaving at $|k| = \infty$, the value of (2.2) depends on those singularities of Z which are situated on the positive real k-axis or below L in the lower half of the k-plane. This is due to the asymptotic behavior of $H_0^{(2)}$ as $\mathrm{Im}\, k \to -\infty$.

3. Preliminary Considerations

3.1 The problem of deriving suitable field expressions by evaluating the integral (2.2) is now reduced to the simpler one of acquiring knowledge of the singularities in the k-plane of the solution $Z(z, z_0, k)$ of (2.3) subject to the boundary condition (2.4).

The theory involved is intimately connected with self adjoint boundary value problems associated with linear second order ordinary differential equations.

In Part I the case was considered of a layer of finite depth with the boundary conditions of vanishing stress at the upper and vanishing displacement at the lower boundary. Thus, the problem was cast in the form of a Sturm-Liouville problem and associated Green's function theory, giving rise to a discrete spectrum of eigenvalues, each of which is related to a normal mode of the radiation problem.

3.2 In the present paper, we study the field in a halfspace, so the limiting case of infinite depth. The end $z = \infty$ of the interval may cause complications as the differential equation may become singular at infinity. The singular or non-singular behavior depends on the asymptotic character of $\mu(z)$ and $\rho(z)$ as $z \to \infty$.

The singular case associated with linear second order ordinary differential equations has been the subject of interest of mathematicians during several decades. Weyl was the first to master the problem. Complete presentations of the theory are given by Titchmarsh (1961) and by Coddington and Levinson (1955).

3.3 In the limit of infinite depth, we can distinguish three essentially different cases.

(a) The regular case.

$z = \infty$ is a regular point of the differential equation (2.3) and there are no further singularities on the real z-axis. This case is equivalent to the Sturm-Liouville problem, on a finite interval. However, in this case, at least one of the quantities Z or $\mu dZ/dz$ must be different from zero at either end point of the interval, in order to obtain a non-trivial solution. As we will require that the displacement as well as the stress vanishes at $z = \infty$, this case is not relevant to our investigation.

(b) The "limit circle" case.

$z = \infty$ is a singular point of the differential equation (2.3), though, if Im $k^2 > 0$, there exist two linearly independent solutions of (2.3) which both are square integrable on an interval containing $z = \infty$. In this case, also, the spectrum is completely discrete.

(c) The "limit point" case.

Here also, $z = \infty$ is a singular point. However, in contrast to the "limit circle" case, if Im $k^2 > 0$, there is only one solution of (2.3) which is square integrable on $(0, \infty)$. In this case, there exists always a continuous spectrum, apart from a possible discrete spectrum of eigenvalues.

3.4 It is difficult to find general criteria for deciding which of the above three cases is pertinent to a differential operator with arbitrary coefficient. In order to be able to get explicit knowledge regarding the character of the spectrum, it is very useful to assume an analytical behavior of the coefficients at infinity in order to obtain an analytical solution for the differential equation, from which explicit knowledge of the spectrum may be obtained.

In the present paper it is assumed that $\mu(z)$ and $\rho(z)$ tend to constant values as $z \to \infty$. This is the same as to assume that the half-space consists of a heterogeneous layer of finite depth laying on top of a homogeneous layer of infinite depth. The depth of the heterogeneous layer may be taken arbitrarily large but finite.

4. Free Love Waves

4.1 We consider the half-space to consist of a layer of finite depth h, on top of a homogeneous layer extending to infinite depth. The shear modular $\mu(z)$ and the density $\rho(z)$ are assumed to be piece-wise continuous, positive and bounded functions. At $z = h$, μ and ρ are assumed to be continuous and for $z \geqq h$ to be constant $\mu(z) = \mu_0$ and $\rho(z) = \rho_0$ leading to a constant shear velocity $\beta_0 = (\mu_0/\rho_0)^{1/2}$ in the lower layer.

Free Love waves are related to solutions of

$$\frac{d}{dz}\left(\mu \frac{d\phi}{dz}\right) + (\rho\omega^2 - \mu k^2)\phi = 0 \tag{4.1}$$

where the horizontal wave number k is real, and where ϕ satisfies the boundary conditions

$$d\phi/dz = 0 \quad \text{at } z = 0 \tag{4.2a}$$

$$\phi = 0 \quad \text{at} \quad z = \infty. \tag{4.2b}$$

For $z \geqq h$, the solution satisfying (4.2b) can be written down immediately:

$$\phi = e^{-(k^2-k_0^2)^{1/2}z} \qquad (z \geqq h) \quad (4.3)$$

(or a constant multiple)

where $k_0 = \omega/\beta_0$, $k^2 > k_0^2$, and the square root in the exponent is assumed positive.

From the last inequality we infer that a necessary condition for the existence of Love waves is given by

$$c < \beta_0; \quad (4.4)$$

i.e. the horizontal phase velocity $c = \omega/k$ must be less than the shear velocity β_0 of the lower layer.

4.2 If c is fixed and $c < \beta_0$, the above problem can be considered as a Sturm-Liouville problem on a finite interval with the boundary conditions

$$d\phi/dz = 0 \quad \text{at} \quad z = 0$$

and

$$d\phi/dz + (k^2 - k_0^2)^{1/2}\cdot\phi = 0 \quad \text{at} \quad z = h.$$

For the existence of a non-trivial solution of this boundary value problem the quantity $(\rho\omega^2 - \mu k^2)$ must be positive on a least a finite range of $(0, h)$, i.e. the condition $c > \beta$ must be satisfied somewhere on $(0, h)$, (Ince, 1927, p. 235). This implies a lower bound for c: $c > \beta_{\min}$ where $\beta_{\min}$ is the minimum value of the shear wave velocity in the upper layer. Hence, combining with (4.4), the existence of Love waves is possible only if

$$\beta_{\min} < c < \beta_0 \quad (4.5)$$

which, in turn, implies that β_0 must be greater than the minimum shear wave velocity in the heterogeneous upper layer.

4.3 For a fixed and real c the Sturm-Liouville theory leads to an infinite sequence of discrete real eigenvalues

$$\omega_0^2 < \omega_1^2 < \omega_2^2 \cdots$$

with

$$\omega_i^2 \rightarrow \infty \quad \text{as} \quad i \rightarrow \infty.$$

The solution corresponding to the eigenvalue ω_i^2 is the i-th eigenfunction $\phi_i(z)$ which has exactly i zeros on $(0, h)$.

However, our interest concerns the spectrum with respect to the parameter k for a fixed and real ω. In this case the spectrum can be derived from the foregoing.

It can be demonstrated that there exists a discrete spectrum of real positive eigenvalues $k^2 = \kappa_i^2$. These eigenvalues, however, have to satisfy

$$k_0^2 < \kappa_i^2 < k_{\max}^2$$

where

$$k_{\max} = \omega/\beta_{\min} .$$

Hence, there is at most a finite number of them such that the conditions for the existence of Love waves are satisfied. Also in this case, the eigenfunction ϕ_i has i zeros on $(0, h)$ and $\kappa_0^2 > \kappa_1^2 > \kappa_2^2 > \cdots > \kappa_n^2$ where n is the total number of eigenvalues for some fixed ω. This total number increases with increasing ω^2.

The cut-off frequency in every mode is determined by the limit $c \to \beta_0$ and in view of the foregoing the cut-off frequency $\omega_{ci} > 0$ corresponding to the eigenvalue κ_i^2 is an increasing sequence. Only the lowest value κ_0^2 is possible for all $\omega > 0$. This can be derived from (4.1) by letting ω approach zero. We then have that ϕ_0 approaches a constant value for all z, which is possible only for ϕ_0 having no zeros on $(0, h)$.

The results of this section were also given by Keilis-Borok *et al* (1965). Their statements were given no proof, however, and conditions for the existence of Love waves were not included.

5. Green's Function

5.1 The solution of the inhomogeneous equation (2.3) subject to the appropriate boundary conditions is essentially the Green's function of the boundary value problem considered.

It was shown in Part I, that the solution could be constructed as follows:

$$\begin{aligned} Z(z, z_0, k) &= \mu(z_0)\phi(z, k^2)\eta(z_0, k^2)/D(\eta, \phi) \qquad z < z_0 \\ &= \mu(z_0)\eta(z, k^2)\phi(z_0, k^2)/D(\eta, \phi) \qquad z > z_0 \end{aligned} \tag{5.1}$$

where ϕ and η are solutions of the homogeneous equation

$$\frac{d}{dz}\left(\mu \frac{d\phi}{dz}\right) + (\rho\omega^2 - \mu k^2)\phi = 0 \tag{5.2}$$

and where ϕ and η satisfy the boundary conditions at $z = 0$ and $z = \infty$ respectively.

The quantity $D(\eta, \phi) = \mu(\eta d\phi/dz - \phi d\eta/dz)$ is (apart from the multiplicative factor μ) the Wronskian of η and ϕ, and can be proved (see Part I) to be independent of z, and so, to be a function of k (and ω) only.

It was also proved that $Z(z, z_0, k)$ is continuous and bounded, except at possible zeros of the denominator D, and has a derivative having a finite jump discontinuity at $z = z_0$. The latter property make Z satisfy the inhomogeneous equation (2.3) with the delta function in the right hand number.

5.2 It is a well-known fact in the theory of ordinary differential equations, that a solution of a homogeneous equation like (5.2) is uniquely determined by its initial conditions at $z = 0$ and that a solution thus determined is an entire (analytic) function of the parameter κ (and ω). These facts can be proved by inverting the equation into an integral equation, from which existence, uniqueness and dependence of initial values and parameters can be established (Titchmarsh, 1962, Ch. 1).

Two solutions ϕ and ψ of (5.2) which satisfy at $z = 0$ the boundary conditions

$$\mu(0)\, d\phi(0)/dz = 0 \qquad \phi(0) = 1$$

$$\mu(0)\, d\psi(0)/dz = 1 \qquad \psi(0) = 0$$

form a set of independent solutions as their conjunct $D(\phi, \psi)$, being independent of z, is identically equal to unity.

Any solution of (5.2) can be obtained as a linear combination of ϕ and ψ. We thus may put for η in (5.1)

$$\eta = \phi + m\psi$$

where the quantity m is a function of k only, which has to be determined such that η satisfies the boundary condition at the other end of the interval, which, in our case is $z = \infty$.

In general theory of singular boundary value problems associated with linear differential equations the function $m(k)$ determines the character of the spectrum ("limit circle" or "limit point" type).

5.3 In the present problem the form of η for $z \geqq h$ is immediately given by

$$\eta = e^{-i(k_0^2-k^2)^{1/2}z}. \tag{5.3}$$

On the path of integration L in (2.2) we have $\text{Im}\, k^2 \geqq 0$. In order to make η one-valued and bounded at $z = \infty$ we must require $\text{Im}\,(k_0^2 - k^2)^{1/2} < 0$ if $\text{Im}\, k^2 > 0$. This can be accomplished by making branch cuts in the k-plane running from $k = k_0$ to $k = k_0 - i\infty$ and from $k = -k_0$ to $k = -k_0 + i\infty$, thus assuring that the path L does not cross the branch cuts.

We note further that for real k, $-k_0 < k < k_0$, η behaves as an outgoing wave at $z = \infty$, and thus satisfies the radiation condition.

For real k, $k^2 > k_0^2$ the quantity $(k_0^2 - k^2)^{1/2}$ is negative imaginary. This corresponds to waves which are exponentially damped with depth, and which do not radiate energy into the lower layer.

The solution $Z(z, z_0, k)$ is now uniquely determined and is finite at $z = \infty$ if $\text{Im}\, k^2 \geqq 0$.

5.4 In virtue of Z being uniquely determined by the branch cuts, the only possible singularities other than branch points must be poles due to the vanishing of the denominator $D(\eta, \phi)$ at certain points in the k-plane. For the values of $k = x_i$ for which this occurs the functions η_i and ϕ_i are linearly dependent and thus are (apart from a constant factor) the same solution of the homogeneous equation (4.1) satisfying the boundary condition $d\phi/dz = 0$ at $z = 0$.

We shall proceed now to locate the zeros of D in the k-plane. If $\phi_i(\eta_i)$ is a complex function its imaginary and real part both must satisfy the differential equation and the boundary condition at $z = 0$. For some fixed k this implies that both are identical, being uniquely determined. This leads to a contradiction as for $z \geqq h$, η_i must be of the form (5.3).

We thus infer that η_i must be real function for $z \geqq h$, which is possible only if η_i can be identified as leading to a Love wave with real wave number κ_i. The constant κ_i corresponding to the i^{th} Love mode, thus is a zero of the denominator $D(\eta, \phi)$ and so is a discrete pole of the solution $Z(z, z_0, k)$. Hence there is at least one pole κ_0 and at most a finite number of poles which are all situated on the real axis and satisfy $k_0^2 < \kappa_i^2 < k_{\max}^2$.

5.5 We still have to determine the residues of Z at the poles.

The Green's formula for arbitrary solutions ϕ_+ and ϕ satisfying the homogeneous equation for k_+^2 and k^2 respectively is given by (Part I)

$$\mu(\phi_+ \, d\phi/dz - \phi \, d\phi_+/dz) \,|_0^z = (k^2 - k_+^2) \int_0^z \mu\phi_+ \phi \, dz.$$

If ϕ_+ is assumed to be the eigenfunction ϕ_i corresponding to the eigenvalue κ_i^2 and we let z go to infinity, we obtain

$$\mu(\phi_i \, d\phi/dz - \phi \, d\phi_i/dz) \,|_0^\infty = (k^2 - k_i^2) \int_0^\infty \mu\phi_i \, \phi \, dz.$$

By assuming $\operatorname{Im} k^2 > 0$, it follows that the upper limit $z = \infty$ does not contribute to the left-hand member as both ϕ_i and ϕ tend to zero as $z \to \infty$. By taking the limit $k^2 \to \kappa_i^2$, the left-hand member reduces to $D(\phi_i, \phi_i)$ evaluated at $z = 0$.

We thus have

$$D(\phi_i, \phi_i) = \lim_{k^2 \to \kappa_i^2} (k^2 - \kappa_i^2) \int_0^\infty \mu\phi\phi_i \, dz.$$

Hence it follows that $D(\eta, \phi)$ has a simple zero at $k = \kappa_i$ and so that D^{-1} has a simple pole at $k = \kappa_i$ with residue $\{2\kappa_i \int_0^\infty \mu\phi_i^2 \, dz\}^{-1}$.

It then follows that $Z(z, z_0, k)$ given by (5.1) has the residue

$$\mu(z_0)\phi_i(z, \kappa_i^2)\phi_i(z_0, \kappa_i^2)/2\kappa_i \int_0^\infty \mu\phi_i^2 \, dz$$

at $k = \kappa_i$, which formula holds for $z \gtrless z_0$. By its very definition of being a continuous function of z, Z is to be a bounded function in the k-plane, except at the poles κ_i. Hence, by Cauchy's theorem and in view of the asymptotic character of $H_0^{(2)}(kr)$ in the lower half of the k-plane, we can replace the integral on L in (2.2) by a residue sum and a branch line integral.

We then arrive at the representation (in which the circular frequency has been restored):

$$v(r, z, \omega) = -2\pi i \sum_i \mu(z_0) H_0^{(2)}(\kappa_i r)\phi_i(z, \kappa_i^2)\phi_i(z_0, \kappa_i^2) \Big/ \int_0^\infty \mu\phi_i^2 \, dz + \int_W H_0^{(2)}(kr) Z(z, z_0, k) k \, dk \tag{5.4}$$

where the sum is taken over the poles on the positive real axis and the path W follows the borders of the branch cut in the fourth quadrant in the clockwise sense.

5.6 In (5.4), the residue contributions appear to be symmetrical in z and z_0, whereas the integrand has a more complicated appearance due to the presence of Z given by (5.1).

Recalling that η can be written as $\eta = \phi + m\psi$, we can write the formula for Z for $z < z_0$ in the form

$$Z(z, z_0, k) = \frac{\mu(z_0)\{\phi(z, k^2)\phi(z_0, k^2) + m(k)\phi(z, k^2)\psi(z_0, k^2)\}}{m \, \mu(\psi \, d\phi/dz - \phi \, d\psi/dz)}$$

$$= \frac{\mu(z_0)\phi(z, k^2)\phi(z_0, k^2)}{m(k) \, D(\psi, \phi)} + \frac{\phi(z, k^2)\psi(z_0, k^2)}{D(\psi, \phi)}.$$

The second term is a one-valued function of k, as m has dropped out and ϕ and ψ are uniquely determined. Hence this term does not contribute to the branch line integral. In the first term, the denominator equals $D(\eta, \phi)$ which is a function of k (and ω) only: $D(\eta, \phi) = \Delta(\omega, k)$.

We thus can transform the field expression (5.4) into

$$v(r, z, \omega) = -2\pi i \cdot \mu(z_0) \sum_i H_0^{(2)}(\kappa_i r)\phi_i(z, \kappa_i^2)\phi_i(z_0, \kappa_i^2) \Big/ \int_0^\infty \mu\phi_i^2 \, dz + \mu(z_0) \int_W H_0^{(2)}(kr)\{\phi(z, k^2)\phi(z_0, k^2)/\Delta(\omega, k)\} k \, dk. \tag{5.5}$$

This formula is symmetrical in z and z_0 and holds for $z \gtrless z_0$. The function ϕ is a solution of the homogeneous equation satisfying the boundary condition $d\phi/dz = 0$ at $z = 0$. The quantity $\Delta(\omega, k)$ appearing in (5.5) is the equivalent of the period function of a layered system. The period equation $\Delta(\omega, k) = 0$ gives the dispersion relations between ω and the wave numbers $\kappa_i(\omega)$ of the free Love modes.

If we assume the eigenfunction ϕ_i to be normalized in the sense that $\int_0^\infty \mu\phi_i^2 \, dz = 1$, the residue-sum in (5.5) takes a still more simple shape.

6. Discussion

6.1 The field representation (5.5) appears to be of a concise and simple form. Digression can be made from (2.1) to solving an ordinary homogeneous differential equation and the corresponding eigenvalue problem. This can be done by analytical means, i.e. by assuming such a dependence of μ and ρ on depth that the differential equation yields analytic closed-form solutions or by finding a numerical solution.

The latter possibility arises if one desires to study wave propagation in a more realistic earth model.

In any case, the advantage of above field expression is apparent from the fact, that for a complicated layered system, one has not to deal ab initio with expressions involving many parameters and functions.

If one assumes the eigenfunction ϕ_i to be known numerically as a function of depth and frequency, the influence of source depth and frequency on the relative excitation of a certain Love mode is immediately derivable from (5.5).

For the far field and an arbitrary time dependence of the action in the source, the usual Fourier-integral techniques and asymptotic approximations are applicable (see Part I).

6.2 A free normal mode has nodal planes parallel to the layering. The number of nodal planes equals the mode number. The phase velocity of a mode is dispersive, i.e. dependent on frequency. If frequency varies, not only phase velocity changes, but also the nodal planes shift up and down in the medium and the depth dependence of amplitude is a function of frequency. Hence, as the relative excitation of a normal mode at the surface enters the expression (5.5) by the factor $\phi_i(z_0, \kappa_i^2)\phi_i(0, \kappa_i^2)$, where $\kappa_i(\omega)$ gives the frequency dependent dispersion, this excitation is in ratio with the amplitude $\phi_i(z_0, \kappa_i^2)$ at $z = z_0$ which is a function of frequency. If for a certain frequency, z_0 is situated on a nodal plane, the mode will not be excited at all at that frequency.

6.3 In the actual seismological context, however, the excitation of surface waves is depending on more frequency dependent factors. These factors are (1) the spectral function at the source, i.e. the Fourier transform of the transient time dependent action at the source, (2) the radiation pattern which may be a function of frequency, and (3) the attenuation of surface waves which is a function of frequency in virtue of the attenuation properties of the medium varying with depth (Anderson *et al.*, 1965).

The last factor can be made explicit through amplitude measurements and is a frequency dependent characteristic of free surface waves, and so, does not involve the source depth.

If we restrict the discussion to the fundamental and first higher normal mode, and assume that the attenuation factors for these modes are given functions of the frequency, then the factors (1), (2) and (3) could be eliminated by considering the quotient of the two modes at the same frequency. The quantity which is decisive for the excitation of the first higher mode at the surface than is given by

$$\phi_1(0, \kappa_1^2)\phi_1(z_0, \kappa_1^2)/\phi_0(0, \kappa_0^2)\phi_0(z_0, \kappa_0^2)$$

if we assume that ϕ_1 and ϕ_0 are normalized. If the structure, and so, the functions ϕ_1 and ϕ_0 are assumed to be known, this formula then is a known function of source depth and frequency and so must be diagnostic for the source depth and the assumptions regarding the structure of the medium. However, sufficient seismological data must be available on the amplitude and the frequency of the modes considered.

If the source is situated in the nodal plane of the first higher mode, then this mode is not excited at all. This condition of vanishing amplitude would lead to a unique relation between source depth and frequency, independent of further frequency dependent factors, i.e.

$$\phi_1(z_0, \kappa_i^2) = f(z_0, \omega) = 0$$

where f is a continuous function of z_0 and ω, which is uniquely determined by the structure of the medium. This equation then could provide a means of the source depth determination with the aid of the first higher Love mode.

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