

SCATTERING OF HIGH VELOCITY ELECTRONS IN HY-
DROGEN AS A TEST OF THE INTERACTION
ENERGY OF TWO ELECTRONS

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ABSTRACT

1. The electronic scattering of high velocity electrons is considered as a test of the formulas for the interaction energy of two electrons. Three formulas for the interaction energy of two Dirac electrons are considered: I, the simple electrostatic interaction, II, the Gaunt formula, III, the Breit formula.

2. The Dirac transition probability method is reviewed briefly.

3. The cross-section of free electrons at rest for scattering high velocity electrons is calculated by the Dirac transition probability method, using each of the three formulas for interaction energy. Random distribution of spins is assumed. Relativistic variation of mass with velocity is taken into account, removing the symmetry about 45° of the classical scattering formula. Expanded to terms in β^2 the formulas obtained are compared with the classical Darwin formula and the recent formula of Mott. They differ from Mott's formula and from each other by terms of order β^2 .

4. For high-velocity electrons, the scattering by hydrogen may be considered as the sum of nuclear and electronic scattering. A criterion for the validity of this procedure is obtained. The advantage of using hydrogen as the scatterer in experiments to test these formulas is discussed.

I. THE scattering of electrons by hydrogen may be considered as the sum of the scattering by free protons and the scattering by free electrons, provided the incident velocity is high and the scattering angle is not too near 0 or $\pi/2$. A criterion for the validity of this approximation is developed in section 4.

The cross-section for scattering depends upon the energy of interaction of the scatterer and the scattered particle and there is no completely justified theoretical formula for the interaction of two electrons or of an electron and a proton. Three formulas have been used,

$$\begin{aligned}
 \text{I} \quad V &= e^I e^{II} / r \\
 \text{II} \quad V &= e^I e^{II} / r [1 - (\alpha^I \cdot \alpha^{II})] \\
 \text{III} \quad V &= e^I e^{II} / r \left[1 - \frac{(\alpha^I \cdot \alpha^{II})}{2} - \frac{(\alpha^I \cdot r)(\alpha^{II} \cdot r)}{2r^2} \right].
 \end{aligned} \tag{1}$$

The α 's, written here as vectors, have components which are the matrices of four rows and columns appearing as coefficients in the Dirac linear Hamiltonian for one particle. r is the distance between the particles. I is just the electrostatic interaction. II includes "spin" terms. III includes also the effect of

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retardation of the potentials. III, which was first derived by Breit,¹ on the analogy $v/c \rightarrow \alpha$, and which is obtained in first approximation from the Heisenberg-Pauli field theory by systematic neglect of the infinite proper energy terms, has the most theoretical justification. However, Breit² has calculated the fine structure of helium with it and his results are no better than those of Gaunt,³ who derived and used II. This failure may be due to the inadequacy of the approximate wave functions used. It is, therefore, of interest to find another place to use these formulas, where it may be possible to determine which gives results in better agreement with experiment.

For scattering of electrons by protons, a calculation similar to that given here for scattering by electrons has been carried out. The result is that the extra terms in II or III have a negligible influence. Massey⁴ concurs in this result. Thus Mott's calculation using I, gives the best formula for nuclear (proton) scattering.⁵

2. The cross-section of an electron for scattering electrons is calculated here in first approximation by the transition probability method of Dirac.⁶ The Hamiltonian for the system, incident electron plus scatterer, is to be written in the form $H = H^\circ + V$, where H° represents the system as it is observed experimentally. The eigenstates $\Psi(\alpha')$ of the unperturbed system whose Hamiltonian is H° , are known and are specified by giving the values, here assumed continuous in range, of a set of observables, $\alpha_1 \cdots \alpha_n$, which are constants of the motion. We shall frequently use α to represent $\alpha_1 \cdots \alpha_n$. The state Ψ of the system whose Hamiltonian is H is to be expanded in the eigen Ψ 's of the unperturbed system,

$$\Psi = \int \Psi(\alpha') d\alpha' a(\alpha'),$$

where $a(\alpha')$ is a function of the time, i.e., the method of variation of constants. Substituting in the equation

$$H\Psi = \frac{i\hbar}{2\pi} \frac{\partial}{\partial t} \Psi,$$

we obtain an integral equation for the time variation of $a(\alpha')$. Assuming that at some initial time our system would certainly be observed in the state α° , i.e., initially $a(\alpha') = \delta(\alpha' - \alpha^\circ)$, this equation reduces to a simple differential equation which can be integrated. The normalization of the $a(\alpha')$ is such that we are concerned not with one system but with an ensemble of similar

¹ Breit, Phys. Rev. **34**, 553 (1929).

² Breit, Phys. Rev. **36**, 383 (1930).

³ Gaunt, Proc. Roy. Soc. **A122**, 513 (1929).

⁴ Massey, Proc. Roy. Soc. **A127**, 666 (1930).

⁵ Mott, Proc. Roy. Soc. **A124**, 425 (1929). He formulates the problem rigorously, using Gordon's solutions of the Dirac equation, and approximates by expanding in powers of $\alpha = Z(2\pi e^2/\hbar c)$, neglecting terms in α^2 .

⁶ Dirac, Proc. Roy. Soc. **A114**, 243 (1927) or Dirac, Principles of Quantum Mechanics, Oxford, 1930, section 54.

systems. The number of systems in our ensemble is $|a^\circ|^2 \int dx |\psi_{\alpha^\circ}(x)|^2$, where x stands for the whole set of coordinates and $\psi_{\alpha^\circ}(x)$ is a solution of the wave-equation for the unperturbed system in the initial state. Then $|a(\alpha')|^2 d\alpha'$ is the number of systems at time t such that α lies between α' and $\alpha' + d\alpha'$. The expression for $|a(\alpha')|^2$ shows that essentially only those transitions occur in which energy is conserved. If we transform to a new set of parameters, $W, \gamma_1 \cdots \gamma_{u-1}$, which are arbitrary independent functions of the α 's, the number of transitions per second from the initial state to states in which the γ 's have values between γ' and $\gamma' + d\gamma'$ is found, by integration with respect to W' , to be

$$4\pi^2/h^2 |a^\circ|^2 |(W^\circ, \gamma^\circ | V | W', \gamma')|^2 J(W^\circ, \gamma^\circ) d\gamma_1' \cdots d\gamma_{u-1}' \quad (2)$$

$J(W^\circ, \gamma^\circ)$ is the Jacobian, $\partial(\alpha_1' \cdots \alpha_u')/\partial(W', \gamma_1' \cdots \gamma_{u-1}')$, with the values $W^\circ, \gamma_1^\circ \cdots \gamma_{u-1}^\circ$ for the variables. $(W^\circ, \gamma^\circ | V | W', \gamma')$ is the matrix element $(\alpha' | V | \alpha^\circ)$, where the α 's are so determined that the parameters, $W, \gamma_1 \cdots \gamma_{u-1}$, have the values $W^\circ, \gamma_1^\circ \cdots \gamma_{u-1}^\circ$ in the initial state and $W', \gamma_1' \cdots \gamma_{u-1}'$ in the final state. If $V = V(x, p)$ is a function of the coordinates and conjugate momenta, then

$$(\alpha' | V | \alpha^\circ) = \int N_{\alpha'} \psi_{\alpha'}(x) dx V \left(x, \frac{\hbar}{2\pi i} \frac{\partial}{\partial x} \right) \psi_{\alpha^\circ}(x). \quad (3)$$

All of our expressions assume that the wave functions have the continuous spectrum normalization, equivalent to the Dirac δ normalization. $N_{\alpha'}$ is the normalizing factor for the final state wave functions and is given⁷ by

$$N_{\alpha'}^2 \int dx \bar{\psi}_{\alpha'}(x) \int_{\alpha' - \Delta\alpha'}^{\alpha' + \Delta\alpha'} d\alpha'' \psi_{\alpha''}(x) = 1. \quad (4)$$

For our purposes, the normalization of the initial state wave functions is quite arbitrary. This is because we are interested in the cross-section for scattering, σ_s , which is defined from the classical treatment of collision scattering by

$$\sigma_s = \frac{\text{Number of transitions per sec. of the specified type, i.e. (2)}}{(\text{Flux of incident particles}) \cdot (\text{number of scatterers})}. \quad (5)$$

Both $|a^\circ|^2$ and the initial state normalizing factors occur in both numerator and denominator and cancel out.

3. In our problem the Hamiltonian H° is the sum of the Dirac linear Hamiltonians for two free electrons. The wave equation for the unperturbed system is

$$\left[W - \frac{\hbar c}{2\pi i} \{ (\alpha^{\text{I}} \cdot \text{grad}^{\text{I}}) + (\alpha^{\text{II}} \cdot \text{grad}^{\text{II}}) \} - mc^2 (\alpha_0^{\text{I}} + \alpha_0^{\text{II}}) \right] \psi = 0.$$

This equation has solutions of the form $\psi = \psi^{\text{I}} \psi^{\text{II}}$ with $W = W_1 + W_2$, where ψ^{I} is a solution of

⁷ Oppenheimer, Phys. Rev. **31**, 66 (1928).

$$\left[W_1 - \frac{hc}{2\pi i}(\alpha^I \cdot \text{grad}^I) - mc^2\alpha_0^I \right] \psi^I = 0$$

and similarly ψ^{II} . The solution in the form of plane waves has been given by Darwin.⁸ The four components of ψ^I are

$$\begin{aligned} \psi_3 &= AS_1^I & \psi_1 &= \frac{-A p_{1z} - B(p_{1x} - i p_{1y})}{mc + W_1/c} S_1^I, \\ \psi_4 &= BS_1^I & \psi_2 &= \frac{-A(p_{1x} + i p_{1y}) + B p_{1z}}{mc + W_1/c} S_1^I \end{aligned} \quad (6)$$

where

$$\begin{aligned} W_1 &= c(m^2c^2 + p_{1x}^2 + p_{1y}^2 + p_{1z}^2)^{1/2} \\ S_1^I &= \exp \left[\frac{2\pi i}{h}(p_{1x}x_1 + p_{1y}y_1 + p_{1z}z_1 - W_1t) \right]. \end{aligned}$$

Darwin shows that the orientation of the spin axis, as given by its colatitude f , measured from the z axis, and longitude ω , is given by the relation $-B/A = \cot f/2e^{i\omega}$. Accordingly, $A=0, B=1$, means that the spin is "parallel" to the z axis, and the corresponding four components of the wave function may be represented by the one symbol Ψ_α . Likewise, $B=0, A=1$, means that the spin is "antiparallel" to the z axis and for this case we may write ψ_β . The constants of the motion, whose values designate the eigenstates of the unperturbed system (the α 's of section 2), are the components of momentum of the two particles $p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}$. We shall use a lower index 1 or 2 to indicate that the corresponding momentum is (p_{1x}, p_{1y}, p_{1z}) or (p_{2x}, p_{2y}, p_{2z}) . Thus the symbol $\psi_{2\alpha}^I$ means the four component wave function in the coordinates (x_1, y_1, z_1) for momentum (p_{2x}, p_{2y}, p_{2z}) with $A=0, B=1$.

To take account of the random distribution of spins of the two electrons before the collision, we use wave functions for the initial state which represent equal probabilities for each electron to have its spin parallel or antiparallel to the z axis. As pointed out by Oppenheimer,⁹ our wave functions must be antisymmetric to conform to the Pauli principle, since we are dealing with identical particles with spin. Accordingly, our initial state wave functions may have the form

$$\begin{aligned} \psi_a &= 1/(2)^{1/2}(\psi_{1\alpha}^I \psi_{2\alpha}^{II} - \psi_{2\alpha}^I \psi_{1\alpha}^{II}) \\ \psi_b &= 1/(2)^{1/2}(\psi_{1\beta}^I \psi_{2\beta}^{II} - \psi_{2\beta}^I \psi_{1\beta}^{II}) \\ \psi_c &= \frac{1}{2}(\psi_{1\alpha}^I \psi_{2\beta}^{II} - \psi_{2\beta}^I \psi_{1\alpha}^{II} + \psi_{1\beta}^I \psi_{2\alpha}^{II} - \psi_{2\alpha}^I \psi_{1\beta}^{II}) \\ \psi_d &= \frac{1}{2}(\psi_{1\alpha}^I \psi_{2\beta}^{II} - \psi_{2\beta}^I \psi_{1\alpha}^{II} - \psi_{1\beta}^I \psi_{2\alpha}^{II} + \psi_{2\alpha}^I \psi_{1\beta}^{II}). \end{aligned} \quad (7)$$

where the first three of these are symmetric in the spins and the fourth is antisymmetric in the spins.

The spin distribution in the final state is not to be measured and is deter-

⁸ Darwin, Proc. Roy. Soc. A120, 621 (1928).

⁹ Oppenheimer, Phys. Rev. 32, 361 (1928).

mined only by the transition probabilities. Wave functions representing any spin orientations can be built up by superposition of four wave functions of the type (7). We shall use a ϕ to represent the complex conjugate of a final state wave function, only the complex conjugates occurring in our formulas. The final state momenta will be designated by $p_{1x}', p_{1y}', p_{1z}', p_{2x}', p_{2y}', p_{2z}'$. The total transition probability out of any one of the four initial spin states is the sum of the transition probabilities to four final spin states of type (7). We shall have taken account of the initial random spin orientations if we take the unweighted average of the total probabilities of transition out of the four initial spin states. Specifically, this means to calculate sixteen cross-sections for scattering, using the matrix elements obtained from all combinations of $\phi_a, \phi_b, \phi_c, \phi_d$ with $\psi_a, \psi_b, \psi_c, \psi_d$, and to take one fourth of the sum.

We shall transform from $p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}$ to a new set of parameters $W, P_x, P_y, P_z, \theta, \phi$ (the $W, \gamma_1 \cdots \gamma_{u-1}$ of section 2), where (P_x, P_y, P_z) is the momentum of the center of gravity and θ, ϕ are the polar angles of the momentum (p_{1x}, p_{1y}, p_{1z}) , the z axis being taken as pole. The equations of transformation are

$$\begin{aligned} W &= c(m^2c^2 + p_1^2)^{1/2} + c(m^2c^2 + p_2^2)^{1/2} & P_x &= p_{1x} + p_{2x} \\ \cos \theta &= p_{1z}/p_1 & P_y &= p_{1y} + p_{2y} \\ \tan \phi &= p_{1y}/p_{1x} & P_z &= p_{1z} + p_{2z} \end{aligned}$$

It is convenient also to use center of gravity coordinates X, Y, Z , relative coordinates x, y, z and relative momenta p_x, p_y, p_z , the equations of transformation being

$$\begin{aligned} X &= \frac{1}{2}(x_1 + x_2) & x &= x_1 - x_2 & p_x &= \frac{1}{2}(p_{1x} - p_{2x}) \\ Y &= \frac{1}{2}(y_1 + y_2) & y &= y_1 - y_2 & p_y &= \frac{1}{2}(p_{1y} - p_{2y}) \\ Z &= \frac{1}{2}(z_1 + z_2) & z &= z_1 - z_2 & p_z &= \frac{1}{2}(p_{1z} - p_{2z}). \end{aligned}$$

Note that

$$\frac{\partial(x_1, y_1, z_1, x_2, y_2, z_2)}{\partial(X, Y, Z, x, y, z)} = 1.$$

Let us now consider the matrix elements $(\alpha' | V | \alpha^0)$. They are of the form

$$\int \cdots \int dx_1 \cdots dz_2 \phi_i V \psi_j$$

where i and j take on the values a, b, c, d . V is considered as an operator on ψ_j . The integrand will consist of a sum of terms of the form

$$\phi_{l's'}^I \phi_{n't'}^{II} V \psi_{ls}^I \psi_{nt}^{II}$$

where l, n , may be 1, 2 or 2, 1; l', n' may be 1, 2 or 2, 1; s, t, s', t' may be α or β . The only operators contained in V are the Dirac α 's and the result of their operation is just to permute the four components of a ψ function and to introduce some factors -1 or i . Also α^I operates¹⁰ only on ψ^I and α^{II} on ψ^{II} so that, for example,

¹⁰ The two electrons are independent in the unperturbed system and all matrices associated with one commute with all matrices associated with the other.

$$\phi^I \phi^{II} \alpha_1^I \alpha_3^{II} \psi^I \psi^{II} = (\phi^I \alpha_1^I \psi^I) (\phi^{II} \alpha_3^{II} \psi^{II}).$$

We write the expressions (6) for the four components of $\phi_{i'}^I$ in the form $(\phi_{i'}^I)_\lambda = a_\lambda S_{i'}^I$, and for the four components of ψ_i^I in the form $(\psi_i^I)_\lambda = a_\lambda S_i^{II}$, where $\lambda = 1, 2, 3, 4$, and define

$$\begin{aligned} \Omega_4 &= a_1' a_1 + a_2' a_2 + a_3' a_3 + a_4' a_4 \\ \Omega_1 &= a_1' a_4 + a_2' a_3 + a_3' a_2 + a_4' a_1 \\ \Omega_2 &= a_1' a_4 - a_2' a_3 + a_3' a_2 - a_4' a_1 \\ \Omega_3 &= a_1' a_3 - a_2' a_4 + a_3' a_1 - a_4' a_2. \end{aligned} \quad (8)$$

We define Ω' 's similarly in terms of the components of $\phi_{n'}^{II}$ and ψ_n^{II} . Further we write

$$S = \exp [2\pi i/h \{ (p_1^0 \cdot r_1) + (p_n^0 \cdot r_2) - (p_{i'} \cdot r_1) - (p_{n'} \cdot r_2) \}].$$

Then with our three V 's of (1), we have for our typical term in the integrand

$$\begin{aligned} &\phi_{i's}^I \phi_{n't}^{II} V \psi_{i's}^I \psi_{n't}^{II} = \\ &\text{I} \quad e^2/r [\Omega_4 \Omega_4'] S \\ &\text{II} \quad e^2/r [\Omega_4 \Omega_4' - \Omega_1 \Omega_1' + \Omega_2 \Omega_2' - \Omega_3 \Omega_3'] S \\ &\text{III} \quad e^2/r [\Omega_4 \Omega_4' - \frac{1}{2} \{ \Omega_1 \Omega_1' - \Omega_2 \Omega_2' + \Omega_3 \Omega_3' \} \\ &\quad - 1/2r^2 \{ x \Omega_1 - iy \Omega_2 + z \Omega_3 \} \{ x \Omega_1' - iy \Omega_2' + z \Omega_3' \}] S. \end{aligned} \quad (9)$$

Since

$$\begin{aligned} (p_1 \cdot r_1) + (p_2 \cdot r_2) &= (P \cdot R) + (p \cdot r) \\ (p_1 \cdot r_2) + (p_2 \cdot r_1) &= (P \cdot R) - (p \cdot r), \end{aligned} \quad (10)$$

it follows that in every case our integrand contains a factor

$$\exp \left[\frac{2\pi i}{h} \{ (P^0 \cdot R) - (P' \cdot R) \} \right].$$

Consequently, if we carry out the integration in the variables X, Y, Z, x, y, z instead of $x_1, y_1, z_1, x_2, y_2, z_2$, every matrix element has the factor

$$\int \int \int dX dY dZ \exp \left[\frac{2\pi i}{h} \{ (P^0 \cdot R) - (P' \cdot R) \} \right].$$

This integral is of the nature of a δ function of $(P^0 - P')$ so that we have a transition probability only between initial and final states such that the momentum of the system is conserved, as well as the energy. Consequently, we shall be interested only in the number of transitions from the initial state to final states in which θ lies between θ and $\theta + d\theta$ and ϕ lies between ϕ and $\phi + d\phi$. This means going back to (2), the γ 's being $P_x', P_y', P_z', \theta, \phi$, and integrating with respect to P_x', P_y', P_z' as we had previously integrated with respect to W' . Although P' occurs in other factors, e.g., in J , besides

$$\left| \int \int \int dX dY dZ \exp \left[\frac{2\pi i}{h} \{ (P^0 \cdot R) - (P' \cdot R) \} \right] \right|^2,$$

this factor predominates and assures us that we shall have contributions to the integral only for $P' = P^0$. Accordingly, we insert $P' = P^0$ in all other factors and take them outside the integration. We note that the remaining integral is just the cube of

$$\int_{-\infty}^{\infty} dP_{x'} \left| \int_{-\infty}^{\infty} dX \exp \left[\frac{2\pi i}{h} (P_{x^0} - P_{x'}) X \right] \right|^2.$$

We replace $\int_{-\infty}^{\infty} dX$ by $\lim_{X \rightarrow \infty} \int_X^X dX$ and carry out the limiting process after integrating with respect to $P_{x'}$, obtaining $\lim_{X \rightarrow \infty} \int_X^X 2hX$. Thus the factor obtained from integration over $P_{x'}$, $P_{y'}$, $P_{z'}$ is just $h^3 V$, where V is the infinite volume of space.

For the remainder of our considerations, we may note that instead of using the parameters $W, P_x, P_y, P_z, \theta, \phi$ and inserting $W' = W^0$ and $P' = P^0$, it is more convenient to use $p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}$, where the values assigned are to be determined by the principles of conservation of energy and momentum, together with the requirement that $p_{1'}$ shall have the direction $\theta\phi$. For the initial state, we shall take

$$p_{1z}^0 = p_1^0 = \frac{mv}{(1 - \beta^2)^{1/2}}; \quad p_{1x}^0 = p_{1y}^0 = p_{2x}^0 = p_{2y}^0 = p_{2z}^0 = 0 \quad (11)$$

which means that we consider a plane wave of electrons of velocity $v = \beta c$, incident in the z direction on electrons at rest. For the final state, since we have symmetry about the z axis, we are independent of ϕ and may most conveniently use $\phi = 0$. We shall also introduce the notation

$$\gamma = 1/(1 - \beta^2)^{1/2}, \quad \sigma = \sin \theta, \quad \rho = \cos \theta.$$

Then we have

$$\begin{aligned} p_{1x}' &= p_1^0 \frac{2\sigma\rho}{2 + (\gamma - 1)\sigma^2} & p_{2z}' &= -p_1^0 \frac{2\sigma\rho}{2 + (\gamma - 1)\sigma^2} \\ p_{1y}' &= 0 & p_{2y}' &= 0 \\ p_{1z}' &= p_1^0 \frac{2\rho^2}{2 + (\gamma - 1)\sigma^2} & p_{2z}' &= p_1^0 \frac{(\gamma + 1)\sigma^2}{2 + (\gamma - 1)\sigma^2}. \end{aligned} \quad (12)$$

The density of particles being $\sum_{\lambda=1}^4 |\psi_{\lambda}|^2$, we have by (6) a density $2\gamma/\gamma + 1$ electrons per cc in our incident beam and a density of one electron per cc of scatterers. Accordingly, the incident flux is $2\gamma v/\gamma + 1$ and the number of scatterers is V , the infinite volume of space. The Jacobian, J , of (2) has the value

$$J = \frac{4m^2 v \gamma \rho \sigma [2\gamma - (\gamma - 1)\sigma^2] [\gamma(\gamma + 1)\sigma^2 + 2\rho^2]}{[2 + (\gamma - 1)\sigma^2]^4}.$$

From the normalization of the final state wave functions by (4), we find for the factor $N_{\alpha'}$.

$$N^2 = \frac{(\gamma + 1)[(\gamma + 1)^2\sigma^2 + 4\rho^2]}{2h^6[2\gamma - (\gamma - 1)\sigma^2][\gamma(\gamma + 1)\sigma^2 + 2\rho^2]}.$$

With this value of J in (2) and of N in (3), we insert in (5) and obtain

$$\sigma_s = \frac{4\pi^2 m^2 \rho \sigma (\gamma + 1)^2 [(\gamma + 1)^2 \sigma^2 + 4\rho^2] d\theta d\phi}{h^4 [2 + (\gamma - 1)\sigma^2]^4 F^2},$$

where F^2 is one fourth of the sum of the squares of the factors obtained by integration over the relative coordinates in the sixteen matrix elements of the interaction energy, V .

To evaluate F^2 , we must go back to (9), which is the typical term of our integrand if we write S_r instead of S and understand by S_r the remainder of the exponential factor, S , after taking out

$$\exp\left[\frac{2\pi i}{h}\{(P^0 \cdot R) - (P' \cdot R)\}\right].$$

As a consequence of (10) and using the values of p^0 and p' given by (11) and (12), this remaining factor has the following form, determined by values of l and l' ,

$$\begin{aligned} l = 1, \quad l' = 1 \quad S_r &= \exp\left[\frac{2\pi i}{h}(p_2' \cdot r)\right] \\ l = 2, \quad l' = 2 \quad S_r &= \exp\left[-\frac{2\pi i}{h}(p_2' \cdot r)\right] \\ l = 2, \quad l' = 1 \quad S_r &= \exp\left[\frac{2\pi i}{h}(p_1' \cdot r)\right] \\ l = 1, \quad l' = 2 \quad S_r &= \exp\left[-\frac{2\pi i}{h}(p_1' \cdot r)\right]. \end{aligned} \quad (14)$$

The integration of (9) with respect to the relative coordinates x, y, z , may be carried out in cylindrical coordinates¹¹ or in polar coordinates, the pole being taken in the direction of the momentum p_k' which occurs in S_r . In the latter case, convergence is obtained by taking account of screening at infinite distance. The integrals occurring are

$$\begin{aligned} \int \int \int dx dy dz \frac{1}{r} \exp[2\pi i/h p_k'^z] &= h^2/(\pi p_k'^2) \\ \int \int \int dx dy dz \frac{z^2}{r^3} \exp[2\pi i/h p_k'^z] &= -h^2/(\pi p_k'^2). \end{aligned} \quad (15)$$

¹¹ Oppenheimer, *Zeits. f. Physik* **43**, 413 (1927).

Then the integration of (9) over the relative coordinates gives

$$\begin{aligned}
 \text{I} & \frac{e^2 \hbar^2}{\pi p_k'^2} [\Omega_4 \Omega_4'] \\
 \text{II} & \frac{e^2 \hbar^2}{\pi p_k'^2} [\Omega_4 \Omega_4' - \Omega_1 \Omega_1' + \Omega_2 \Omega_2' - \Omega_3 \Omega_3'] \\
 \text{III} & \frac{e^2 \hbar^2}{\pi p_k'^2} [\Omega_4 \Omega_4' + \Omega_2 \Omega_2' - (\Omega_1 \rho_k - \Omega_3 \sigma_k)(\Omega_1' \rho_k - \Omega_3' \sigma_k)].
 \end{aligned} \tag{16}$$

Here ρ_k and σ_k are the cosine and sine of the angle between $+p_k'$ and the z axis.

The remainder of the task is the evaluation of the Ω 's and Ω' 's, the summing of the various terms of the type (16) that occur in each matrix element, the squaring of the resulting expressions and taking one fourth of their sum. Only the results are given here.

For purposes of comparison with previous formulas, it is useful to expand all our expressions in powers of β^2 , neglecting higher powers than β^2 . The resulting formulas for the cross-section of a free electron at rest for scattering electrons into a solid angle $\sin\theta d\theta d\phi$ are

$$\sigma_s = \frac{e^4}{m^2 v^4} \sin 2\theta \, 2d\theta \, d\phi \left[\frac{1}{\sigma^4} + \frac{1}{\rho^4} - \frac{1}{\sigma^2 \rho^2} - \frac{\beta^2}{4} f(\theta) \right] \tag{17}$$

where $f(\theta)$ is given by

$$\begin{aligned}
 \text{I} \quad f(\theta) &= \frac{4}{\sigma^4} + \frac{2}{\rho^4} - \frac{1}{\sigma^2 \rho^2} \\
 \text{II} \quad f(\theta) &= \frac{4}{\sigma^4} + \frac{2}{\rho^4} - \frac{2}{\sigma^2 \rho^2} \\
 \text{III} \quad f(\theta) &= \frac{4}{\sigma^4} + \frac{2}{\rho^4} - \frac{3}{\sigma^3 \rho^2}.
 \end{aligned} \tag{18}$$

The purely classical formula of Darwin¹² gives the first two terms in the bracket,

$$\frac{1}{\sigma^4} + \frac{1}{\rho^4} = \operatorname{cosec}^4 \theta + \operatorname{cosec}^4 (\pi/2 - \theta).$$

Mott¹³ has calculated the scattering of electrons by electrons, using interaction formula I, rigorously on the wave mechanics, taking account of interchange phenomena but not of the relativistic variation of mass with velocity. For low velocities, his formula is better than ours. In the region in which our formulas apply, we may neglect the square and higher powers of $1/137\beta$, in

¹² Darwin, Phil. Mag. 27, 499 (1914).

¹³ Mott, Proc. Roy. Soc. A126, 259 (1930).

which case Mott's formulas gives just the first three terms in the bracket of (17). It may be noted that $f(\theta)$ is not symmetric about the angle $\theta = \pi/4$, whereas the classical formulas had this symmetry. This was to be expected because, when we take account of the relativistic variation of mass with velocity in setting up the equations for the conservation of energy and momentum, the angle between the two scattered electrons is not $\pi/2$ but is $\arccos \{(\gamma-1)\sigma\rho/(\gamma+1)^2\sigma^2+4\rho^2\}^{1/2}$ or, in the same approximation as the above, $\arccos \{\beta^2/8[1+\beta^2/4(2+\rho^2)]\sin 2\theta\}$.

For comparison with experimental data, we should use the complete formula, rather than the expansion in powers of β^2 . Our F^2 can be written in the form

$$F^2 = \frac{e^4 h^4 [2 + (\gamma - 1)\sigma^2]^4}{\pi^2 m^4 c^4 (\gamma^2 - 1)^2 [4 + (\gamma - 1)(\gamma + 3)\sigma^2]^2} F(\theta). \quad (19)$$

Substituting (19) in (13), we have

$$\sigma_s = \frac{e^4 (\gamma + 1)^2 \sin 2\theta \, 2d\theta \, d\phi}{m^2 v^4 \gamma^4 [4 + (\gamma - 1)(\gamma + 3)\sigma^2]} F(\theta) \quad (20)$$

where the complete expression for $F(\theta)$ is

$$\begin{aligned} \text{II } F(\theta) &= \frac{1}{\sigma^4} + \frac{1}{\rho^4} - \frac{1}{\sigma^2 \rho^2} + \frac{(\gamma - 1)}{\sigma^4 \rho^4} \left[3\sigma^6 + \sigma^4 \rho^6 - \sigma^6 \rho^2 + \frac{2\rho^6(1 + \sigma^2 \rho^2)}{\gamma + 1} \right. \\ &\quad \left. - \frac{2(\gamma + 1)\sigma^2 \rho^4}{4 + (\gamma - 1)(\gamma + 3)\sigma^2} \right] \\ &\quad + \frac{(\gamma - 1)^2}{8\sigma^4 \rho^4} \left[\sigma^6 \{ 12 - (\gamma + 1)(\gamma - 3)\sigma^2 + 2\gamma(\gamma + 2)\sigma^4 \} \right. \\ &\quad + \frac{8\sigma^4 \rho^4 (3 + \sigma^2)}{\gamma + 1} + \frac{24\sigma^4 \rho^4}{4 + (\gamma - 1)(\gamma + 3)\sigma^2} - \frac{16(\gamma - 1)\sigma^2 \rho^6}{(\gamma + 1)[4 + (\gamma - 1)(\gamma + 3)\sigma^2]} \\ &\quad \left. + \frac{8\rho^6(1 + 3\sigma^2 + 3\sigma^4)}{(\gamma + 1)^2} - \frac{32\gamma\sigma^2 \rho^6}{[4 + (\gamma - 1)(\gamma + 3)\sigma^2]^2} \right]. \quad (21) \\ \text{III } F(\theta) &= \frac{1}{\sigma^4} + \frac{1}{\rho^4} - \frac{1}{\sigma^2 \rho^2} + \frac{(\gamma - 1)}{2\sigma^4 \rho^4} \left[5\sigma^6 + \sigma^8 + \frac{4\rho^6}{\gamma + 1} + \frac{4\sigma^2 \rho^6}{(\gamma + 1)^2} \right] \\ &\quad + \frac{(\gamma - 1)^2}{4\sigma^4 \rho^4} \left[\sigma^4 \{ -4 + 23\sigma^2 + (\gamma^2 + 6\gamma - 13)\sigma^4 + (\gamma^2 - 2\gamma - 3)\sigma^6 - (\gamma^2 - 5)\sigma^8 \} \right. \\ &\quad + \frac{2\sigma^4 \rho^4 (5 - 4\sigma^4)}{\gamma + 1} + \frac{8\sigma^2 \rho^4 (2 + 2\sigma^2 + (2\gamma - 3)\sigma^4 - (\gamma - 1)\sigma^6)}{4 + (\gamma - 1)(\gamma + 3)\sigma^2} \\ &\quad \left. + \frac{4\rho^6(1 + 3\sigma^2 + 2\sigma^4)}{(\gamma + 1)^2} - \frac{16(\gamma^2 - 1)\sigma^4 \rho^4}{[4 + (\gamma - 1)(\gamma + 3)\sigma^2]^2} \right]. \end{aligned}$$

4. *Criterion for neglect of orbital motion of electron in hydrogen.* The scattering of electrons in hydrogen may be considered as the sum of the scattering by free protons and the scattering by free electrons, both at rest, provided the error in the scattering by the electrons, due to their orbital motion, is negligible. This error is of the order

$$\epsilon = \Delta\sigma_s/\sigma_s$$

where $\Delta\sigma_s$ is the change in σ_s if the square of the momentum of the incident electron is changed by the square of the orbital momentum of the electron in the atom or molecule. Using our formulas for σ_s , we obtain a value for ϵ which, for fast electrons and small angles may be written

$$\epsilon \sim \frac{1 - 3/2\beta^2}{10^4\beta^2 \sin \theta}$$

and for angles near $\pi/2$ may be written

$$\epsilon \sim \frac{1 - \frac{1}{2}\beta^2}{10^4\beta^2 \cos^2 \theta}$$

This is just equivalent to considering the error in the nuclear scattering formula of Wentzel when we drop the $1/k^2a^2$ term¹⁴ and obtain $1/\sin^4\theta/2$ instead of $1/[\sin^2\theta/2 + (1/ka)^2]^2$. For a 140,000 volt electron, the error is 1 per cent at 7° angle of scattering.

5. Since scattering at angles greater than $\pi/2$ is purely nuclear and should be given by Mott's formula,¹⁵ while scattering at angles less than $\pi/2$ is both nuclear and electronic, it is hoped that the part of the scattering due to the electrons may be separated off experimentally and compared with the formulas derived here. Hydrogen is suggested as the scatterer for this test, in spite of the experimental difficulties of working with gases, first because there, only, is the electronic scattering, which increases with the atomic number Z , comparable with the nuclear scattering, which increases with Z^2 , and second because the separation of nuclear and electronic scattering is possible at lower voltages with hydrogen than with any other element, the necessary voltage going up roughly with Z . Mr. H. V. Neher, at this institute, is hoping to carry out experiments in the scattering of high velocity electrons in hydrogen in the near future.

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¹⁴ Sommerfeld, Wave Mechanics, Dutton, 1930, p. 197.

¹⁵ Reference 5.