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THE EFFFCT OF TORSIONAL OSCILLATIONS ON EARTHQUAKE STRESSES

By George W. Housner and Hannu Outinen

ABSTRACT

A comparison is made between the maximum stresses induced by earthquake-excited vibrations of an unsymmetrical structure and those induced in a symmetrical structure. These stresses are compared with the commonly used equivalent static method of analysis, and it is shown that the static method underestimates significantly the magnitude of the maximum stresses. Although the maximum stresses are quite sensitive to changes in the relative rigidity of the walls, an energy analysis of the vibration problem shows that the relative rigidities are not crucially important in determining the ultimate strength of the building.

During an earthquake, buildings that are not symmetrical in mass and stiffness will undergo torsional oscillations in addition to the normal horizontally directed oscillations. The nonsymmetrical features will affect the induced stresses, and it is customary to compute the resulting stresses by means of a static analysis. For example, in the case of a one-story building a certain equivalent static force is assumed to act through the center of mass of the structure, and the forces in the resisting members are computed on the basis of an elastic analysis. For more complicated structures it can become very difficult to conceive of a static method of analysis, and even for a simple structure the accuracy of the static method has not been investigated.

The present paper reports the results of an investigation of the accuracy of the static method. The results obtained by this method are compared with the results of a dynamic analysis. The investigation was concerned only with rectangular, one-story buildings. Similar studies should be made of multistory buildings.

Dynamic analysis of the structure.—The building has plan dimensions as indicated in figure 1*a*. The roof of the building is assumed to be a rigid slab and the walls to be linearly elastic with spring constants k_1, k_2, k_3. The center of mass of the structure is assumed to coincide with the geometric center of the building. The equivalent vibration problem is then as shown in figure 1*b*, where four springs restrain a block having mass m and moment of inertia I about the mass center. Under the action of ground acceleration $\ddot{y}$ the block's center of mass will be excited into horizontal oscillations, and the block will also execute torsional oscillations about the center of mass. These motions can be described by the horizontal displacements y_1 and y_2 as shown in figure 1*b*. The forces in the springs (or walls) will be given by k_1y_1 and k_2y_2 so that y_1 and y_2 are the displacements relative to the ground. The differential equations describing the motion are

$$\left(\frac{m}{4} + \frac{I}{4a^2}\right) \ddot{y}_1 + \left(\frac{m}{4} - \frac{I}{4a^2}\right) \ddot{y}_2 + \left(k_1 + \frac{b^2}{2a^2} k_3\right) y_1 - \left(\frac{b^2}{2a^2} k_3\right) y_2 = -\frac{m}{2} \ddot{y} \tag{1}$$

$$\left(\frac{m}{4} - \frac{I}{4a^2}\right) \ddot{y}_1 + \left(\frac{m}{4} + \frac{I}{4a^2}\right) \ddot{y}_2 - \frac{b^2}{2a^2} k_3 y_1 + \left(k_2 + \frac{b^2}{2a^2} k_3\right) y_2 = -\frac{m}{2} \ddot{y} \tag{2}$$

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The roots of the frequency equations are

$$p_{1,2}^2 = \frac{k_1 + k_2}{2\alpha m} [(1 + \alpha + \beta) \pm \{(1 - \alpha + \beta)^2 + 4\alpha(1 - R)^2\}^{\frac{1}{2}}] \tag{3}$$

where

$$\alpha = \frac{I}{ma^2}$$

$$\beta = \frac{k_3 b^2}{2a^2(k_1 + k_2)}$$

$$R = \frac{2k_1}{k_1 + k_2}$$

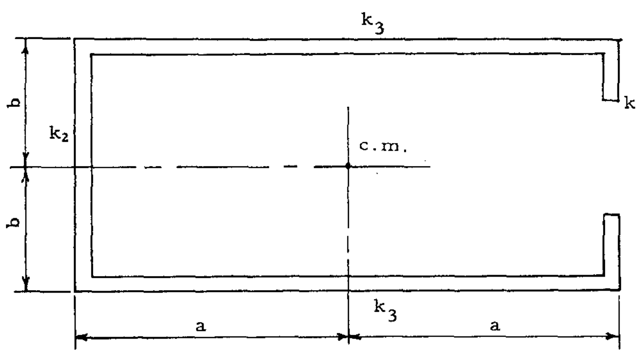

Fig. 1*a*.

The displacements are given by

$$y_1 = \frac{2a_{11}A_1}{p_1} S_1 + \frac{2a_{12}A_2}{p_2} S_2 = C_1S_1 + D_1S_2 \tag{4}$$

$$y_2 = \frac{2a_{21}A_1}{p_1} S_1 - \frac{2a_{22}A_2}{p_2} S_2 = C_2S_1 + D_2S_2 \tag{5}$$

where

$$a_{11} = \beta + \lambda_1^2(1 - \alpha) \qquad A_1 = \frac{a_{11} + a_{21}}{(1 + \alpha)(a_{11} + a_{21})^2 - 4\alpha a_{12}a_{21}}$$

$$a_{12} = \beta + \lambda_2^2(1 - \alpha)$$

$$a_{21} = 2R + \beta - \lambda_1^2(1 + \alpha) \qquad A_2 = \frac{a_{12} + a_{22}}{(1 + \alpha)(a_{12} + a_{22})^2 - 4\alpha a_{12}a_{22}}$$

$$a_{22} = 2R + \beta - \lambda_2^2(1 + \alpha)$$

$$\lambda_1 = p_1 \left(\frac{m}{k_1 + k_2}\right)^{\frac{1}{2}}$$

$$S_{1,2} = \int_0^t \ddot{y} \sin p_{1,2} (t - \tau) d\tau$$

$$\lambda_2 = p_2 \left(\frac{m}{k_1 + k_2}\right)^{\frac{1}{2}}$$

The forces in the walls are given by

$$F_1 = k_1 y_1$$

$$F_2 = k_2 y_2$$

Analysis of strong earthquake records shows that the maximum value attained by the integral S during the ground motion may be considered, for practical purposes, to be constant over a range of periods from approximately 0.3 second to 2.5 seconds.[1] Therefore, in this range S_1 (max) $= S_2$ (max) $= S$.

The highest stresses will be produced when the maximum values of S_1 and S_2 occur at the same time with additive signs. There is always a possibility that this

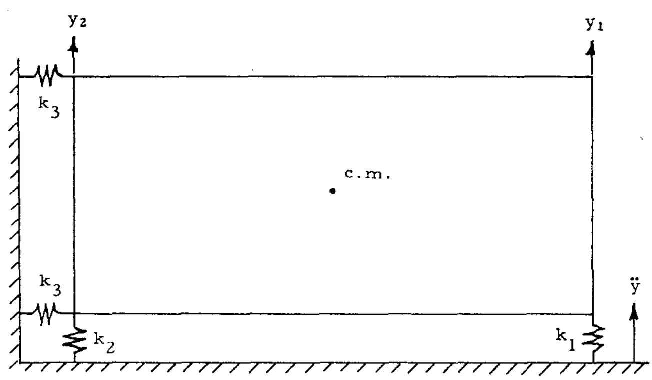

Fig. 1*b*.

condition will occur, although usually the two maxima will not concur exactly. Studies have shown that assuming the maximum values of S_1 and S_2 to occur at the same time will usually result in an overestimation of the order of 5 to 10 per cent.

It is here assumed that the maximum values of S_1 and S_2 are additive, and hence the maximum displacements are given by

$$\begin{aligned} Y_1 &= (C_1 + D_1)S \\ Y_2 &= (C_2 + D_2)S \end{aligned} \tag{6}$$

The maximum forces corresponding to y_1 and y_2 will be compared with the maximum forces that would be obtained if the building were symmetrical with $k_1' = k_2'$ and $k_1' + k_2' = k_1 + k_2$, that is, the total stiffness $k_1 + k_2$ is divided equally between the two walls while everything else remains unchanged. In the symmetrical case the maximum forces are

$$F_1' = F_2' = \frac{1}{2}\left(\frac{m}{k_1 + k_2}\right)^{\frac{1}{2}} S$$

[1] D. E. Hudson, *Response Spectrum Techniques in Engineering Seismology*, Proceedings of the World Conference on Earthquake Engineering, June, 1956; Earthquake Engineering Research Institute, San Francisco.

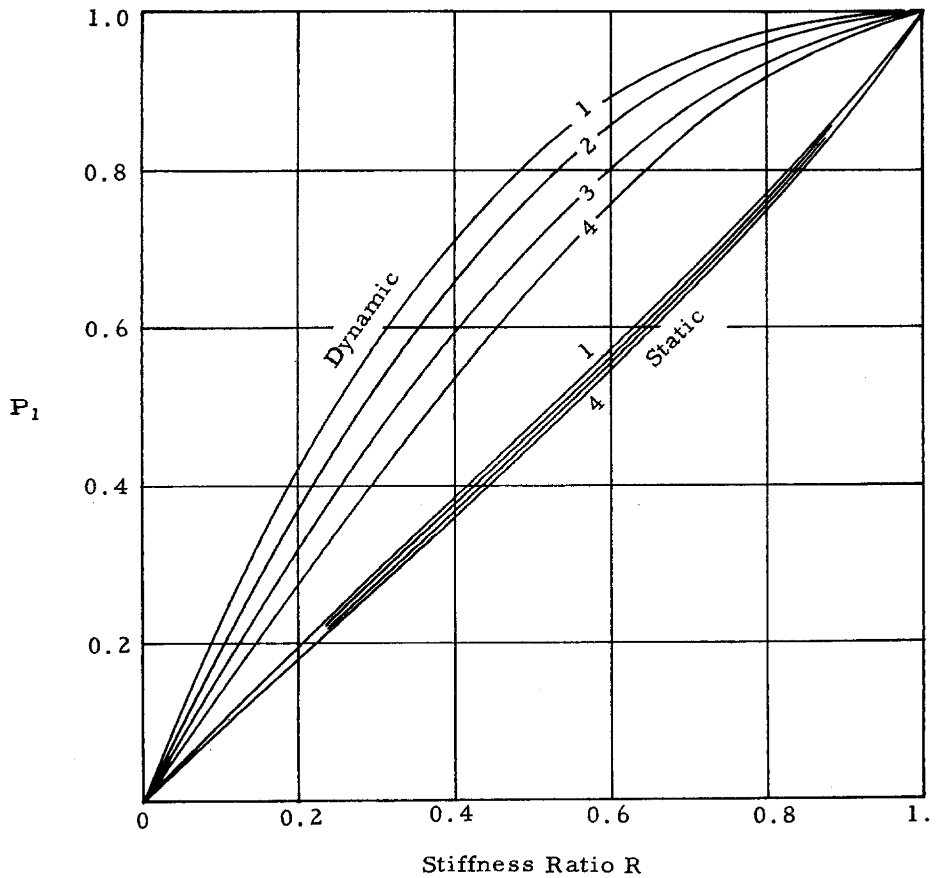

Fig. 2.
P_1 ratio of maximum force in flexible wall of unsymmetrical structure to maximum force in wall of symmetrical structure.

and the ratio of maximum forces are

$$P_1 = \frac{F_1}{F_1'} = 2R\left[\frac{a_{11}A_1}{\lambda_1^{\frac{1}{2}}} + \frac{a_{12}A_2}{\lambda_2^{\frac{1}{2}}}\right] \tag{7}$$

$$P_2 = \frac{F_2}{F_2'} = 2(2 - R)\left[\frac{a_{21}A_1}{\lambda_1^{\frac{1}{2}}} + \frac{a_{22}A_2}{\lambda_2^{\frac{1}{2}}}\right] \tag{8}$$

If the same comparison is made for the customary static method of analysis the ratios are

$$P_{1_s} = R\left(1 - \frac{1 - R}{2(2 - R + \beta/R)}\right) \tag{9}$$

$$P_{2_s} = 2 - P_{1_s} \tag{10}$$

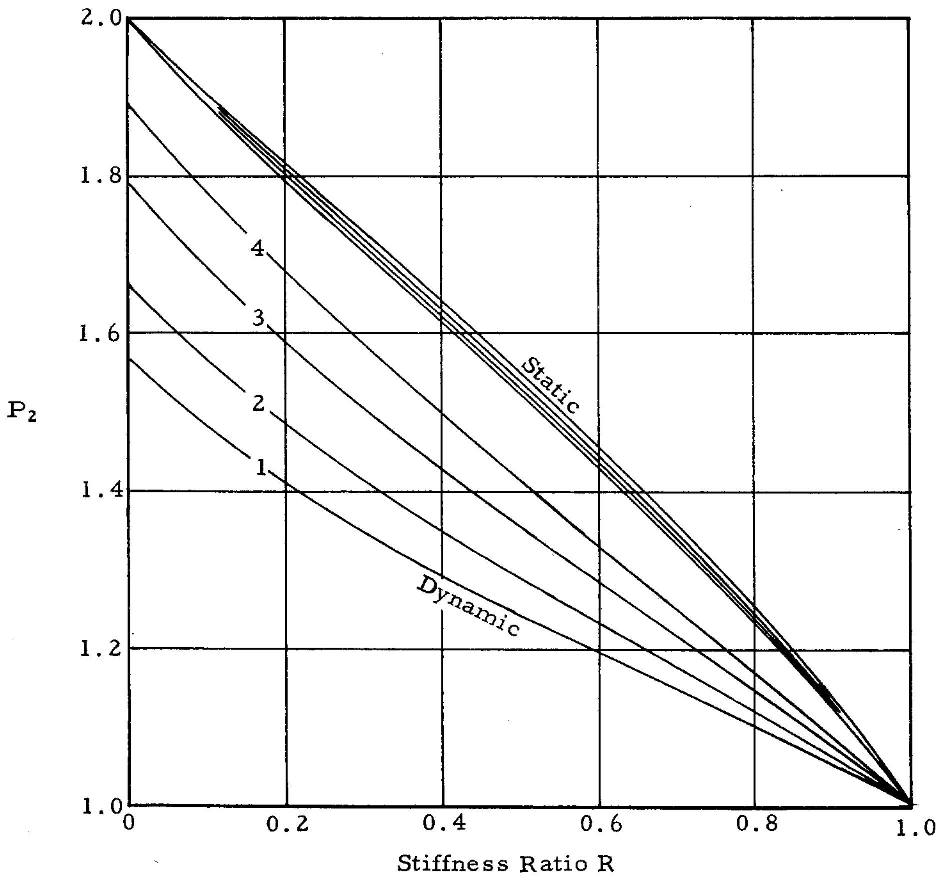

Fig. 3.
P_2 ratio of maximum force in stiff wall of unsymmetrical structure to maximum force in wall of symmetrical structure.

Numerical results.—To exhibit the results, calculations have been made for some typical structures. In order to associate the parameters with physical dimensions the buildings may be thought of as having rigidities that are proportional to the lengths of the walls. In this case the structures are as follows:

Structure 1. $b/a = 1/4; \alpha = 0.3; \beta = 0.5; 0 < R < 1$

Structure 2. $b/a = 1/2; \alpha = 0.42; \beta = 0.8; 0 < R < 1$

Structure 3. $b/a = 1; \alpha = 0.66; \beta = 1.6; 0 < R < 1$

Structure 4. $b/a = 2; \alpha = 1.66; \beta = 2.3; 0 < R < 1$

Structure 1 is a long, narrow building with one of the short walls more flexible than the other. At the other extreme structure 4 has one of the long walls more flexible than the other. The results of the calculations are plotted in figures 2 and 3. The parameter R represents the unbalance in rigidities; if $R = 1$ the two walls have the same rigidity and the structure is symmetrical; if $R = 0$ one of the walls has 0 rigidity.

It will be noted that the static analysis gives an almost linear variation with R and that the force in the flexible wall goes from one-half the lateral load to zero as R goes from 1 to 0. The force in the rigid wall goes from one-half the lateral load to 100 per cent of the lateral load as R goes from 1 to 0.

The dynamic analysis shows the force in the flexible wall to be larger than that given by the static analysis, and the force in the stiff wall to be less than in the static case. In particular, when $R = 0$ the force in the stiff wall is appreciably less. This result could have been foretold, for when $R = 0$ the structure has only three walls and it behaves like a horizontal pendulum and the torsional oscillations are severe whereas the lateral oscillations are only moderate. Hence, the usual concept of equivalent static lateral load is not applicable.

Figures 2 and 3 show that the greater the eccentricity between the center of mass and the center of rigidity the greater is the deviation between the dynamic and the static analysis. For eccentricities giving $R = 0.6$ the flexible wall has a maximum force approximately 50 per cent larger than computed by the static method and the stiff wall has a maximum force approximately 80 per cent as large as computed by the static method. These differences are due to the torsional oscillations of the structure whose effect is not included in the static analysis. The foregoing analysis was for a one-story structure, but it can be concluded that unsymmetrical multistory structures will behave similarly and that the flexible walls will sustain larger forces than a static analysis would indicate.

Energy considerations.—The foregoing remarks are pertinent to the stresses developed during an earthquake, but it is not always feasible to design a structure so that the maximum stresses do not exceed the normally allowable working stresses for the material. In such a case some of the vibrational energy may be absorbed by cracking, stressing beyond the yield point, etc., while still maintaining an adequate factor of safety against serious damage. The elastic stresses are then not reliable indicators of the ultimate strength of the structure, but rather it is the energy-absorbing capacity of the structure that determines the ultimate strength. In this regard it is possible to draw some general conclusions about the vibrational energy of structures during earthquakes.

Consider a structure having normal modes of vibration. For convenience, only undamped structures will be considered, though the same results are obtained if the structure has linear damping. Only horizontal planar vibrations will be considered, but it will be seen that the same results hold for more complex structures that vibrate with three-dimensional motion. The displacement and velocity of such a multi-degree-of-freedom structure during free vibrations that are excited by a unit impulse are described by equations having the forms

$$
\begin{aligned}
Y &= \sum_n B_n \frac{\phi_n}{p_n} \sin p_n t \\
\dot{Y} &= \sum_n B_n \, \phi_n \cos p_n t \qquad (11) \\
B_n &= \frac{\int \phi_n \, d\rho}{\int \phi_n^2 \, d\rho}
\end{aligned}
$$

where $d\rho$ is an element of mass and the integrals are taken over the mass of the structure, and ϕ_n is the normal mode shape and is a function of the space coördinates. Let each element of mass $d\rho$ be acted upon by a horizontal force equal to $(-\ddot{y}d\rho)$. The resulting motion will then be the same as if the base of the structure were shaken with horizontal acceleration $\ddot{y}$. The displacement generated at time t is

$$\begin{aligned} Y &= \sum_n B_n \frac{\phi_n}{p_n} \int_0^t \ddot{y} \sin p_n(t - \tau) d\tau \\ &= \sum_n B_n \frac{\phi_n}{p_n} \left\{ \sin p_n t \int_0^t \ddot{y} \cos p_n \tau d\tau - \cos p_n t \int_0^t \ddot{y} \sin p_n \tau d\tau \right\} \end{aligned} \tag{12}$$

The kinetic energy in the structure at time t is

$$\begin{aligned} T &= \int \tfrac{1}{2} \dot{Y}^2 \, d\rho \\ &= \frac{1}{2} \sum_n B_n^2 \int \phi_n^2 \, d\rho \left\{ \cos p_n t \int_0^t \ddot{y} \cos p_n \tau d\tau + \sin p_n t \int_0^t \ddot{y} \sin p_n \tau d\tau \right\}^2 \end{aligned} \tag{13}$$

Using D'Alambert's principle, the potential energy at time t is

$$\begin{aligned} V &= -\int \tfrac{1}{2} Y(\ddot{Y} - \ddot{y}) d\rho \\ &= \frac{1}{2} \sum_n B_n^2 \int \phi_n^2 \, d\rho \left\{ \sin p_n t \int_0^t \cos p_n \tau d\tau - \cos p_n t \int_0^t \ddot{y} \sin p_n \tau d_\tau \right\} \\ &\qquad \cdot \left\{ \sin p_n t \int_0^t \ddot{y} \cos p_n \tau d\tau - \cos p_n t \int_0^t \ddot{y} \sin p_n \tau d\tau \right\} \end{aligned} \tag{14}$$

The total energy in the structure at time t is

$$E = T + V = \frac{1}{2} \sum_n B_n^2 \int \phi_n^2 \, d\rho \left\{ \left(\int_0^t \ddot{y} \sin p_n \tau d\tau \right)^2 + \left(\int_0^t \ddot{y} \cos p_n \tau d\tau \right)^2 \right\} \tag{15}$$

As shown in figure 4, the expression within the brackets in equation (15) is just the envelope of the integral, S, appearing in equation (12). Hence for practical purposes the maximum value of S may be equated to the square root of the maximum value of the expression in brackets. Letting S_n represent this maximum value, the maximum energy attained by the structure may be written

$$E = \frac{1}{2} \sum_n \left(B_n^2 S_n^2 \int \phi_n^2 \, d\rho \right)$$

If the value of S_n is independent of the period of vibration ($S_1 = S_2 = \cdots = S_n = S$) the energy is

$$E = \frac{1}{2} S^2 \sum_n B_n^2 \int \phi_n^2 \, d\rho$$

As may be seen from the equations of free vibration the coefficient B_n is determined so that $\dot{Y} = 1$ at $t = 0$ and hence

$$\sum_n B_n \phi_n = 1$$

and from the orthogonality relations

$$\sum_n B_n^2 \int \phi_n^2 \, d\rho = \int d\rho = M$$

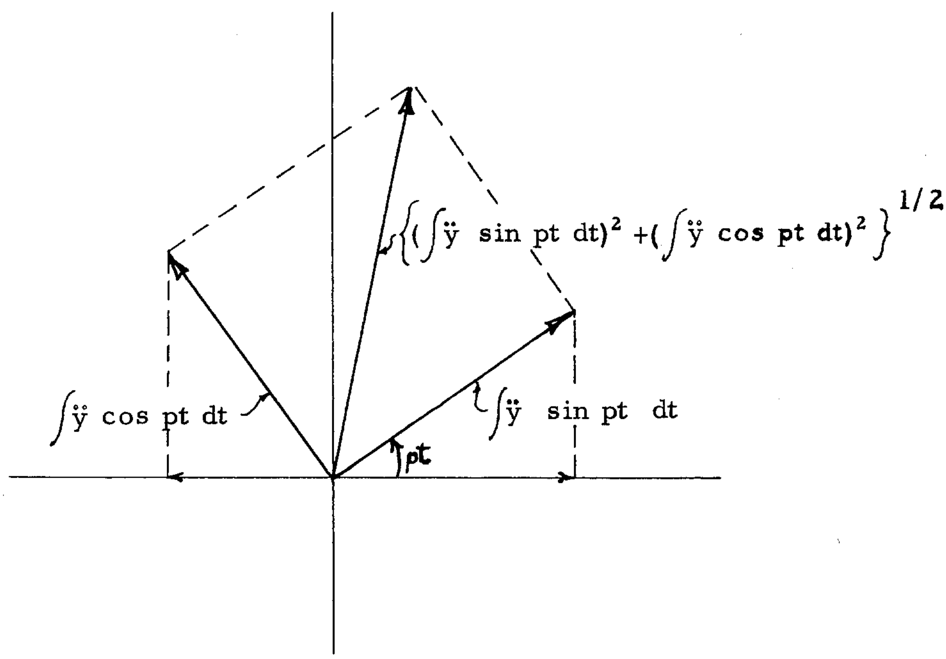

Fig. 4.

where M is the total mass of the structure. The maximum energy attained by the structure is thus

$$E = \tfrac{1}{2} M S^2$$

The maximum energy thus depends only on the mass and on S and is independent of how the mass is distributed and how the stiffness is distributed.

The value of S computed from strong ground motion is not itself a constant, but varies a bit. However, for a number of earthquakes the average value of S may be taken to have a constant value for periods from approximately 0.3 second to 2.5+ seconds. It can thus be said that if the modes of a structure are in this period range the expected value of the maximum energy will be independent of the distribution of mass and stiffness.

The foregoing value of $E = \tfrac{1}{2}MS^2$ represents the maximum energy attained by the undamped, elastic, structure. In an actual structure this energy will be partly stored as potential and kinetic energy and partly dissipated by the damping. If the amount of energy involved is sufficiently large the stresses will go beyond the yield point, or cracks may form with consequent energy loss. In the case of torsional oscillations of an unsymmetrical building it can thus be concluded that the asymmetry does not appreciably affect the total energy received by the building, but only affects the magnitude of the maximum stress. However, if the flexible wall were designed to be relatively too weak it would be the first to be overstressed and it would absorb the excess energy and the damage would tend to be concentrated there.

Summary

Summary.—It has been shown that an unsymmetrical building undergoing torsional accelerations can be expected to sustain stresses in the more flexible wall that are higher than those predicted by the customary static method of analysis. If the objective of the design is to keep the maximum stress within the usual allowable limits, a correction should be made to the results given by the usual static method of analysis. On the other hand, if the objective of the design is to provide a certain ultimate strength, the relative rigidity of the wall is not so important a factor as is the ability of the wall to absorb energy.

Division of Engineering,
California Institute of Technology,
Pasadena, California.