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COMMUNICATION AND EQUILIBRIUM IN DISCONTINUOUS GAMES OF INCOMPLETE INFORMATION

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This paper offers a new approach to the study of economic problems usually modeled as games of incomplete information with discontinuous payoffs. Typically, the discontinuities arise from indeterminacies (ties) in the underlying problem. The point of view taken here is that the tie-breaking rules that resolve these indeterminacies should be viewed as part of the solution rather than part of the description of the model. A solution is therefore a tie-breaking rule together with strategies satisfying the usual best-response criterion. When information is incomplete, solutions need not exist; that is, there may be no tie-breaking rule that is compatible with the existence of strategy profiles satisfying the usual best-response criteria. It is shown that the introduction of incentive compatible communication (cheap talk) restores existence.

KEYWORDS: Auctions, cheap talk, discontinuous games, sharing rules, tie-breaking rules.

1. INTRODUCTION

ECONOMICS IS REPLETE WITH SITUATIONS in which privately informed agents behave strategically; such situations are usually modeled as games of incomplete information. As Harsanyi showed, the equilibrium analysis of such games is no more complicated than the equilibrium analysis of games of complete information—provided the set of possible types of agents and the set of actions available to agents are finite. However, in many familiar situations—including Bertrand price competition, Cournot quantity competition, Hotelling spatial competition, games of timing, and auctions—actions are naturally modeled as continuous variables. Strategic analysis of such situations is difficult because tie-breaking rules—prescribing behavior of the auctioneer when agents submit the same bid for instance—lead to payoff functions that are discontinuous in actions.

[1] This paper merges "Cheap Talk and Discontinuous Games of Incomplete Information," by Simon and Zame, and "Existence of Equilibrium in Auctions and Discontinuous Bayesian Games: Endogenous and Incentive Compatible Sharing Rules," by Jackson and Swinkels. We are grateful for comments from Kim Border, Martin Cripps, Bryan Ellickson, Preston McAfee, Roger Myerson, John Nachbar, Phil Reny, John Riley, Larry Samuelson, Mark Satterthwaite, Tianxiang Ye, seminar audiences at Minnesota, Northwestern, Rochester, Stanford, the Stony Brook Game Theory Conference, Texas, UCLA, and UCSD. We especially thank a co-editor and three referees for helpful comments. Jackson is grateful for financial support from the National Science Foundation. Zame is grateful for the hospitality of the UC Berkeley Economics Department in Winter 1996, when much of this work was done, and for financial support from the Ford Foundation, from the National Science Foundation, and from the UCLA Academic Senate Committee on Research. Swinkels thanks the Boeing Center for Technology, Information, and Manufacturing for financial support.

Much of the existing analysis of such situations avoids the consequences of discontinuity by imposing conditions (such as private values, symmetric information, nonatomicity of prior distributions, etc.) that guarantee ties do not occur at equilibrium and hence that discontinuities do not matter.[2] As soon as we leave the simplest environments, however, we find situations in which ties do occur and discontinuities do matter—indeed, we find situations in which equilibrium does not exist.

For contexts in which information is complete, Simon and Zame (1990) (henceforth SZ) argued that such situations should be modeled, not as games in which payoffs are discontinuous, but rather as games in which payoffs are only partially determined, and that the tie-breaking rule that leads to discontinuities in payoffs should be viewed as part of the solution, rather than as part of the data. SZ show that (with natural conditions), such a solution (a tie-breaking rule together with a strategy profile satisfying the usual best-response criteria) always exists. In this paper, we extend this point of view and result to situations in which information is incomplete.

It might seem at first glance that this extension would be routine, following Harsanyi's method of analyzing a game Γ of incomplete information by transforming it into a game Γ^* of complete information—but it is not. One difficulty is that it is not clear how indeterminacies in the game Γ should be transformed into indeterminacies in the game Γ^*; another is that it is not clear what assumptions on Γ will guarantee that Γ^* has a solution. Most importantly, it is not clear how a solution for Γ^* (if it exists) should be interpreted as a solution to Γ.

That these difficulties reflect real problems with existence of a solution, and not merely with a particular approach, can be seen in a simple example. Consider a sealed-bid auction with two bidders, whose private valuations v_1, v_2 for a single indivisible object are drawn from a joint distribution as follows:

- with probability 1/2, $v_1 = 1$ and v_2 is drawn from the uniform distribution on [1, 2];
- with probability 1/2, $v_2 = 1$ and v_1 is drawn from the uniform distribution on [1, 2].

It is easy to see that the only tie-breaking rules that admit any equilibrium at all have the property that when both bidders bid 1, the object is awarded to the bidder whose valuation is higher. Given any such tie-breaking rule, it is an equilibrium for both bidders always to bid 1. Of course, such a tie-breaking rule cannot be implemented by an auctioneer who does not observe the valuations of the bidders, and allowing such observation would hardly seem consistent with the presumption that this information is private.

Suppose, however, we allow the bidders to announce their types (their true valuations) as well as their bids. If the auctioneer is constrained to sell the object at the highest bid and breaks ties by awarding the object to the bidder who announces the higher valuation, then it is an equilibrium for both bidders to bid 1

[2] Milgrom (1989) provides excellent background reading; for recent work, see LeBrun (1995, 1999), Maskin and Riley (2000), Bajari (2001), Reny (1999), Athey (2001), and Bresky (2000).

and to truthfully announce their types.[3] The key insight of this paper is that, in considerable generality, this communication is necessary and sufficient for the existence of a solution.

It is instructive to think about this auction in an environment in which bids must be in multiples of a smallest monetary unit δ; for simplicity assume that $1/\delta$ is an integer. Independently of the tie-breaking rule, it is an equilibrium for both players to follow the bidding strategy

$$b(v) = \begin{cases} 1-\delta & \text{if } v = 1, \\ 1 & \text{if } v > 1. \end{cases} \tag{1}$$

(Because $1/\delta$ is an integer, 1 is a multiple of δ, hence an admissible bid.) For every $\delta > 0$ the bids convey the information as to which bidder has the higher valuation, but in the limit when $\delta = 0$ this information is lost; allowing bidders to announce their valuations restores this lost information. (This echoes a theme of Christopher Harris.)

As in this simple example, our approach is to extend the model so that individuals may announce their private information. Such announcements need not be truthful and do not directly affect payoffs; their only role is to aid in breaking ties. In at least one interpretation (discussed further below), these announcements can be viewed as “cheap talk.” Our main result (Theorem 1) is that (with natural conditions) this extension always has at least one solution (a tie-breaking rule together with a strategy profile satisfying the usual best-response criterion) in which individuals truthfully announce their private information. Type announcements are thus incentive compatible. We emphasize that the tie-breaking rule will be determined as part of the solution, and not prescribed exogenously, that the tie-breaking rule may prescribe different divisions at different ties, and that the tie-breaking rule may depend on announcements as well as on actions. As the previous example and others in the text demonstrate, if we are not satisfied with such a tie-breaking rule then we will be faced with many situations in which no solution exists.

Although the proof of our main result is parallel to the proof of the main result of SZ, it is by no means a routine extension. (Our analysis would be much simpler if we restricted attention to finite type spaces, but it would seem contrived to insist on continuous action spaces and discrete type spaces.) The text discusses the differences between the present argument and that in SZ in some detail.

As in SZ, we might interpret an endogenous tie-breaking rule as a proxy for “actions taken by unseen agents whose behavior is not modeled explicitly.” For example, although we would commonly model a sealed-bid auction among N bidders as a simultaneous-move game with N players, it might also (and perhaps more properly) be modeled as a two-stage game with $N+1$ players. In the first

[3] The bidder whose value is 1 has no incentive to lie since he derives no surplus from obtaining the object at that price; the bidder whose value is above 1 has no incentive to lie since by telling the truth she obtains the object for her bid of 1.

stage of this latter game, the N bidders submit simultaneous bids; in the second stage the auctioneer chooses the winner. If the auctioneer is constrained (by law, for instance) to choose among the high bidders, then the auctioneer's strategy in the two-stage game corresponds precisely to an endogenous tie-breaking rule in the simultaneous-move game, and the subgame perfect equilibria of the two-stage game correspond precisely to solutions of the simultaneous-move game.[4,5] In general, the two-stage game will not admit any (subgame perfect) equilibrium unless we allow the bidders to communicate their private information. These communications do not affect payoffs—utilities depend only on private information, on bids, and on the auctioneer's actions—but the auctioneer conditions his actions on these communications. Hence communications in the two-stage game—which correspond precisely to announcements in the simultaneous-move game—are "cheap talk" in a familiar sense. Manelli (1996) provides a very similar use of cheap talk to guarantee the existence of equilibrium in signalling games.

Alternatively, an endogenous tie-breaking rule might be interpreted as a proxy for the outcome of an unmodeled second stage game. Thus, in their analysis of first-price sealed-bid auctions for a single indivisible item, Maskin and Riley (2000) adjoin to the sealed-bid stage a second stage in which the bidders who submitted the high bids in the first stage participate in a Vickrey auction. In the private value setting, it is a dominant strategy for bidders in this second stage Vickrey auction to bid their true values. Thus the second stage auction induces a tie-breaking rule that awards the item to the bidder who values it the most.

We should emphasize that our results concern only the existence of solutions in *mixed strategies*; we have little to say about the existence of solutions in *pure strategies*. For recent work on pure strategy equilibrium in auctions and similar environments, see Maskin and Riley (2000), Reny (1999), and Athey (2001).

Of course, the difficulties that arise because of discontinuities are a consequence of our insistence on a model in which action spaces are continuous. Restricting attention to discrete action spaces, either as an assumption about the situations to be modeled or as a modeling strategy for approximating continuous action spaces by discrete action spaces, will yield a game to which familiar fixed point theorems may be applied. However, there are a number of reasons why models with continuous action spaces may be more satisfactory than models with discrete action spaces.

(i) The equilibria of the game with discrete action spaces may depend very sensitively on the particular discretization chosen—but it may not be obvious that

[4] Of course, the auctioneer chooses among actions that leave him indifferent. One might ask why the auctioneer should choose in any particular way, but the answer, to quote SZ, is that "... equilibrium theory *never* explains why *any* agents would act in *any* particular way. Equilibrium theory is intended to explain *how* agents behave, not *why*."

[5] In general, equilibrium play for the $N+1$st player might depend on the valuation functions of bidders and on the distribution of types, data that a real auctioneer might not have. It seems an important challenge to identify circumstances in which uniform auction rules—not depending on such information—suffice to guarantee the existence of equilibrium. The two-stage auctions of Maskin and Riley (2000) are encouraging first steps in such a program.

any particular discretization is "correct." When van Gogh's "Irises" was sold at auction, bids were required to be in multiples of \$100,000, but other auctions frequently allow bids in multiples of \$10 or \$1. Indeed, when bids are prices per unit, they may well be in multiples of \$.01 or less. When the strategic variable is time, the issue is more subtle. Discretization amounts to an assumption that players can move only at some pre-specified speed, but there may be no reason to suppose that all players can move at the same speed—especially if a great deal can be gained by moving just a little more quickly.

Of course, continuous action spaces are an idealization, and would not be of much interest if equilibria in models with continuous action spaces did not correspond to limits of equilibria in models with discrete action spaces. Our convergence result (Theorem 2) shows that this is the case: if we restore information lost in the limit, then equilibria of the discrete action games converge to equilibria of the continuous action games.

(ii) The decision to model choice variables as continuous can greatly simplify the analysis of equilibrium. In private value auctions, for example, it is frequently the case that modeling bids as discrete variables leads to a multiplicity of equilibria, while modeling bids as continuous variables allows the conclusion that equilibrium is (essentially) unique.[6] Moreover, as in Maskin and Riley (2000), modeling bids as continuous variables allows equilibrium to be characterized as the solutions to differential equations.

(iii) Game theory usually simplifies the study of strategic interactions by assuming that choice variables are discrete; general equilibrium theory usually simplifies the study of markets by assuming that commodities are divisible. If we want to think about strategic interactions in markets, it seems necessary to accommodate continuous choice variables in game theory, just as indivisible goods have been accommodated in general equilibrium theory.

Applications are largely beyond the scope of the present paper, but we do give one simple application to private value auctions to show how a solution with communication may sometimes be used as a starting point from which to derive a solution without communication. Jackson and Swinkels (1999) and Simon and Zame (1999) provide more extensive elaborations on the same theme, extending some results of LeBrun (1995, 1999) and Maskin and Riley (2000).

Following this Introduction, Section 2 presents several examples that illustrate some of the difficulties we face and the way in which communication resolves them. Section 3 presents the general model. Section 4 discusses the extension to allow communication, and discusses our general existence result and a convergence theorem that follows as a straightforward consequence. Section 5 presents the application to private value auctions. Proofs are collected in Section 6.

[6] Our convergence result shows that equilibria of the auction games with discrete bids converge to a communication equilibrium of the auction game with continuous bids. As in Example 3, we can show that such communication equilibria are behaviorally equivalent to equilibria without communication. It follows that the equilibria of the auction games with discrete bids converge to the unique equilibrium of the auction game with continuous bids. Thus, the multiplicity of equilibria disappears in the limit.

2. EXAMPLES

In the Introduction, we have described a first price auction with private values which has the property that no tie-breaking rule that is independent of private information is compatible with any equilibrium. The analysis and conclusion depends crucially on the fact that marginal distributions have atoms (see Section 5). Lest the reader suspect that atoms play a crucial role in general, we give here a simple example to show that type-dependent tie-breaking rules may be required as soon as valuations have a common component.[7]

EXAMPLE 1: Consider a sealed-bid first price auction for a single indivisible object. There are two bidders; each bidder i observes a private signal t_i (which we identify as i's type). Types are independently and uniformly distributed on $[0, 1]$. Given types t_1, t_2, valuations are

$$\begin{aligned} &v_1(t_1, t_2) = 5 + t_1 - 4t_2, \\ &v_2(t_1, t_2) = 5 - 4t_1 + t_2. \end{aligned} \tag{2}$$

After observing private signals, bidders simultaneously submit bids; the high bidder wins and pays his bid. Ties are resolved according to some specified tie-breaking rule.

We claim that *no* type-independent tie-breaking rule is compatible with the existence of equilibrium. We defer the messy analysis to Section 6, but the intuition is not hard to convey.[8] Because types are independently distributed and valuations are increasing in own type, we can use standard arguments (see Maskin and Riley (2000) for instance) to show that if there is any equilibrium at all, then there is an equilibrium in which bid functions are weakly increasing and continuous at 0. For intuition, suppose there is an equilibrium in which bid functions b_1, b_2 are *strictly* increasing. Suppose $b_1(0) < b_2(0)$. Then the lowest types of bidder 2 sometimes win the object, which has an expected value less than 5, so $b_2(0) < 5$. But then the lowest types of bidder 1 *never* win the object and would prefer to bid slightly above $b_2(0)$ and win against the lowest types of bidder 2; this would be a contradiction. Hence $b_1(0) \geq b_2(0)$; reversing the roles of bidders 1 and 2 we conclude that $b_1(0) = b_2(0)$. If $b_1(0) = b_2(0) < 5$, then the lowest types of bidder 1 would prefer to bid slightly more than $b_1(0)$ and win more often when bidder 2's type is low and bidder 1's valuation is therefore high; this would be a contradiction. The only remaining possibility is that $b_1(0) = b_2(0) \geq 5$, but in that case the winning bid is always at least 5, and hence the expected payoff to the winner is negative. That means that the ex ante expectation of one of the bidders must be negative; that bidder would prefer to bid 0. Again, this would be a contradiction.

[7] Our example is a close relative of Example 3 of Maskin and Riley (2000), but in our example information has a continuous distribution, rather than an atomic distribution.

[8] Our analysis is aided by the fact that each bidder's valuation is decreasing in the other bidder's type, but our experience with discrete examples suggests to us that similar examples could be constructed in which valuations are increasing in both types.

Suppose, however, that we allow the bidders to announce their types as well as their bids and allow the auctioneer to use these announcements when breaking ties. Suppose, for instance, that the auctioneer breaks ties so that a bidder who announces a type above .5 always wins against a bidder who announces a type below .5, but randomizes with equal probabilities following all other pairs of announcements. Given this tie-breaking rule, it is an equilibrium for both bidders to bid 3.5, independent of their type, and to announce their type truthfully.[9] (Verifying that this is an equilibrium is straightforward but illuminating. Say a bidder is of high type if his signal is above .5, and a low type otherwise. If bidder 1 is a high type, bidding above 3.5 wins more often than bidding 3.5 *only* when bidder 2 is also a high type—in which case bidder 1 would (on average) prefer to lose. Thus, this is not an improving deviation. On the other hand, if bidder 1 is a low type, bidding above 3.5 guarantees than bidder 1 wins *all* the time—which yields negative expected utility. Since the putative equilibrium play yields positive expected utility to all types of bidder 1, this is not an improving deviation either. Finally, if bidder 1 bids below 3.5 he never wins, and hence obtains 0 expected utility. Because putative equilibrium play yields positive expected utility to all types of bidder 1, this is again not an improving deviation.)

The example above illustrates that without type-dependent tie-breaking rules, equilibria need not exist. Even when equilibria do exist, however, it may happen that the only equilibria are trivial or degenerate in some sense. The following example from Jackson (1999) illustrates the point.

EXAMPLE 2: Consider a sealed-bid second price auction for a single indivisible object. There are two bidders. Valuations have both a personal component and a common component; if i's personal component is x_i and the common component is q, bidder i's valuation is $ax_i + (1-a)q$. Note that $a = 1$ is the case of pure private values, while $a = 0$ is the case of pure common values; we assume $0 < a < 1$.

Prior to the auction, bidder i observes a private signal (x_i, t_i), which we identify as his type; x_i is i's personal component of the valuation and $t_i \in \{L, M, H\}$ is correlated with the common component q:

- if $t_i = L$, then $q = 0$;
- if $t_i = H$, then $q = v$;
- if $t_i = M$, then $q = 0, v$ with equal probability (so this signal is uninformative).

Personal components x_i are distributed independently and take on values $0, 1$ with equal probability. The true common component q is distributed independently of x_1, x_2 and takes on values $0, v$ with equal probability. If the common component $q = 0$, signals are drawn independently from the distribution that puts probability .5 on each of the alternatives L, M; if the common component $q = v$, signals are drawn independently from the distribution that puts probability .5 on each of the alternatives M, H.

[9] Neither the cut point .5 nor the common bid 3.5 is uniquely determined.

This auction always admits trivial asymmetric equilibria: one bidder always bids $1+v$ while the other bidder always bids 0. But if the tie-breaking rule is type-independent, then for some values of a and v this auction admits no symmetric equilibrium; indeed, for some values of a and v it admits *only* equilibria in which at least one type of one player bids above his maximum possible valuation (conditional on his information). Equilibria in which such strategies are employed would be ruled out by any sensible notion of perfection, such as that offered by Simon and Stinchcombe (1995).

3. GAMES WITH INDETERMINATE OUTCOMES

A *game with indeterminate outcomes* consists of:

- a finite set of players $N = \{1, \ldots, n\}$;
- for each player i, a space A_i of *actions*; write $A = \times A_i$ for the space of *action profiles*;
- for each player i, a space T_i of *types*; write $T = \times T_i$ for the space of *type profiles*;
- a probability measure τ on T;
- a space Ω of *outcomes*;
- an *outcome correspondence* Θ: $A \to \Omega$;
- a utility mapping u: graph $\Theta \times T \to \mathbb{R}^N$.

Note that outcomes depend on actions *but not on types*. On the other hand, utilities depend on actions, on outcomes, and on types. The interpretation intended is that for an action profile $a \in A$, the set $\Theta(a)$ represents the set of outcomes that *might* result if the action profile a is taken. In the usual first price auction for a single indivisible object, the structure is particularly simple: $\Theta(a)$ is a singleton unless a involves ties, in which case $\Theta(a)$ represents the probability with which each of the high bidders receives the object. (Assuming that $\Theta(a)$ assigns positive probability only to the high bidders represents a natural constraint on the auctioneer. Of course one might certainly imagine situations in which the auctioneer was subject to fewer constraints, leading to a different specification of Θ.) In an auction for k (not necessarily identical) divisible objects, actions might be demand schedules, and $\Theta(a)$ might represent physical or probabilistic divisions of each of the various objects and charges to the various bidders.

As usual, we write t_{-i} for a profile of types of all players other than i, and T_{-i} for the space of such type profiles. We adopt similar notation for action profiles, strategy profiles, marginals, etc.

We assume throughout that:

- action spaces A_i and type spaces T_i are compact metric;
- τ is a Borel measure;
- τ is absolutely continuous with respect to the product $\times \tau_i$ of its marginals;[10]

[10] So information is absolutely continuous in the sense of Milgrom and Weber (1985).

- Ω is a compact convex[11] metrizable subset of a locally convex topological vector space $\mathscr{E}$;
- the outcome correspondence Θ is upper-hemi-continuous, with nonempty compact values;
- the utility mapping is continuous.[12]

The game Γ is *affine* if the correspondence Θ has convex values and, for each action profile $a \in A$ and type profile $t \in T$ the function

$$(3) \qquad u(a, \cdot, t): \Theta(a) \longrightarrow \mathbb{R}^N$$

is affine. The meaning of affineness can be seen easily in a simple perfect information, first-price auction for a cake of size 1. Suppose each bidder i is a von Neumann–Morgenstern expected utility maximizer, whose utility for cake and money is $U_i(c, m) = u_i(c) + m$; without loss normalize so that $u_i(0) = 0$, $u_i(1) = 1$. If outcomes are physical divisions of cake and payment, then bidder i's utility when both bidders bid b and i gets the physical share θ is $u_i(\theta) - \theta b$. Hence the game is affine exactly when the functions u_i are affine; with our normalization this means that $u_i(c) = c$. On the other hand, if outcomes are probabilistic divisions of cake and payment, then (because i is an expected utility maximizer) bidder i's utility when i bids b and gets the probabilistic share θ is $\theta[u_i(1) - b]$. Hence the game is always affine. In particular, when outcomes are probabilistic divisions, affineness is compatible with risk aversion or with any other attitude toward risk.

If type spaces are singletons, an affine game with indeterminate outcomes is equivalent to what SZ calls a *game with an endogenous sharing rule*.

Following Milgrom and Weber (1985), a *distributional strategy* for player i is a probability measure σ_i on $T_i \times A_i$ whose marginal on T_i is τ_i. If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a profile of distributional strategies, we write $\bar{\sigma}$ for the joint distribution on $T \times A$. If f is the Radon-Nikodym derivative of τ with respect to the product $\prod_i \tau_i$ of its marginals, then $\bar{\sigma} = f \prod_i \sigma_i$.

If θ: $A \to \Omega$ is a selection from Θ (that is, $\theta(a) \in \Theta(a)$ for each $a \in A$) that is universally measurable,[13] and $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a profile of distributional strategies, we define expected utilities

$$(4) \qquad Eu_i(\sigma|\theta) = Eu_i(\sigma_i|\sigma_{-i}, \theta) = \int_{T \times A} u_i(a, \theta(a), t)\, d\bar{\sigma}.$$

[11] Convexity of Ω itself is merely a convenient technical assumption; all we need is that the range of Θ be contained in *some* compact convex metrizable set.

[12] Assuming that type spaces are compact metric and that utility is continuous in types—rather than simply measurable—may involve some small loss of generality.

[13] Recall that a set G is *universally measurable* if it is measurable with respect to the completion of every Borel measure; i.e., for every Borel measure ν there are Borel sets G', G'' such that $(G \backslash G') \cup (G' \backslash G) \subset G''$ and $\nu(G'') = 0$. Universal measurability of the selection is the weakest measurability requirement consistent with the desideratum that expected utility be well-defined for all strategy profiles. As the reader will see, the selections constructed in Theorems 1 and 2 of this paper will in fact be Borel measurable.

A *solution* of Γ consists of a universally measurable selection θ from Θ and distributional strategies $\sigma_1, \ldots, \sigma_n$ that satisfy the usual best response criterion: for each i and each distributional strategy σ_i' on $T_i \times A_i$ we have

$$Eu_i(\sigma_i | \sigma_{-i}, \theta) \geq Eu_i(\sigma_i' | \sigma_{-i}, \theta). \tag{5}$$

We emphasize that we allow the tie-breaking rule θ to depend on actions but not on types—which will typically be unobservable.

Alternatively, given a universally measurable selection θ from Θ, we define a Bayesian game Γ^θ by specifying players, action spaces, type spaces, and priors as for Γ, and defining utilities by $u^\theta(a, t) = u(a, \theta(a), t)$. A solution for Γ may therefore be identified as a universally measurable selection θ from Θ and a profile of distributional strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ which constitute a Bayesian Nash equilibrium for Γ^θ. If θ is a continuous selection, then Γ^θ has continuous payoffs and the existence of equilibrium follows from familiar results. However, the correspondence Θ may not admit *any* continuous selections, so that the game Γ^θ will typically have discontinuous payoffs.

4. COMMUNICATION

As our examples show, a game with indeterminate outcomes may not admit any solutions. In this section we show how to expand the game to allow players to communicate their private information. In the presence of natural assumptions, this communication guarantees the existence of equilibrium.

Let Γ be a game with indeterminate outcomes. For mnemonic purposes, set $S_i = T_i$ for each i; we will view elements of S_i as announcements and elements of T_i as true types. The *communication extension* Γ^c is the game with indeterminate outcomes defined by:

- player set $N = \{1, \ldots, n\}$;
- action spaces $S_i \times A_i$;
- type spaces T_i;
- outcomes Ω;
- outcome correspondence Θ^c: $S \times A \to \Omega^c$ defined by $\Theta^c(s, a) = \Theta(a)$;
- prior τ;
- utility mapping u^c: graph $\Theta^c \times T \to \mathbb{R}^N$ defined by $u^c(s, a, \omega, t) = u(a, \omega, t)$.

That is, Γ^c differs from Γ only in that we allow players to announce their types—and hence allow the auctioneer to condition on these announcements—but the announcements are not payoff relevant.

THEOREM 1: *If Γ is an affine game with indeterminate outcomes, then the extension Γ^c admits a solution in which the tie-breaking rule is Borel measurable and type announcements are truthful.*[14]

[14] That is, equilibrium strategies σ_i, which are probability distributions on $S_i \times A_i \times T_i$, are supported on $\{(s_i, a_i, t_i) : s_i = t_i\}$.

If information is complete, the extension Γ^c coincides with Γ. As we have noted earlier, when information is complete, a game with indeterminate outcomes is equivalent to a game with an endogenous sharing rule in the sense of SZ. In this case, therefore, Theorem 1 reduces to the main result of SZ.

The proof of Theorem 1 is in six steps, paralleling the proof of the main result of SZ, but with substantial differences, indicated below.

Step 1—Finite Approximations: Given a game Γ and the communication extension Γ^c, we construct families $\{\Gamma^r\}$, $\{\Gamma^{cr}\}$ of games with finite action spaces (but the same type spaces as Γ) which "approximate" Γ, Γ^c. For each of these games we choose an arbitrary selection q^r from the outcome correspondence, and use these selections to define a Bayesian game. Each of these Bayesian games admits a Bayesian Nash equilibrium σ^r in distributional strategies, having the property that type announcements are truthful. Each q^r, which is a function on A, has trivial extensions $\theta^r: S \times A \to \Omega$, $\tilde{\theta}^r: S \times A \times T \to \Omega$ that are independent of announcements and types. (The construction here is more elaborate than in SZ—because we must take account of outcomes and types—but very similar.)

Step 2—Limits: The strategy profiles σ^r correspond to joint distributions $\bar{\sigma}^r$, and induce outcome-valued vector measures $\tilde{\theta}^r \bar{\sigma}^r$. Passing to a subsequence as necessary, we show that (σ^r) converges to a strategy profile σ for the game Γ^c, and $(\tilde{\theta}^r \bar{\sigma}^r)$ converges to an outcome-valued vector measure ν of the form $\nu = \tilde{\theta}\bar{\sigma}$ where θ is a selection from the outcome correspondence Θ^c and $\tilde{\theta}$ is the trivial extension to $S \times A \times T$ that does not depend on types.

Step 3—Convergence of Utilities: Convergence of strategy profiles and selections implies convergence of utilities. (This step, which is required because we work in outcome space, has no analog in SZ, which works entirely in utility space.)

Step 4—Identifying Better Responses: The desired solution strategy profile is σ; the tie-breaking rule will be a perturbation of θ. Perturbation may be necessary because $\tilde{\theta}$ (hence θ) is only determined up to sets of $\bar{\sigma}$-measure 0, leaving open the possibility that there are profitable deviations. For each player i we identify a set $H_i \subset A_i$ where perturbations may be necessary to prevent deviations by that player, and use the absence of profitable deviations in the games Γ^r to show that H_i has measure 0. (The argument here is different than in SZ because the dependence of utilities on types requires that we be substantially more careful in the construction of the corresponding deviations in the finite games.)

Step 5—Perturbation: We construct the necessary perturbations on the measure 0 sets H_i. (The argument here is much different than in SZ and more subtle in several ways. The perturbations are constructed to punish the potential deviator. In SZ all that is necessary is to choose the worst possible outcome for the deviator. Here, however, utilities depend on types, so there need be no outcome that is "worst possible" for all types of the potential deviator—indeed, there need be no outcome that is uniformly bad for all types of the potential deviator. We therefore use punishment outcomes constructed as limits of outcomes in the finite games. The argument is subtle because these limits must be taken in the weak sense of convergence of vector valued measures, and because these various punishments must be assembled in a measurable way. The construction relies on an infinite dimensional extension of a measurable selection theorem of Dellacherie and Meyer (1982).)

Step 6—Equilibrium: We verify that the perturbed selection θ' and strategy profile σ constitute a solution for Γ^c.

The same argument that proves Theorem 1 establishes a convergence result for equilibria of sequences of games. In order to give a precise statement, we first need to describe the relevant notion of convergence of games.

For $r = 0, 1, \ldots$, let $\Gamma^r = \langle N^r, (A_i^r), (T_i^r), \tau^r, \Omega^r, \Theta^r, u^r \rangle$ be a game with indeterminate outcomes. We assume that all the games in question have the same set of players $N^r = N = \{1, \ldots, n\}$, that the action spaces A_i^r lie in a fixed compact metric space A_i, that the type spaces T_i^r lie in a fixed compact metric space T_i, and that the outcome spaces Ω^r lie in a fixed compact metric space Ω. We say that the sequence of games $\{\Gamma^r\}$ *converges to* Γ^0 if:

- for each i, $A_i^r \to A_i^0$ in the Hausdorff metric;
- for each i, $T_i^r \to T_i^0$ in the Hausdorff metric;
- $\Omega^r \to \Omega^0$ in the Hausdorff metric;
- the graph of Θ^r: $A^r \to \Omega^r$ converges to the graph of Θ^0: $A^0 \to \Omega^0$ in the Hausdroff metric;
- $\tau^r \to \tau$ in the total variation norm (as measures on $T = T_1 \times \cdots \times T_n$);
- for every $\varepsilon > 0$ there is a $\delta > 0$ and an index r_0 such that if:

  (i) $r \geq r_0$;
  (ii) $(a^r, \omega^r, t^r) \in \text{graph } \Theta^r$;
  (iii) $(a^0, \omega^0, t^0) \in \text{graph } \Theta^0$;
  (iv) $\text{dist}((a^r, \omega^r, t^r), (a^0, \omega^0, t^0)) < \delta$;

then $|u^r(a^r, \omega^r, t^r) - u^0(a^0, \omega^0, t^0)| < \varepsilon$.

Note that we require convergence of priors in the variation norm—not in the weak* topology—and uniform convergence of utilities.

Our convergence theorem can be formulated in the following way.

THEOREM 2: *Let $\{\Gamma^r\}$ be a sequence of affine games with indeterminate outcomes, converging to the affine game with indeterminate outcomes Γ^0. For every $r \geq 1$, let θ^r; $\sigma_1^r, \ldots, \sigma_n^r$ be a solution for the communication extension Γ^{rc} in which type announcements are truthful. Then there is a solution θ^0; $\sigma_1^0, \ldots, \sigma_n^0$ for the communication extension Γ^{0c} in which the type announcements are truthful and a subsequence $\{\Gamma^{r^j}\}$ such that*:

- *for each i, $\sigma_i^{r^j} \to \sigma_i^0$ weak*;*
- *if $\bar{\sigma}^{r^j}$, $\bar{\sigma}^0$ are the joint distributions of actions, then $\theta^{r^j}\bar{\sigma}^{r^j} \to \theta^0\bar{\sigma}^0$ weak*;*
- *for each i, $Eu_i^{r^j c}(\sigma^{r^j}, \theta^{r^j}) \to Eu_i^{0c}(\sigma^0, \theta^0)$.*

Some consequences of this convergence theorem are worth noting.

(i) Theorem 1 itself is a direct application of Theorem 2 (as sketched above).

(ii) If Γ is a symmetric game with indeterminate outcomes, then the communication extension Γ^c has a symmetric solution. (Write Γ^c as the limit of symmetric finite games Γ^r. Each Γ^r has a symmetric solution, which induces a symmetric solution of the communication extension Γ^{rc}. Some subsequence of these symmetric solutions converges to a solution of the communication extension Γ^c, and any limit of such a subsequence is a symmetric solution.) Note that symmetry entails that the tie-breaking rule depends on actions and type announcements but not on names of the players.

(iii) The communication extension of a game with indeterminate outcomes admits a solution that is "perfect" in the sense of Simon and Stinchcombe (1995). This is useful because, as we have noted earlier, perfection rules out trivial equilibria.

We have focused on solutions in which type announcements are truthful, but this probably involves little loss. Indeed, if Γ is a game with indeterminate outcomes, Γ^c is the communication extension, and θ, σ is a solution of Γ^c for which type announcements are not truthful, we can use a familiar revelation argument to construct a solution θ', σ' of Γ^c that prescribes the same actions, and truthful type announcements, and that induces the same outcome distribution.[15]

5. PRIVATE VALUE AUCTIONS

Our existence result (Theorem 1) guarantees that the communication extension of a game admits solutions in which players communicate their private information. However, in many circumstances it is possible to turn a solution with communication into a solution without communication, and hence obtain a solution to the game without communication. The following simple private value auction will illustrate the point. For more on private value auctions, see LeBrun (1995, 1999), Maskin and Riley (2000), Athey (2001), Jackson and Swinkels (1999), Simon and Zame (1999), and Bresky (2000).

EXAMPLE 3: Consider a sealed bid first price auction for a single indivisible item. Risk neutral bidders $i = 1, \ldots, n$ draw a private value t_i according to a joint distribution τ on $T = T_1 \times \cdots \times T_n = [0, 1]^n$; i's utility if he wins the item and pays b is $t_i - b$. Write τ_i for the marginal of τ on T_i. We assume that each of the marginals τ_i is nonatomic, and that τ is absolutely continuous with respect to the product $\tau_1 \times \cdots \times \tau_n$ of the marginals. Note that, subject to the limitation that the joint distribution be absolutely continuous with respect to the product of its marginals, we allow valuations to be correlated or affiliated to an arbitrary extent.

If bids are constrained to be multiples of a smallest monetary unit, the game corresponding to this auction admits a perfect equilibrium, and such an equilibrium has the property that bidders never submit bids above their true values. Theorem 2 therefore guarantees that when arbitrary bids are allowed, the communication extension of the game has a solution θ; $\sigma_1, \ldots, \sigma_n$ with the property that bidders never submits bids above their true values. We assert that in any such solution, the probability that a tie for the high bid occurs is 0. Because the

[15] We thank Phil Reny for pointing this out.

details are a little fussy, we merely indicate the argument here, referring to Jackson and Swinkels (1999) or Simon and Zame (1999) for details.

(i) Whenever a tie for the highest bid occurs, there is at most one bidder whose value exceeds the bid, and this bidder must win the item with probability 1. (Otherwise, some bidder who does not win such ties with probability 1 would gain by bidding a little bit more.)

(ii) Ties for the highest bid occur with probability 0. (To see this, fix a bidder i and a type t_i of bidder i, and condition on bidder i being type t_i, bidding below his true value, being the highest bidder, and winning a tie. In view of (i), this can only happen if the other bidders who are tied with i are bidding their true value and i is bidding precisely this value. Absolute continuity of information and nonatomicity of marginal distributions imply that this is a set of probability 0. Integrating and applying Fubini's theorem guarantees that ties for the highest bid occur with probability 0.)

Since ties for the high bid occur with probability 0, the tie-breaking rule is irrelevant; in particular, if φ is the tie-breaking rule that randomizes equally among all high bidders, then $\varphi; \sigma$ is also a solution for the game with communication, whence $\varphi; \sigma$ is a solution for the game without communication. (To see this, suppose that $\varphi; \sigma$ were not a solution, so that some bidder, say bidder 1, would prefer to follow a strategy $\sigma_1' \neq \sigma_1$. If ties occur with positive probability when bidders follow $\sigma_1', \sigma_2, \ldots, \sigma_n$, then we could construct another strategy σ_1'' for bidder 1, in which 1 bids slightly more than in σ_1', which 1 still prefers to σ_1, and which has the property that ties occur with 0 probability when bidders follow $\sigma_1'', \sigma_2, \ldots, \sigma_n$. But if ties occur with 0 probability when bidders follow $\sigma_1'', \sigma_2, \ldots, \sigma_n$, then payoffs will be the same as when bidders follow σ. That is, σ_1'' is not preferred by bidder 1, hence σ_1' is not preferred by bidder 1.)[16]

An additional point is worth noting. Assume in addition that valuations are independently distributed, and write $b_i(t_i)$ for i's (perhaps mixed) equilibrium bidding strategy conditional on observing the valuation t_i. It is easily seen that b_i is strictly monotone, in the sense that if $t_i > t_i'$, then every bid in the support of $b_i(t_i)$ is at least as large as every bid in the support of $b_i(t_i')$. It follows that $b_i(t_i)$ is a pure strategy for almost all valuations t_i. Since ties occur with probability 0, we can change the bidding strategies on a set of measure 0 and obtain a pure strategy equilibrium.

6. PROOFS

6.1. *Details of Example 1*

As promised, we provide here the details that no type-independent tie-breaking rule is compatible with the existence of any equilibrium. To see this, fix a type-independent tie-breaking rule, and assume that b_1, b_2 constitute an equilibrium in mixed behavioral strategies. Because signals are independent and valuations

[16] The existence of such a solution could also be obtained from the results of Reny (1999).

are strictly increasing in own signal, the proof of Proposition 1 in Maskin and Riley (2000) is easily adapted to show that there is no loss in assuming the bidding strategies b_1, b_2 are monotone, in the sense that if $t_i' > t_i$, then every bid in the support of $b_i(t_i')$ is at least as large as every bid in the support of $b_i(t_i)$. It follows immediately that there is an at most countable set of signals t_i for which the support of $b_i(t_i)$ is not a singleton. For such t_i, replace b_i by the infimum of the support of $b_i(t_i)$. It is easily checked that the modified bid functions b_1, b_2 again constitute an equilibrium. Thus we have an equilibrium in monotone, pure behavioral strategies. Altering bids following signals 0, 1 if necessary, there is no loss in assuming that b_1, b_2 are continuous at 0, 1.

Let $\underline{b} = \max\{b_1(0), b_2(0)\}$, and for each i, let $\tau_i = \sup\{t | b_i(t) \leq \underline{b}\}$. Suppose that $\tau_2 = 0$. Then, $b_1(0)$ wins with probability 0, and so earns 0. But, a bid of $\underline{b}+\varepsilon$ by 1 wins with positive probability, and for ε small, does so only when $t_2 \cong 0$ (using that $\tau_2 = 0$) so that $v_1 \cong 5+0-4(0) = 5$. For there not to be a profitable deviation of this form, it must thus be that $\underline{b} \geq 5$, and hence that the winning bid is always at least 5. But, the average value of the object, even if allocated optimally to the player with larger t, is $5+2/3-4(1/3) < 5$, since 2/3 and 1/3 are the expected higher and lower values of two draws from the uniform distribution. So, someone is losing money on average, and would be better off to bid 0 always. This is a contradiction, and so $\tau_2 > 0$. Arguing symmetrically, $\tau_1 > 0$.

Assume both players use $\underline{b}$ with positive probability, and assume that ties at $\underline{b}$ are broken with probability $p \in (0,1)$ in favor of player 1. Let t' and t'', $t' < t''$, be two values of t for which $b_1(t) = \underline{b}$. It follows that $5+t'-4E(t_2 | b_2(t_2) = \underline{b}) \geq \underline{b}$, else 1 would be better to bid $\underline{b}-\varepsilon$ with t'. But then $5+t''-4E(t_2 | b_2(t_2) = \underline{b}) > \underline{b}$, and so 1 should deviate to $\underline{b}+\varepsilon$ with t''. This is a contradiction. There are thus two remaining possibilities.

(i) One player, w.l.o.g. player 2, does not use $\underline{b}$ with positive probability (since $\tau_2 > 0$, this implies $b_2(0) < \underline{b}$, and also that player 1 bids $\underline{b}$ with positive probability since $\tau_1 > 0$, and by definition of $\underline{b}$). Now, with $t = \tau_2 - \varepsilon$, player 2 never wins, but by bidding $\underline{b}+\varepsilon$ wins with positive probability for an expected value of $5+\tau_2 - \varepsilon - 4(\tau_1/2)$. So, for 2 not to want to deviate, it must be that $\underline{b} \geq 5+\tau_2-4(\tau_1/2) > 5+\tau_2-4\tau_1$.

(ii) Both players use $\underline{b}$ with positive probability. Then, by the above, one player, again w.l.o.g. player 2, always has ties at $\underline{b}$ decided against him. Let (γ_1, τ_1) be the (nonempty) interval over which player 1 bids $\underline{b}$. Then, with $t = \tau_2 - \varepsilon$, player 2 wins only when $t_1 < \gamma_1$, while by bidding ε more, he can also win when $t_1 \in (\gamma_1, \tau_1)$. For this not to be a profitable deviation, it must be that $\underline{b} \geq 5+\tau_2-4(\gamma_1+\tau_1)/2 > 5+\tau_2-4\tau_1$.

Assume that $\tau_2 < 1$. Pick $t = \tau_2+\varepsilon$, and consider replacing $b_2(t)$ (which is by definition greater than $\underline{b}$) by any bid in $(\underline{b}, b_2(t))$. This bid pays less in the (positive probability, since $\tau_1 > 0$) event that it still wins, and when it changes a win into a loss, $t_1 \geq \tau_1$, and hence v_2 is at best $5+\tau_2+\varepsilon-4\tau_1 < \underline{b}$. So, this is a profitable deviation, a contradiction. Thus, $\tau_2 = 1$.

Since $\tau_2 = 1$, it follows that 1 wins with probability 1 (since 2 does not win when $t_2 < 1$), and hence that he always bids $\underline{b}$ (he bids at least this by definition, and need not bid any more since $\tau_2 = 1$). Hence, $\underline{b} \leq 3 = 5 + 0 - 4(1/2)$; otherwise 1 is better to bid 0 with t_1 near 0. But then, player 2 can profitably bid $\underline{b} + \varepsilon$ when he has t above 0, a contradiction. Thus, there is no equilibrium to this game.

6.2. *Proofs of Theorems 1 and 2*

We need some preliminary results. We begin by establishing some facts about weak* convergence and marginals.

LEMMA 1: *Let X, Y be compact metric spaces, let λ be a positive measure on X, let f be a positive λ-integrable function on X, and let (γ_n) be a sequence of positive measures on $Y \times X$ whose marginals on X are λ. If $\gamma_n \to \gamma$ in the weak* topology, then*
(i) *the marginal of γ on X is λ;*
(ii) *$f\gamma_n \to f\gamma$ in the weak* topology.*

PROOF: To see (i), fix an open set $U \subset X$ and $\varepsilon > 0$. Because X is compact and metrizable, U is the increasing union of compact sets. Thus, we can choose a compact set $K \subset U$ such that

$$\lambda(U) - \lambda(K) < \varepsilon \quad \text{and} \quad \gamma(Y \times U) - \gamma(Y \times K) < \varepsilon. \tag{6}$$

Use Urysohn's Lemma to choose a continuous function g: $X \to [0, 1]$ that is 1 on K and 0 on the complement of U. Then

$$\gamma(Y \times K) \leq \int g \, d\gamma \leq \gamma(Y \times U). \tag{7}$$

Because the marginal of γ_n on X is λ and g is independent of Y, it follows that

$$\lambda(K) = \gamma_n(Y \times K) \leq \int g \, d\gamma_n \leq \gamma_n(Y \times U) = \lambda(U) \tag{8}$$

for each n. Weak* convergence of (γ_n) to γ guarantees that for n sufficiently large

$$\left| \int g \, d\gamma_n - \int g \, d\gamma \right| < \varepsilon. \tag{9}$$

Combining these inequalities, we conclude that $|\gamma(Y \times U) - \lambda(U)| < \varepsilon$. Because $\varepsilon > 0$ is arbitrary, it follows that $\gamma(Y \times U) = \lambda(U)$. Because U is an arbitrary open set, it follows that the marginal of γ on X is λ, as asserted.

To see (ii), fix a continuous real-valued function h on $X \times Y$ and $\varepsilon > 0$. Write $M = \sup_{X \times Y} |h|$. Because f is integrable, there is a $\delta > 0$ such that $\int_E f \, d\lambda < \varepsilon$ whenever $\lambda(E) < \delta$; without loss we may assume $\delta < \varepsilon$. Use Lusin's Theorem to choose a compact set $K \subset X$ such that $\lambda(X \setminus K) < \delta$ and the restriction of f to K is continuous. Choose an open set $U \supset K$ such that $\lambda(U \setminus K) < \varepsilon / \max_K f$. Use Urysohn's Lemma and the Tietze Extension Theorem to choose a continuous function $\bar{f}$ on X that agrees with f on K, vanishes off U, and for which $\max_X \bar{f} = \max_K f$.

Recalling that the marginal of γ on X is λ, that $\int_E f\, d\lambda < \varepsilon$ whenever $\lambda(E) < \delta$, that $\lambda(X \setminus K) < \delta$, and that $\lambda(U \setminus K) < \varepsilon / \max_X \bar{f}$, yields

$$(10) \qquad \left| \int h\bar{f}\, d\gamma - \int hf\, d\gamma \right| \leq \int_{Y\times(U\setminus K)} |h\bar{f} - hf|\, d\gamma + \int_{Y\times(X\setminus U)} |h\bar{f} - hf|\, d\gamma$$
$$\leq M\left[\int_{Y\times(U\setminus K)} \bar{f}\, d\gamma + \int_{Y\times(U\setminus K)} f\, d\gamma + \int_{Y\times(X\setminus U)} f\, d\gamma \right]$$
$$= M\left[\int_{U\setminus K} \bar{f}\, d\lambda + \int_{U\setminus K} f\, d\lambda + \int_{X\setminus U} f\, d\lambda \right]$$
$$\leq 3\varepsilon M.$$

Similarly

$$(11) \qquad \left| \int h\bar{f}\, d\gamma_n - \int hf\, d\gamma_n \right| \leq 3\varepsilon M.$$

Weak* convergence of (γ_n) to γ entails that for n sufficiently large

$$(12) \qquad \left| \int h\bar{f}\, d\gamma_n - \int h\bar{f}\, d\gamma \right| < \varepsilon.$$

Combining these inequalities, we conclude that for n sufficiently large

$$(13) \qquad \left| \int hf\, d\gamma_n - \int hf\, d\gamma \right| < \varepsilon + 6\varepsilon M.$$

Because $\varepsilon > 0$ is arbitrary, it follows that $\int hf\, d\gamma_n \to \int hf\, d\gamma$. Because h is arbitrary, we conclude that $f\gamma_n \to f\gamma$ weak*, as asserted. *Q.E.D.*

The proof makes use of the theory of vector measures and integration of vector-valued functions. An excellent reference is Diestel and Uhl (1977); we collect the basic information here. Let X be a set and $\mathcal{F}$ a sigma-algebra of subsets of X. (When X is a compact metric space we take $\mathcal{F}$ to be the sigma-algebra of Borel sets.) Let $\mathcal{E}$ be a Hausdorff, locally convex topological vector space and let $\mathcal{E}^*$ be its dual, the space of continuous linear functionals. For $\omega \in \mathcal{E}, \varphi \in \mathcal{E}^*$, we write $\varphi \cdot \omega$ for the value of φ at ω. A *vector measure on X with values in $\mathcal{E}$ (an $\mathcal{E}$-valued measure)* is a (weakly) countably-additive function $\mu: \mathcal{F} \to \mathcal{E}$.[17] For μ an $\mathcal{E}$-valued measure and $\varphi \in \mathcal{E}^*$, we write $\varphi \cdot \mu$ for the real-valued measure defined by $\varphi \cdot \mu(E) = \varphi \cdot (\mu(E))$.

The function $f: X \to \mathcal{E}$ is *weakly measurable* if the real-valued composition $\varphi \cdot f$ is measurable for each $\varphi \in \mathcal{E}^*$. (Equivalently, the inverse image of every

[17] Weak countable additivity means that if $\{E_n\}$ is a countable disjoint collection of Borel measurable subsets of X, then $\mu(\cup E_n) = \sum \mu(E_n)$, convergence of the summation being in the weak topology of $\mathcal{E}$; equivalently, $\varphi \cdot \mu(\cup E_n) = \sum \varphi \cdot \mu(E_n)$ for each $\varphi \in \mathcal{E}^*$.

weakly open set is measurable.) If λ is a measure on $\mathscr{F}$, the weakly measurable function f is *Pettis integrable* (or *weakly integrable*) if for each $E \in \mathscr{F}$ there is an element $\omega_E \in \mathscr{E}$ such that $\varphi \cdot \omega_E = \int_E \varphi \cdot f \, d\lambda$ for all $\varphi \in \mathscr{E}^*$. If this is the case, we define $\int_E f \, d\lambda = \omega_E$ to be the *Pettis integral of f on E*. If f is Pettis integrable then the function μ: $\mathscr{F} \to \mathscr{E}$ defined by $\mu(E) = \int_E f \, d\lambda$ is an $\mathscr{E}$-valued measure. In this circumstance, we say that f is the *Radon-Nikodym derivative of μ with respect to λ* and write $\mu = f\lambda$.

Recall that weak* convergence of scalar measures on a compact space X is defined by

(14) $$\lambda_\alpha \longrightarrow \lambda \iff \int f \, d\lambda_\alpha \longrightarrow \int f \, d\lambda \text{ for all continuous } f.$$

Weak* convergence of $\mathscr{E}$-valued measures on X is defined by

(15) $$\mu_\alpha \longrightarrow \mu \iff \varphi \cdot \mu_\alpha \longrightarrow \varphi \cdot \mu \text{ (weak}^*\text{) for all } \varphi \in \mathscr{E}^*.$$

If $\Omega \subset \mathscr{E}$ and Θ: $X \to \Omega$ is a correspondence, we write $M(X, \Omega)$ for the space of $\mathscr{E}$-valued measures μ for which $\mu(E) \in \Omega$ for each $E \in \mathscr{F}$, and $AC(X, \Theta)$ for the space of $\mathscr{E}$-valued measures μ for which there exists a probability measure λ and a Pettis integrable selection z from Θ such that $\mu = z\lambda$. The first part of the next lemma is a standard result for which there seems to be no convenient reference; the second part extends Lemma 2 of SZ to the infinite dimensional context.

LEMMA 2: *If X is a compact metric space, $\mathscr{E}$ is a Hausdorff locally convex topological vector space, $\Omega \subset \mathscr{E}$ is a compact convex metrizable subset, and Θ: $X \to \Omega$ is an upper-hemi-continuous correspondence with nonempty compact convex values, then*:

(i) *$M(X, \Omega)$ is a compact metric space (in the weak* topology)*;

(ii) *$AC(X, \Theta)$ is a closed subset of $M(X, \Omega)$.*

PROOF: To establish (i), we first show that $M(X, \Omega)$ is compact. To this end, let $(\mu_\alpha) \subset M(X, \Omega)$ be a net. For each $\varphi \in \mathscr{E}^*$, $(\varphi \cdot \mu_\alpha)$ is a net of scalar measures, and hence has a convergent subnet. We may therefore extract a single subnet (μ_β) of (μ_α) with the property that for each $\varphi \in \mathscr{E}^*$ there is a scalar measure λ_φ such that $\varphi \cdot \mu_\beta \to \lambda_\varphi$ weak*. For each Borel set $E \subset X$, compactness and convexity of Ω guarantees that we may implicitly define a unique element $\mu(E) = \Omega$ by requiring that $\varphi \cdot \mu(E) = \lambda_\varphi(E)$ for every $\varphi \in \mathscr{E}^*$. The definitions imply immediately that $\mu \in M(X, \Omega)$ and $\mu_\beta \to \mu$ weak*.

To see that $M(X, \Omega)$ is metrizable, use compactness and metrizability of X to choose a countable dense subset $\{f_i\}$ of the space $C(X)$ of real-valued continuous functions on X and use compactness and metrizability of Ω to choose a countable family $(\varphi_j) \subset \mathscr{E}^*$ of linear functionals that distinguishes points of Ω. (That is, $\omega = \omega'$ if and only if $\varphi_j(\omega) = \varphi_j(\omega')$ for each j.) Write

(16) $$\|f_i\| = \sup_{x \in X} |f_i(x)|, \quad \|\varphi_j\|_\Omega = \sup_{\omega \in \Omega} |\varphi \cdot \omega|.$$

Define a distance function on $M(X, \Omega)$ by

$$(17) \qquad d(\mu, \mu') = \sum_{i,j} \frac{2^{-i-j}}{\|f_i\| + \|\varphi_j\|_\Omega} \left| \int f_i d(\varphi_j \cdot \mu) - \int f_i d(\varphi_j \cdot \mu') \right|.$$

This is easily seen to be a metric and it is easily checked that the metric topology is weaker than the weak* topology. Since the weak* topology is compact, the metric topology coincides with the weak* topology. This completes the proof of (i).

To establish (ii), we must show first that $AC(X, \Theta)$ is a subset of $M(X, \Omega)$. To see this, let $\mu \in AC(X, \Theta)$ and write $\mu = z\lambda$ for some probability measure λ and some selection z of Θ. By definition, for each Borel set $E \subset X$ and each $\varphi \in \mathscr{E}^*$ we have $\varphi \cdot \mu(E) = \int \varphi \cdot z \, d\lambda$. This is the integral of a scalar function with respect to a probability measure, so lies in the closed convex hull of 0 and the range of $\varphi \cdot z$, which is a subset of $\varphi \cdot \Omega$.[18] Since $\varphi \cdot \mu(E) \in \varphi \cdot \Omega$ for each $\varphi \in \mathscr{E}^*$ and Ω is convex, the Separation Theorem guarantees that $\mu(E) \in \Omega$. Since E is arbitrary, it follows that $\mu \in AC(X, \Theta)$ as desired.

To complete the proof, let $(\mu^n) \subset AC(X, \Theta)$ be a sequence converging weak* to $\mu \in M(X, \Omega)$; we must show $\mu \in AC(X, \Theta)$. For each n, choose a probability measure λ^n and a selection z^n from Θ such that $\mu^n = z^n \lambda^n$. Passing to a subsequence if necessary, we may assume that there is probability measure λ such that $\lambda^n \to \lambda$ weak*; we construct a selection z from Θ such that $\mu = z\lambda$.

It is convenient to imbed Ω in $\mathbb{R}^\infty$. To accomplish this, let $\{\varphi_j\}$ be the countable family of linear functionals chosen above and define a linear mapping Φ: $\mathscr{E} \to \mathbb{R}^\infty$ by $\Phi(x) = (\varphi_1(x), \dots,)$. If $\mathbb{R}^\infty$ is endowed with the product topology, this mapping is continuous. Because the collection (φ_j) distinguishes points of Ω, the restriction of Φ to Ω is one-to-one; because Ω is compact, the restriction of Φ to Ω is a homeomorphism. Because we can now replace $\Omega, \Theta, \mathscr{E}$ by $\Phi(\Omega)$, $\Phi \circ \Theta, \mathbb{R}^\infty$, we may assume without loss that $\mathscr{E} = \mathbb{R}^\infty$.

For each positive integer k, let Π_k: $\mathbb{R}^\infty \to \mathbb{R}^k$ be the projection on the first k coordinates and let ρ_k: $\mathbb{R}^k \to \mathbb{R}^{k-1}$ be the projection on the first $k-1$ coordinates. Fix an index k. The composition $\Pi_k \circ \mu^n$ is a vector measure with values in $\mathbb{R}^k$ and the composition $\Pi_k \circ z^n$ is a selection from the correspondence $\Pi_k \circ \Theta$. Continuity of Π_k implies that the sequence $(\Pi_k \circ \mu^n)$ of $\mathbb{R}^k$-valued measures converges weak* to $\Pi_k \circ \mu$, so Lemma 2 of SZ guarantees that there is a selection z_k from $\Pi_k \circ \Theta$ such that $\Pi_k \circ \mu = z_k \lambda$.

Note that $\rho_k \circ \Pi_k \circ \mu = \Pi_{k-1} \circ \mu$. Uniqueness of the Radon-Nikodym derivative implies that $\rho_k \circ z_k = z_{k-1}$ almost everywhere (with respect to λ). We can therefore choose a set $X_0 \subset X$ such that $\lambda(X \setminus X_0) = 0$ and $\rho_k \circ z_k(x) = z_{k-1}(x)$ for every $x \in X_0$ and every index k. For each index k and $x \in X_0$, write $\Theta_k(x) = \Pi_k^{-1}(z_k(x)) \cap \Theta(x)$. Compactness of $\Theta(x)$ and continuity of Π_k guarantee that $\Theta_k(x)$ is compact, and the construction of X_0 guarantees that $(\Theta_k(x))$ is a decreasing sequence of compact sets, so the intersection $\cap \Theta_k(x)$ is not empty.

[18] Note that $0 = \mu(\varnothing) \in \Omega$.

Our construction guarantees that this intersection consists of a single point, which we define to be $z(x)$. By construction, z is a selection from Θ on X_0. The graph of z is the intersection of the graphs of the measurable correspondences $\Pi_k^{-1}(z_k(\cdot)) \cap \Theta(\cdot)$, so z is measurable. Extend z arbitrarily to a measurable selection on all of X.

Our construction guarantees that, for each k, $z_k = \Pi_k \circ z$ is the Radon-Nikodym derivative of $\Pi_k \circ \mu$. Linearity of Π_k guarantees that for every Borel subset $G \subset X$ we have

$$\begin{aligned} \Pi_k[\mu(G)] &= (\Pi_k \circ \mu)(G) \\ &= \mu_k(G) \\ &= \int_G z_k d\lambda \\ &= \int_G (\Pi_k \circ z) d\lambda \\ &= \Pi_k\left(\int_G z\, d\lambda\right). \end{aligned} \tag{18}$$

Since this is true for every k, we conclude that $\mu(G) = \int_G z\, d\lambda$, as desired. *Q.E.D.*

The next lemma is an extension of a result of Dellacherie and Meyer (1982) to the present context.[19]

LEMMA 3: *Let X, Y be a compact metric spaces, let $\mathcal{E}$ be a locally convex topological vector space, let Ω be a compact convex metrizable subset of $\mathcal{E}$, and let $\Theta: X \to \Omega$ be an upper-hemi-continuous correspondence with nonempty, compact, convex values. Let $y \mapsto \mu_y$ be a weak* measurable family of $\mathcal{E}$-valued measures on X having the property that for every $y \in Y$ there is a selection z_y from Θ such that $\mu_y = z_y \lambda$. Then there is a Borel measurable function $Z: X \times Y \to \mathcal{E}$ such that for each $y \in Y$, $Z(\cdot, y)$ is a selection from Θ and $\mu_y = Z(\cdot, y)\lambda$. (That is, $Z(\cdot, y) = z_y$ almost everywhere with respect to λ.)*

PROOF: As in the proof of Lemma 2, there is no loss in assuming that $\mathcal{E} = \mathbb{R}^\infty$; we adopt the notation of that proof. For each k, $\{\Pi_k \circ \mu_y : y \in Y\}$ is a weak* measurable family of measures on X with values in $\mathbb{R}^k$, and for every $y \in Y$ the composition $\Pi_k \circ z_y$ is a selection from $\Pi_k \circ \Theta$ such that $\Pi_k \circ \mu_y = (\Pi_k \circ z_y)\lambda$. Applying Theorem V.58 of Dellacherie and Meyer (1982) to each coordinate separately, we may find a Borel function $Z_k: X \times Y \to \mathbb{R}^k$ such that $\Pi_k \circ \mu_y = Z_k(\cdot, \cdot, y)\lambda$ for each $y \in Y$. As in the proof of Lemma 2, we can construct a Borel function $Z^*: X \times Y \to \mathbb{R}^\infty$ such that $\Pi_k \circ Z^* = Z_k$ for each k.

The construction guarantees that $\mu_y = Z^*(\cdot, y)\lambda$ for every $y \in Y$. However, Z^* need not be quite the function we want because it need not be a selection from Θ;

[19] We thank a referee for directing us to Dellacherie and Meyer (1982).

a perturbation will achieve this. To accomplish this perturbation, fix an arbitrary Borel measurable selection z_0 of Θ. Define the Borel function Q: $X \times Y \to X \times \Omega \times Y$ by $Q(x, y) = (x, Z^*(x, y), y)$. Define a Borel measurable selection Z from Θ by

$$Z(x, y) = \begin{cases} Z^*(x, y) & \text{if } (x, y) \in Q^{-1}[(\text{graph}\,\Theta) \times Y], \\ z_0(x) & \text{otherwise.} \end{cases} \tag{19}$$

Uniqueness of the Radon-Nikodym derivative implies that for every $y \in Y$, $Z^*(x, y) = z_y(x)$ for λ-almost all $x \in X$, whence $Z(x, y) = Z^*(x, y) = z_y(x)$ for λ-almost all $x \in X$. Thus Z is the desired mapping. *Q.E.D.*

Finally, it is convenient to isolate a lemma that will be used several times. If Γ is a game with action spaces A_i and type spaces T_i and f is any function or correspondence defined on A, we write $\tilde{f}$ for the trivial extension of f to $A \times T$: $\tilde{f}(a, t) = f(a)$.

LEMMA 4: *Let Γ be an affine game with indeterminate outcomes. Let (z^r) be a sequence of selections from the outcome correspondence Θ and let (γ^r) be a sequence of positive measures on $A \times T$.*[20] *If there is a selection z from the outcome correspondence Θ and a positive measure γ on $A \times T$ such that $\gamma^r \to \gamma$ and $\tilde{z}^r \gamma^r \to \tilde{z}\gamma$ (weak*), then*

$$\int u_i(a, \tilde{z}^r(a, t), t)\, d\gamma^r \longrightarrow \int u_i(a, \tilde{z}(a, t), t)\, d\gamma \tag{20}$$

for each i.

PROOF: Fix i. We begin by constructing an approximation to u_i by a continuously weighted sum of affine functions. Write $\mathscr{E}^*$ for the dual space of $\mathscr{E}$ (the space of continuous linear functionals, equipped with the topology of pointwise convergence).

Fix $\varepsilon > 0$. For $a \in A$, $t \in T$, the function $u_i(a, \cdot, t)$ is affine on $\Theta(a)$, so it can be approximated to within ε by an affine function on $\mathscr{E}$.[21] That is, there are a constant c_{at} and a linear functional $\varphi_{at} \in \mathscr{E}^*$ such that

$$|u_i(a, \omega, t) - c_{at} - \varphi_{at} \cdot \omega| < \varepsilon \tag{21}$$

for each $\omega \in \Theta(a)$. Compactness of A, T, Ω, continuity of u_i and φ_{at}, and upper-hemi-continuity of Θ imply that there are neighborhoods $W(a, t)$ of (a, t) in

[20] We do not assume that γ^r is the joint distribution of any strategy profile.

[21] See Phelps (1966). In the infinite dimensional context, there may be no affine function on $\mathscr{E}$ that *coincides* with $u_i(a, \cdot, t)$ on $\Theta(a)$.

$A \times T$ and $W'(a,t)$ of 0 in Ω such that

$$(a',t') \in W(a,t), \omega' \in \Theta(a') \Longrightarrow \exists \omega \in \Theta(a) \text{ such that } (\omega - \omega') \in W'(a,t);$$

$$(a',t') \in W(a,t), \omega' \in \Theta(a'), (\omega - \omega') \in W'(a,t) \tag{22}$$
$$\Longrightarrow |u_i(\omega,a,t) - u_i(\omega',a',t')| < \varepsilon, |\varphi_{at} \cdot (\omega - \omega')| < \varepsilon.$$

Combining these facts, we conclude that

$$(a',t') \in W(a,t), \omega' \in \Theta(a') \Longrightarrow |u_i(\omega',a',t') - c_{at} - \varphi_{at} \cdot \omega'| < 3\varepsilon. \tag{23}$$

The family $\{W(a,t)\}$ is a cover of $A \times T$ by open sets. Choose a finite subcover $\{W(a^j,t^j)\}$ and a partition of unity $\{f^j\}$ subordinate to this cover; i.e., a family of continuous functions f^j: $A \times T \to [0,1]$ such that:

- $f^j(a',t') = 0$ if $(a',t') \notin W(a^j,t^j)$;
- $\sum_j f^j \equiv 1$.

To simplify notation, write $c_j = c_{a^j t^j}$ and $\varphi_j = \varphi_{a^j t^j}$. Define mappings

$$c_\varepsilon \colon A \times T \longrightarrow \mathbb{R}, \quad \varphi_\varepsilon \colon A \times T \longrightarrow \Omega^* \tag{24}$$

by

$$c_\varepsilon(a,t) = \sum_j f^j(a,t) c_j, \quad \varphi_\varepsilon(a,t) = \sum_j f^j(a,t)\varphi_j. \tag{25}$$

Because $\{f^j\}$ is a partition of unity subordinate to the cover $\{W(a^j,t^j)\}$, it follows that $c_\varepsilon, \varphi_\varepsilon$ are continuous functions and that

$$(a,\omega,t) \in \text{ graph } \Theta \times T \Longrightarrow |u_i(a,\omega,t) - c_\varepsilon(a,t) - \varphi_\varepsilon(a,t) \cdot \omega| < 3\varepsilon. \tag{26}$$

This is the desired approximation to u_i.

It follows from (26) that

$$\left| \int u_i(a, \tilde{z}(a), t)\, d\gamma - \int [c_\varepsilon(a,t) + \varphi_\varepsilon(a,t) \cdot \tilde{z}(a,t)]\, d\gamma \right| < 3\varepsilon, \tag{27}$$
$$\left| \int u_i(a, \tilde{z}^r(a,t), t)\, d\gamma^r - \int [c_\varepsilon(a,t) + \varphi_\varepsilon(a,t) \cdot \tilde{z}^r(a,t)(a,t)]\, d\gamma^r \right| < 3\varepsilon,$$

for every r.

Now apply weak* convergence:

$$
\begin{aligned}
(28)\qquad & \int [c_\varepsilon(a,t) + \varphi_\varepsilon(a,t)\cdot \tilde{z}^r(a,t)]\, d\gamma^r \\
&= \int \sum_j f^j(a,t)[c_j + \varphi_j \cdot \tilde{z}^r(a,t)]\, d\gamma^r \\
&= \sum_j c_j \int f^j(a,t)\, d\gamma^r + \sum_j \int f^j(a,t)\varphi_j \cdot \tilde{z}^r(a,t)\, d\gamma^r \\
&= \sum_j c_j \int f^j(a,t)\, d\gamma^r + \sum_j \int f^j(a,t)\, d(\varphi_j \cdot \tilde{z}^r \gamma^r) \\
&\longrightarrow \sum_j c_j \int f^j(a,t)\, d\gamma + \sum_j \int f^j(a,t)\, d(\varphi_j \cdot \tilde{z}\gamma) \\
&= \sum_j c_j \int f^j(a,t)\, d\gamma + \sum_j \int f^j(a,t)\varphi_j \cdot \tilde{z}(a,t)\, d\gamma \\
&= \int \sum_j f^j(a,t)[c_j + \varphi_j \cdot \tilde{z}(a,t)]\, d\gamma.
\end{aligned}
$$

Combining this with (26) and keeping in mind that ε is arbitrary, we obtain

$$
(29)\qquad \int u_i(a, \tilde{z}^r(a,t), t)\, d\gamma^r \longrightarrow \int u_i(a, \tilde{z}(a,t), t)\, d\gamma,
$$

which is the desired result. *Q.E.D.*

With the preliminaries complete, we turn to the proof of Theorem 1.

PROOF OF THEOREM 1: As indicated, the proof is in six steps. The argument is a bit fussy because we need to keep track of strategies and selections in several games. Recall that, as a mnemonic device to distinguish between announcements and true types, we write $S_i = T_i$, for each i. Without loss of generality, we assume that $0 \le u_i \le 1$ for each i.

Step 1—Finite Approximations: For each $r = 1, 2, \ldots$ and each player i, choose and fix finite subsets $S_i^r \subset S_i$, $A_i^r \subset A_i$ such that every point of S_i is within $1/r$ of some point in S_i^r and every point of A_i is within $1/r$ of some point in A_i^r. For each r, let $q^r : A \to \Omega$ be a Borel measurable selection from Θ. For each r, let Γ^r be the Bayesian game with player set N, action spaces A_i^r, type spaces T_i, prior probability distribution τ, and utility functions $u_i^r(a,t) = u_i(a, q^r(a), t)$. Milgrom and Weber (1985) show that Γ^r has an equilibrium $\alpha^r = (\alpha_1^r, \ldots, \alpha_n^r)$ in distributional strategies.

Let Γ^{cr} be the Bayesian game with player set N, action spaces $S_i^r \times A_i^r$, type spaces T_i, prior probability distribution τ, and utility functions $u_i^r(s,a,t) =$

$u_i(a, q^r(a), t)$. Payoffs in Γ^{cr} are independent of announcements, so announcements are cheap talk. Define

$$d_i\colon A_i \times T_i \longrightarrow S_i \times A_i \times T_i, \quad d\colon A \times T \longrightarrow S \times A \times T \tag{30}$$

by $d_i(a_i, t_i) = (t_i, a_i, t_i)$ and $d(a, t) = (t, a, t)$. Let $\sigma_i^r = d_i \alpha_i^r$, $\bar{\sigma}^r = d\bar{\alpha}^r$ be the direct image measures. Note that the marginal of σ_i^r on T is τ_i, so σ_i^r is a distributional strategy for the game Γ^{cr}. Moreover, $\bar{\sigma}^r$ is the joint distribution on $S \times A \times T$ of the tuple $\sigma^r = (\sigma_1^r, \ldots, \sigma_n^r)$. Because payoffs in Γ^{cr} do not depend on announcements, σ^r is an equilibrium for Γ^{cr}. Write

$$\Delta = \{(s, a, t) \in S \times A \times T : s = t\} \tag{31}$$

for the set of announcement/action/type profiles for which announcements are truthful. By construction, $\bar{\sigma}^r$ is supported on Δ, so gives probability one to truthful announcements.

For each r, define $\theta^r\colon S \times A \to \Omega$ and $\tilde{\theta}^r\colon S \times A \times T \to \Omega$ by $\tilde{\theta}^r(s, a, t) = \theta^r(s, a) = q^r(a)$. Define $\tilde{\Theta}^c\colon S \times A \times T \to \Omega$ by $\tilde{\Theta}^c(s, a, t) = \Theta^c(s, a) = \Theta(a)$. Note that θ^r is a selection from Θ^c and that $\tilde{\theta}^r$ is a selection from $\tilde{\Theta}^c$.

Step 2—Limits: Passing to an appropriate subsequence if necessary, we may assume that, for each i, the sequence (σ_i^r) of scalar measures converges weak* to a scalar measure σ_i on $S_i \times A_i \times T_i$, and the sequence $(\tilde{\theta}^r \bar{\sigma}^r)$ of $\mathscr{E}$-valued measures converges weak* to an $\mathscr{E}$-valued measure ν on $S \times A \times T$. Note that convergence of individual strategies implies convergence of joint distributions; that is, $\bar{\sigma}^r \to \bar{\sigma}$, the joint distribution of $\sigma = (\sigma_1, \ldots, \sigma_n)$.

By Lemma 2, there is a Borel measurable selection ϕ from Θ such that $\nu = \phi\bar{\sigma}$. Define $\theta(s, a) = \phi(s, a, s)$ for every $(s, a) \in S \times A$. Note that $\bar{\sigma}$ is supported on Δ, the set of truthful profiles, so $\tilde{\theta} = \phi$ almost everywhere (with respect to $\bar{\sigma}$), whence $\nu = \tilde{\theta}\bar{\sigma}$.

Step 3—Convergence of Utilities: Applying Lemma 4 to the game Γ^c, we conclude that utilities converge. That is, for each i

$$Eu_i(\sigma^r | \theta^r) \longrightarrow Eu_i(\sigma | \theta). \tag{32}$$

Step 4—Identifying Better Responses: The selection ϕ is only determined up to sets of measure 0, so we may have chosen the wrong selections $\theta, \tilde{\theta}$. This leaves open the possibility that players may have profitable deviations. We will construct perturbations of $\theta, \tilde{\theta}$ to eliminate these deviations. In order to do this, we first identify the places where perturbation is required.

By assumption, τ is absolutely continuous with respect to the product of its marginals; let $F\colon T \to \mathbb{R}$ be the Radon-Nikodym derivative (which we can assume is a Borel function), so that $\tau = F(\times \tau_i)$. Notice that the conditionals are $\tau(\cdot | t_i) = F(\cdot | t_i)(\times \tau_i)$.

Fix a player i. Consider the maps $\Upsilon\colon T_i \to \mathbb{R}$, $\Psi\colon T_i \to L^1(\tau_{-i})$ defined by $\Upsilon(t_i) = Eu_i(\sigma | \theta, t_i)$, $\Psi(t_i) = F(\cdot | t_i)$. The maps Υ, Ψ are Borel measurable,[22]

[22] We give $L^1(\tau_{-i})$ the norm topology.

so Lusin's Theorem allows us to find an increasing sequence (T_i^k) of compact subsets of T_i such that the restrictions of Υ, Ψ to each T_i^k are continuous and $\tau_i(T_i \backslash T_i^k) < 2^{-k}$. There is no loss of generality in assuming that the support of $\sigma_i | T_i^k$ is T_i^k; equivalently, every relatively open subset of T_i^k has positive τ-measure. Set $T_i^* = \cup T_i^k$, so that $\tau_i(T_i \backslash T_i^*) = 0$. Let H_i be the set of pairs $(s_i, a_i) \in S_i \times A_i$ for which there is a type $t_i \in T_i^*$ such that

$$(33) \qquad Eu_i(s_i, a_i | \sigma_{-i}, t_i, \theta) > Eu_i(\sigma_i | \sigma_{-i}, t_i, \theta).$$

That is, player i of type t_i prefers to announce s_i and play a_i rather than to follow σ_i, given that others are following σ_{-i} and that the selection is θ. The continuity properties of Υ and Ψ and the continuity of utility functions guarantees that H_i is a Borel set. We assert that $\sigma_i(H_i \times T_i^*) = 0$.

To see that this is so, we suppose not, and construct a profitable deviation in Γ^{cr} for r sufficiently large. To this end, let ζ_i be the marginal of σ_i on $A_i \times T_i$, so $\zeta(H_i) = \sigma_i(H_i \times T_i) = \sigma_i(H_i \times T_i^*) > 0$. For each j, let H_i^j be the set of pairs $(s_i, a_i) \in S_i \times A_i$ for which there is a type $t_i \in T_i^*$ such that

$$(34) \qquad Eu_i(s_i, a_i | \sigma_{-i}, t_i, \theta) > Eu_i(\bar{\sigma}_i | \sigma_{-i}, t_i, \theta) + \frac{1}{j}.$$

Since $H_i = \cup H_i^j$, we can find some j so that $\zeta_i(H_i^j) > 0$.

We can find a Borel measurable map $h : H_i^j \to T_i^*$ such that

$$(35) \qquad Eu_i(s_i, a_i | \sigma_{-i}, h(s_i, a_i), \theta) > Eu_i(\sigma_i | \sigma_{-i}, h(s_i, a_i), \theta) + \frac{1}{j}$$

for every $(s_i, a_i) \in H_i^j$. Applying Lusin's Theorem to h and recalling that $T_i^* = \cup T_i^k$, we can find a compact subset $H \subset H_i^j$ such that $\zeta_i(H) > 0$, the restriction of h to H is continuous, and $h(H) \subset T_i^k$. There is no loss in assuming that supp $(\zeta_i | H) = H$, so every relatively open subset of H has positive ζ_i-measure.

Fix an arbitrary $(s_i^*, t_i^*) \in H$. Continuity of u_i and of Ψ on T_i^k guarantees that the map

$$(36) \qquad (s_i, a_i, t_i) \longmapsto Eu_i(s_i, a_i | \sigma_{-i}, t_i, \theta)$$

is continuous on $S_i \times A_i \times T_i^k$. Hence there are compact neighborhoods K of (s_i^*, t_i^*) in H and L of $h(s_i^*, t_i^*)$ in T_i^k so that

$$(37) \qquad Eu_i(s_i, a_i | \sigma_{-i}, t_i, \theta) > Eu_i(\sigma_i | \sigma_{-i}, t_i, \theta) + \frac{1}{2j}$$

whenever $(s_i, a_i) \in K$, $t_i \in L$. Our construction guarantees that $\zeta_i(K) > 0$ and $\tau_i(L) > 0$. Shrinking K, L if necessary, we may find a real number R such that

$$(38) \qquad (s_i, a_i) \in K, t_i \in L$$
$$\Longrightarrow Eu_i(s_i, a_i | \sigma_{-i}, t_i, \theta) > R + \frac{1}{2j} > R > Eu_i(\sigma_i | \sigma_{-i}, t_i, \theta).$$

Because the restriction of Ψ to T_i^k is continuous, shrinking L further if necessary we may guarantee

$$t_i, t_i' \in L \Longrightarrow \|F(\cdot|t_i) - F(\cdot|t_i')\| < \frac{1}{16j}. \tag{39}$$

Finally, continuity of u_i guarantees that, shrinking L still further if necessary, we may guarantee

$$t_i, t_i' \in L \Longrightarrow |u_i(s, a, \omega, t) - u_i(s, a, \omega, t')| < \frac{1}{16j}. \tag{40}$$

Let $\varepsilon > 0$. Choose an open set Q with $K \subset Q \subset S_i \times T_i$ such that $\zeta_i(Q \backslash K) < \varepsilon$, and a continuous function $\varphi : S_i \times A_i \to [0, 1]$ that is identically 1 on K, and identically 0 off U. Let χ be the characteristic function of L in T_i. View φ, χ as functions on $S \times A \times T$ that are independent of other components. Write $M = \int \varphi\chi d\bar{\sigma}$, and set $\Phi = (1/M)\varphi\chi$. Note that

$$\int 1 d(\Phi\bar{\sigma}) = 1. \tag{41}$$

Lemma 1 guarantees that $\Phi\bar{\sigma}^r \to \Phi\bar{\sigma}$ and that $\tilde{\theta}^r\Phi\bar{\sigma}^r \to \tilde{\theta}\Phi\bar{\sigma}$ (weak*). Weak* convergence guarantees that

$$\int 1\, d(\Phi\bar{\sigma}^r) \longrightarrow \int 1\, d(\Phi\bar{\sigma}) = 1. \tag{42}$$

Lemma 4 guarantees that

$$\int u_i(s, a, \tilde{\theta}^r(s, a), t)\, d(\Phi\bar{\sigma}^r) \longrightarrow \int u_i(s, a, \tilde{\theta}(s, a), t)\, d(\Phi\bar{\sigma}). \tag{43}$$

Together, the inequalities (38), (41) guarantee that

$$\int u_i(s, a, \tilde{\theta}(s, a), t)\, d(\Phi\bar{\sigma}) > R \tag{44}$$

so

$$\int u_i(s, a, \tilde{\theta}^r(s, a), t)\, d(\Phi\bar{\sigma}^r) > R \tag{45}$$

for r sufficiently large.

For each r, the marginal of $\bar{\sigma}^r$ on $S_i \times A_i \times T_i$ is σ_i^r; let $\nu^r(\cdot|(s_i, a_i, t_i))$ be the conditionals. Because Φ depends only on s_i, a_i, t_i, the marginal of $\Phi\bar{\sigma}^r$ on $S_i \times A_i \times T_i$ is $\Phi\sigma_i^r$ and the conditionals are $\nu^r(\cdot|(s_i, a_i, t_i))$. Hence we can write

$$\begin{aligned} &\int u_i(s, a, \tilde{\theta}^r(s, a, t), t)\, d(\Phi\bar{\sigma}^r) \\ &\qquad = \int \Big[\int u_i(s, a, \tilde{\theta}^r(s, a, t), t)\, d\nu^r \Big] d(\Phi\sigma_i^r). \end{aligned} \tag{46}$$

In view of (42), $\int 1\, d(\Phi\bar{\sigma}^r)$ is arbitrarily close to 1 for r sufficiently large. Hence, for each r sufficiently large, there is some (s_i^r, a_i^r, t_i) for which

$$\text{(47)} \qquad \int u_i\big[(s_i^r, s_{-i}), (a_i^r, a_{-i}), \tilde{\theta}^r\big((s_i^r, s_{-i}), (a_i^r, a_{-i}), (t_i, t_{-i})\big), (t_i, t_{-i})\big]\, d\nu^r > R + \frac{1}{4j}.$$

Note that this integral is the expected payoff to player i in the game Γ^{cr} if he is type t_i, announces s_i^r, and plays a_i^r. Taken together, (39) and (40) guarantee that the expected payoff to player i in the game Γ^{cr} if he is type $t_i' \in L$, announces s_i, and plays a_i is at least $R + (1/8j)$. In particular, the expected payoff to player i in the game Γ^{cr} when his type lies in L, he announces s_i, and plays a_i is at least $[R + (1/8j)]\tau_i(L)$.

On the other hand, applying Lemma 4 to $\chi\bar{\sigma}^r \to \chi\bar{\sigma}$ guarantees that the expected payoff to player i in the game Γ^{cr} if his type lies in L and he plays according to σ_i^r converges to the expected payoff to player i in the game Γ^c if his type lies in L and he plays according to σ_i^r. In view of (38), this is at most $R\tau_i(L)$. Taken together, these last three facts constitute a contradiction (for r large enough). We conclude that $\zeta_i(H_i) = 0$ as asserted.

Step 5—Perturbation: We now correct the selection θ on H_i. Intuitively speaking, the correction is to give player i the limit of what he would obtain in the games Γ^{cr}; the details are complicated because we must put these limits together in a manner that is consistent across actions of others and measurable in i's own actions.

Write

$$\text{(48)} \qquad \Delta_{-i} = \{(s_{-i}, a_{-i}, t_{-i}) \in S_{-i} \times A_{-i} \times T_{-i} : s_{-i} = t_{-i}\}.$$

Fix $(s_i, a_i) \in H_i$. Define B: $\Delta_{-i} \to \Omega$ by $B(s_{-i}, a_{-i}, s_{-i}) = \Theta(s_i, a_i, s_{-i}, a_{-i})$. For each r, define β^r: $\Delta_{-i} \to \Omega$ by $\beta^r(s_{-i}, a_{-i}, s_{-i}) = \theta^r(s_i, a_i, s_{-i}, a_{-i})$; note that β^r is a selection from B.

Write $\bar{\sigma}_{-i}^r, \bar{\sigma}_{-i}$ for the joint distributions on $S_{-i} \times A_{-i} \times T_{-i}$ of announcements/actions/types of players other than i. Because announcements (of players other than i) are truthful, $\bar{\sigma}_{-i}^r$, $\bar{\sigma}_{-i}$ are supported on Δ_{-i}; we abuse notation and view them as measures on this space.

Let $\Xi(s_i, a_i)$ be the set of $\mathscr{E}$-valued measures ξ on Δ_{-i} for which there is a sequence of integers (r_m) and points $(s_i^{r_m}, a_i^{r_m}) \in S_i \times A_i$ such that:

- $(s_i^{r_m}, a_i^{r_m}) \to (s_i, a_i)$;
- $\beta^{r_m}\bar{\sigma}_{-i}^r \to \xi$.

Lemma 2 guarantees that $\Xi(s_i, a_i)$ is a nonempty compact set of $\mathscr{E}$-valued measures, and that for each $\xi \in \Xi(s_i, a_i)$ there is a selection β from B so that $\xi = \beta\bar{\sigma}_{-i}$. It is easily checked that the correspondence $(s_i, a_i) \mapsto \Xi(s_i, a_i)$ is weak* upper-hemi-continuous, so it admits a weak* Borel measurable selection $(s_i, a_i) \mapsto \xi_{(s_i, a_i)}$.

Lemma 3 guarantees that there is a Borel function R_i: $H_i \times \Delta_{-i} \to \Omega$ such that $\xi_{(s_i, a_i)} = R_i(s_i, a_i, \cdot)\bar{\sigma}_{-i}$ for each (s_i, a_i). Define

$$(49) \qquad \theta'(s, a) = \begin{cases} R_i(s_i, a_i, s_{-i}, a_{-i}, s_{-i}) & \text{if there is a unique } i \text{ with } (s_i, a_i) \in H_i, \\ \theta(s, a) & \text{otherwise.} \end{cases}$$

Note that $\theta' = \theta$ except on

$$(50) \qquad H = \bigcup_i [H_i \times T_i \times S_{-i} \times A_{-i} \times T_{-i}].$$

The construction of Step 4 guarantees that $\bar{\sigma}(H) = 0$. In particular, $Eu_i(\sigma|\theta) = Eu_i(\sigma|\theta')$ for all i.

Step 6—Equilibrium: We assert that the selection θ' and strategy profile σ constitute a solution for the game Γ^c. To see this, fix a player i. We must show that for almost all $t_i \in T_i$, the strategy σ_i is a best response to σ_{-i}, given that agent i is type t_i. We only have to worry about types $t_i \in T_i^*$, because the complementary set of types has measure 0. Let $t_i \in T_i^*$ and suppose there is an announcement s_i and action a_i so that, given he is type t_i, player i would strictly prefer to play (s_i, a_i) rather than follow σ_i. By construction, $t_i \in T_i^k$ for some k. Continuity of payoffs and information on T_i^k implies there is a relatively open subset $W \subset T_i^k$ such that for every $t_i' \in W$, player i of type t_i' would strictly prefer to play (s_i, a_i) rather than follow σ_i. Thus, if we write $Eu_i(s_i, a_i|\sigma_{-i}, W, \theta')$ for the expected utility of player i when his type is in W, he plays s_i, a_i, others follow their components of σ, and the tie-breaking rule is θ' (and similarly for σ_i in place of s_i, a_i), we find

$$(51) \qquad Eu_i(s_i, a_i|\sigma_{-i}, W, \theta') > Eu_i(\sigma_i|\sigma_{-i}, W, \theta').$$

On the other hand, we can estimate $Eu_i(s_i, a_i|\sigma_{-i}, W, \theta')$ directly from payoffs in the games Γ^{cr}. By definition, $(s_i, a_i) \in H_i$. By construction, for each $t_i' \in W$

$$(52) \qquad \tilde{\theta}'(s_i, a_i, s_{-i}, a_{-i}, t_i', t_{-i}) = R_i(s_i, a_i, s_{-i}, a_{-i}, s_{-i}) = \beta(s_i, a_i)$$

except for $(s_{-i}, a_{-i}, t_i', t_{-i})$ belonging to a set of $\bar{\sigma}_{-i}$-measure 0. Hence, using Lemma 4 exactly as in Step 4 above and recalling the equilibrium conditions in the games Γ^{cr}, we see that

$$(53) \qquad \begin{aligned} Eu_i(s_i, a_i|\sigma_{-i}, W, \theta') &= \lim_{r_m \longrightarrow \infty} Eu_i(s_i^{r_m}, a_i^{r_m}|\sigma_{-i}^r, W, \theta^{r_m}) \\ &\leq \lim_{r_m \longrightarrow \infty} Eu_i(\sigma_i^{r_m}|\sigma_{-i}^r, W, \theta^{r_m}) \\ &= Eu_i(\sigma_i|\sigma_{-i}, W, \theta'). \end{aligned}$$

This contradicts (51), so we conclude that θ', σ is a solution, as desired. *Q.E.D.*

PROOF OF THEOREM 2: The argument follows by substituting the given solutions for the solutions constructed in Step 1 of the proof of Theorem 1, and continuing as in Steps 2–6. *Q.E.D.*

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