Bulletin of the Seismological Society of America. Vol. 63, No. 1, pp. 9–48. February 1973

DYNAMICS OF BUILDING-SOIL INTERACTION

BY PAUL C. JENNINGS AND JACOBO BIELAK

ABSTRACT

In this study of the dynamics of building-soil interaction, the soil is modeled by a linear elastic half-space, and the building structure by an n-degree-of-freedom oscillator. Both earthquake response and steady-state response to sinusoidal excitation are examined. By assuming that the interaction system possesses $n+2$ significant resonant frequencies, the response of the system is reduced to the superposition of the responses of damped linear oscillators subjected to modified excitations. The results are valid even though the interaction systems do not possess classical normal modes. For the special cases of single-story systems and the first modes of n-story systems, simplified approximate formulas are developed for the modified natural frequency and damping ratio and for the modified excitation. Example calculations are carried out by the approximate and more exact analysis for one-story, two-story and ten-story interaction systems.

The results show that interaction tends to decrease all resonant frequencies, but that the effects are often significant only for the fundamental mode for many n-story structures and are more pronounced for rocking than for translation. If the fixed-base structure has damping, the effects of interaction on the earthquake responses are not always conservative, and an increase or decrease in the response can occur, depending on the parameters of the system.

INTRODUCTION

There are two aspects of building-foundation interaction during earthquakes which are of primary importance to earthquake engineering. First, the response to earthquake motion of a structure founded on a deformable soil can be significantly different from that of a structure supported on a rigid foundation. Second, the motion recorded at the base of a structure or in the immediate vicinity can be different in important details from that which would have been recorded had there been no building. Observations of the response of buildings during earthquakes have shown that the response of typical structures can be markedly influenced by the soil properties if the soils are sufficiently soft (Ishizaki and Hatakeyama, 1960). Furthermore, for relatively rigid structures such as nuclear reactor containment structures, interaction effects can be important even for relatively firm soils because the important parameter apparently is not the stiffness of the soil, *per se*, but the relative stiffness of the building and its foundation. From the point of view of engineering, it is important to determine the conditions under which soil-structure interaction is practically significant, and to develop methods that can be used in design for calculating interaction effects.

In terms of the dynamic properties of the building foundation system, past studies have shown that interaction will, in general, reduce the fundamental frequency of systems from that of the structure on a rigid base, dissipate part of the vibrational energy of the building by wave radiation into the foundation medium (there will also be energy losses from internal friction in the soil), and modify the base motion of the structure in comparison to the free-field motion. Although all of these effects may be present in some degree for every structure, the important point is to determine the conditions under which the effects are of practical significance.

The complex material properties of soils, the involved geometries of building foundations, and the complicated nature of earthquake ground motions combine to make the soil-structure interaction problem extremely complicated, and it is necessary, in general, to make major simplifying assumptions in all these aspects of the problem before calculations can be made. In most studies, the soil is idealized as a linear, homogeneous, isotropic, elastic half-space (Sato and Yamaguchi, 1960; Parmelee, 1967; Sarrazin, 1970; Scavuzzo *et al.*, 1971), and in many instances the dynamic properties of the half-space are further approximated by discrete springs and dashpots. A still further approximation often made is that the discrete elements have properties that do not vary with frequency (Merritt and Housner, 1954; Thomson, 1960; Parmelee *et al.*, 1969). The building foundation is usually simplified by assuming that the building-soil interface is at the ground surface and that the cross-section of the contact area can be represented by a circle (Thomson, 1960). The earthquake excitation is typically idealized as vertically propagating, horizontally incident, planar motion (Hradilek and Luco, 1970). In addition to earthquake motion, studied among others by Housner (1957), Parmelee *et al.* (1969) and Castellani (1970), steady-state response to sinusoidal excitation has been studied extensively to clarify the basic features of the problem (Sato and Yamaguchi, 1960; Thomson, 1960; Parmelee, 1967). The application of the finite element method to the problem can avoid some of the above assumptions which are primarily geometrical, but simplified models of the soil and excitation are still required and, unless a three-dimensional approach is used, it is necessary to make a two-dimensional idealization of the problem. Thus, using a finite element formulation for plane strain problems, Isenberg (1970) has studied the effects of interaction for elastic buildings embedded into elastic/perfectly plastic soils.

The representation of the foundation system by springs and dashpots is an attractive approach for design because the resulting system is similar to the usual representation of a fixed-base structure. It is important to realize, however, that the representation of the foundation by constant springs and dashpots is not consistent with using an elastic half-space as an idealization of the soil. If the springs and dashpots are to be equivalent to the elastic half-space, their properties must be frequency-dependent (Hsieh, 1962). Fortunately, it seems possible in many instances to approximate the frequency-dependence reasonably well by representative constant values, an approximation that results in a system of linear differential equations with constant coefficients. Thus, some of the standard methods of analysis can be applied to interaction systems when the foundation medium is modeled this way (Parmelee *et al.*, 1969). Normal mode methods of structural dynamics cannot be applied to such systems, however, because the foundation dashpots are such that the building-foundation model does not possess classical normal modes. Unlike these methods, operational methods of analysis can be used even when the properties of the springs and dashpots are frequency-dependent (Sandi, 1960; Rosenberg, 1965).

There are two major efforts required to make the theory of soil-structure interaction a better tool for use in earthquake-resistant design. First, methods of calculation must be developed which are accurate within the framework of simplifying assumptions, and second, more experimental studies and earthquake-response measurements are needed to establish the range of validity of the various methods of simplifying the problem. The present study is directed toward the first of these efforts and is performed under the scope of the assumptions outlined above. As is common in structural analysis, the building itself is modeled by a linear, viscously damped, multi-degree-of-freedom oscillator. In the first part of the paper an analytical examination of the n-story building-soil interaction problem is made to show that under the assumption that $n+2$ resonant frequencies exist, the earthquake response of the interaction system reduces to the linear superposition of the responses of damped, linear, single-degree-of-freedom oscillators subjected to modified excitations. This result is shown to be valid even for systems that do not possess classical normal modes. The major advantages of the approach are that it makes the calculations involved equivalent to those for simple, rigid-base structures, and that it gives physical insight into the dynamics of building-foundation systems.

The second portion of the study is devoted to the examination of the effects of the important parameters on the dynamics of single- and multi-degree-of-freedom soil-structure systems, and to the presentation of examples of earthquake response and steady-state response to sinusoidal excitation. Simplified formulas for natural frequency changes, radiation damping values, and other response parameters of interest in design are developed from the analysis and the examples.

ANALYSIS OF THE SYSTEM

General information. The system under investigation is shown in Figure 1. It consists of a linear, viscously damped n-story structure with one degree of freedom per floor, resting on the surface of an elastic half-space with density ρ, shear modulus μ, and Poisson's ratio, σ. For fixed-base response, the superstructure has a stiffness matrix K, mass matrix M, and damping matrix C, satisfying the condition $M^{-1}KM^{-1}C = M^{-1}CM^{-1}K$. O'Kelly (1964) has shown this to be a necessary and sufficient condition for the superstructure to admit decomposition into classical normal modes. (The assumption of classical normal modes for the superstructure can be removed, but because buildings seem to possess such modes over significant range of amplitudes, this simplifying assumption is retained.) The structural base is assumed to be a rigid plate of radius a and negligible thickness, and no slippage is allowed between the base and the soil. Formulated this way, the building-foundation system has $n+2$ significant degrees of freedom, namely, horizontal translation of each floor mass, horizontal translation of the base mass, and rotation of the system in the plane of motion.

The system, initially at rest, will be subjected to seismic motion or harmonic excitation represented by plane, horizontal shear waves traveling vertically upward. No scattering will result as the waves are normally incident on the flat foundation. In this idealization of the excitation, the free-field acceleration at the surface is twice the amplitude of the incoming wave, and the motion at depth is the sum of the incident and reflected waves.

The model for the building-foundation system shown in Figure 1 has also been studied by Tajimi (1967), Parmelee *et al.* (1969) and others. Because the superstructure by itself has classical normal modes, there is a simple physical model which is equivalent to the building-foundation system under study. This model, shown in Figure 2, consists of n simple, damped oscillators attached to a base identical to that of the system shown in Figure 1. Each oscillator is described by its natural frequency ω_j, critical damping ratio, η_j, mass M_j and height H_j defined by the corresponding modal quantities (given in the Appendix). In addition, the sum of the centroidal moments of inertia of the n masses is the same for both systems.

Assuming small displacements, the equations of motion of the building-foundation model shown in Figure 1 are

$$M\ddot{\mathbf{v}}^t + C\dot{\mathbf{v}} + K\mathbf{v} = 0 \tag{1a}$$

$$\sum_{j=1}^{n} m_j \ddot{v}_j{}^t + m_0(\ddot{v}_0 + \ddot{v}_g) + P(t) = 0 \tag{1b}$$

$$\sum_{j=1}^{n} m_j h_j \ddot{v}_j{}^t + I_t \ddot{\varphi} + Q(t) = 0. \tag{1c}$$

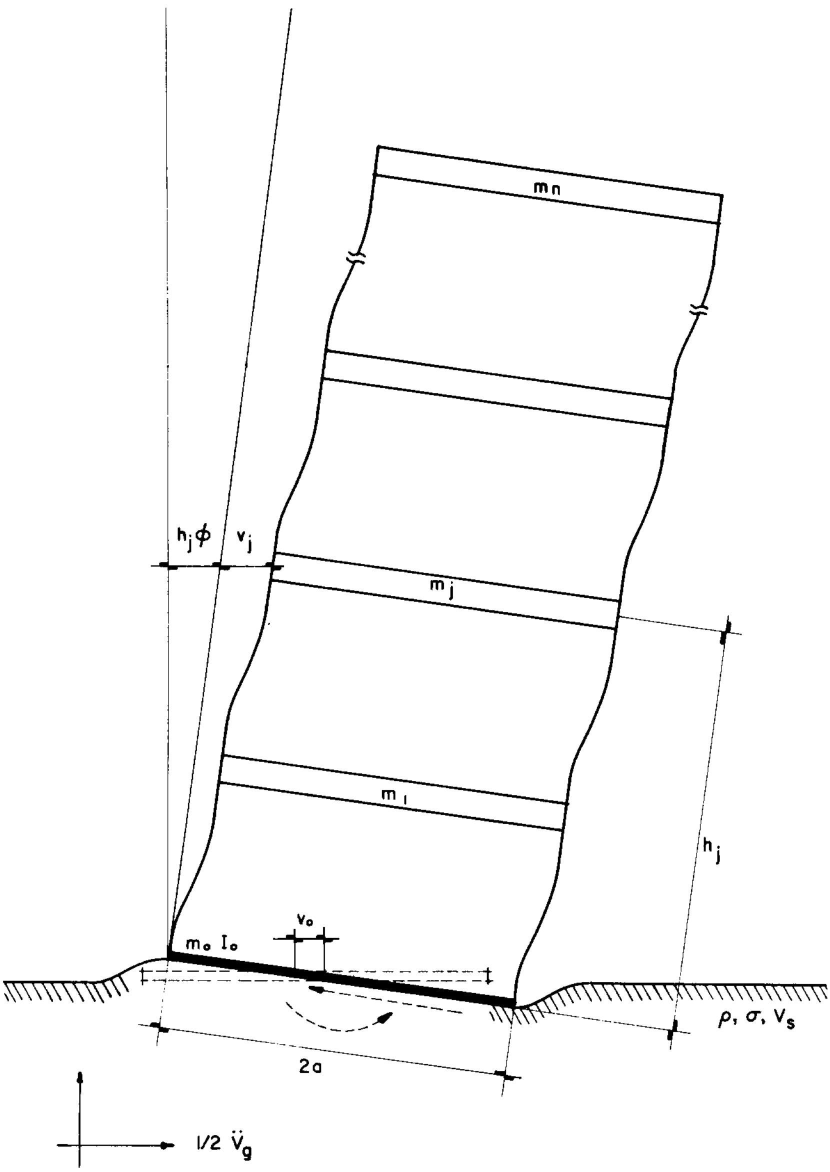


FIG. 1. Model of building-foundation system.

In these equations, $\mathbf{v} = \{v_j\}$, a column vector ; v_j is horizontal displacement of the superstructure at the jth floor relative to the base mass, excluding rotations; v_g is free-field, surface displacement resulting from the incident earthquake wave and its total reflection; v_0 is translation of the base mass relative to the free-field motion; φ is rotation of the base mass; h_j is height of the jth story above the base mass; v_j^t is total horizontal displacement of the jth mass with respect to a fixed vertical axis, i.e., $v_j^t = v_g + v_0 + h_j\varphi + v_j$; m_j is mass of the jth floor, m_0 is base mass, I_t is the sum of the centroidal moments of inertia of the $n+1$ masses, and $P(t)$ and $Q(t)$ are the interaction force and moment, respectively, between the base mass and the soil.

The required interaction forces $P(t)$ and $Q(t)$ may be obtained from the solution of a mixed boundary-value problem in elastodynamics, that is, the forced horizontal and rocking vibration of a massless, rigid disc resting on the surface of the elastic half-space. Thus, $P(t)$ and $Q(t)$ may be expressed in terms of the displacements $v_0(t)$ and $\varphi(t)$ as (Bielak, 1971; Luco and Westmann, 1971; Veletsos and Wei, 1971).

$$\begin{Bmatrix} \dfrac{\bar{P}(s)}{\mu a^2} \\ \\ \dfrac{\bar{Q}(s)}{\mu a^3} \end{Bmatrix} = \begin{bmatrix} K_{hh}(s_0, \sigma) & K_{hm}(s_0, \sigma) \\ \\ K_{mh}(s_0, \sigma) & K_{mm}(s_0, \sigma) \end{bmatrix} \begin{Bmatrix} \dfrac{\bar{v}_o(s)}{a} \\ \\ \bar{\varphi}(s) \end{Bmatrix} \tag{2}$$

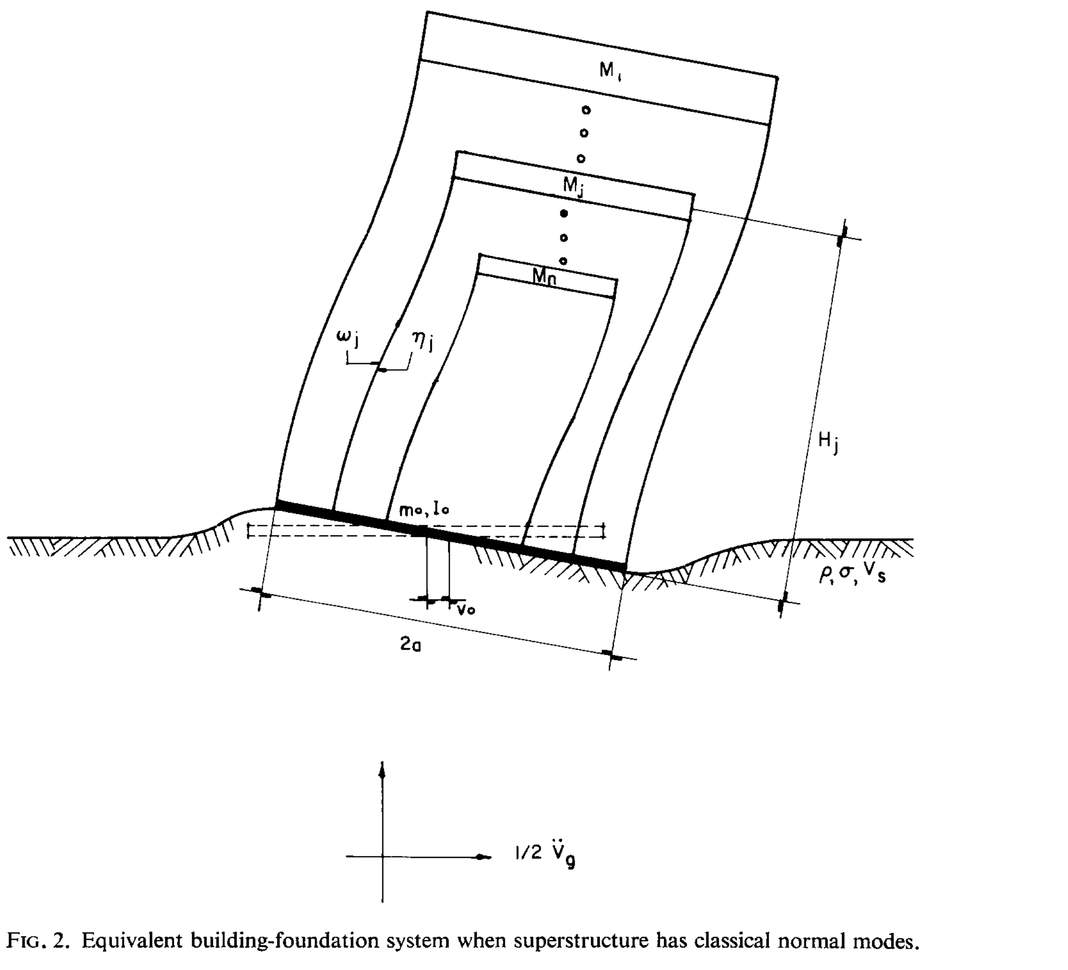


FIG. 2. Equivalent building-foundation system when superstructure has classical normal modes.

In equation (2), a bar over a function denotes the Laplace transform of that function and s is the complex parameter of the transform; $s_0 = sa/V_s$ where $V_s = (\mu/\rho)^{1/2}$ is the shear-wave velocity of the foundation medium. The functions K_{hh}, K_{hm}, K_{mh} and K_{mm} are the dimensionless impedances of the problem; K_{hm} and K_{mh} are equal, a consequence of reciprocity theorems.

If the inverse transformation of equation (2) is taken, it will be seen that $P(t)$ and $Q(t)$ are related to $v_0(t)$ and $\varphi(t)$ through convolution integrals which, upon substitution into equations (1) lead to a system of linear integrodifferential equations of motion. These equations do not lend themselves to direct numerical integration but can be solved by the Laplace operational method.

Because the superstructure has classical normal modes, the transformed version of equation (1a) may be uncoupled and solved explicitly for the displacements $\bar{v}_j$ in terms of the free-field acceleration $\bar{\ddot{v}}_g$ and the unknown displacements $\bar{v}_0$ and $\bar{\varphi}$, which are additional excitations of the superstructure in this formulation. Replacing the resulting expression for $\bar{v}_j$ together with equation (2) into the transformed versions of equations (1b) and (1c) results in a system of two linear algebraic equations in the unknown functions $\bar{v}_0$ and $\bar{\varphi}$, which can then be solved explicitly. The displacements $\bar{v}_j$ are obtained by substituting the solutions for $\bar{v}_0$ and $\bar{\varphi}$ into the initial expression for $\bar{v}_j$. This process leads to

$$\bar{v}_0(s) = \bar{\ddot{v}}_g(s)\,\frac{\Delta_o(s)}{\Delta(s)} \tag{3a}$$

$$\bar{\varphi}(s) = \bar{\ddot{v}}_g(s)\,\frac{\Delta_\varphi(s)}{\Delta(s)} \tag{3b}$$

$$\bar{v}_j(s) = \bar{\ddot{v}}_g(s)\,\frac{\Delta_j(s)}{\Delta(s)}; \qquad j = 1, 2, \ldots, n. \tag{3c}$$

Formulas for the transfer functions Δ_0/Δ, Δ_φ/Δ, and Δ_j/Δ are given in the Appendix. Hence, equations (3) provide a closed-form solution in transform space to the equations of motion (1) in terms of the transform of the incident excitation, the physical properties of the building and its foundation, and the functions K_{hh}, K_{hm} and K_{mm}.

The desired displacements $v_0(t)$, $\varphi(t)$ and $v_j(t)$ may now be found by inverting their corresponding Laplace transforms. Thus, from equation (3)

$$\begin{Bmatrix} v_0(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \frac{1}{2\pi i}\int_C \frac{\bar{\ddot{v}}_g(s)}{\Delta(s)} \begin{Bmatrix} \Delta_0(s) \\ \Delta_\varphi(s) \\ \Delta_j(s) \end{Bmatrix} e^{st} ds \tag{4}$$

where the integrals are evaluated over C, the Bromwich contour.

By a direct application of the convolution theorem of Laplace transformations (Carrier *et al.*, 1966), equation (4) becomes

$$\begin{Bmatrix} v_0(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \int_0^t \begin{Bmatrix} h_0(t-\tau) \\ h_\varphi(t-\tau) \\ h_j(t-\tau) \end{Bmatrix} \ddot{v}_g(\tau)\, d\tau \tag{5a}$$

in which

$$h_k(t) = \frac{1}{2\pi i}\int_C \frac{\Delta_k(s)}{\Delta(s)} e^{st} ds; \qquad k = 0, \varphi, j; j = 1, \ldots, n \tag{5b}$$

are the impulse response functions of the system.

The integrals in equation (5b) are next solved by contour integration. For purposes of exposition, these integrals will be evaluated first for the case of linear impedances (equivalent to a discrete foundation of linear springs and dashpots). Second, the elastic half-space will be considered.

Analysis for a Discrete Foundation

Hsieh (1962) showed that the steady-state, forced harmonic motion of a rigid, circular plate on an elastic half-space can be modeled by a simpler system. For each of the four degrees of freedom of the plate, the elastic medium can be replaced by a linear spring whose stiffness depends upon the frequency of oscillation, and a linear, viscous damper which is also frequency-dependent.

Available numerical results (Bycroft, 1956; Thomson and Kobori, 1963; Luco and Westmann, 1971; Veletsos and Wei, 1971) indicate that most of the dynamic properties of the springs and dashpots representing the elastic half-space remain nearly constant within the frequency range of interest for typical buildings. It is then reasonable to assume as a first approximation that the linear springs and viscous dampers have constant properties. This is the approach used by Parmelee *et al.* (1969) to study, by numerical integration of the equations of motion, the earthquake response of selected multi-story buildings resting on elastic half-spaces.

With the assumption of constant properties, the functions K_{hh}, K_{hm}, and K_{mm} become linear in s_0 (or s). The functions $\Delta_0(s)$, $\Delta_\varphi(s)$, and $\Delta_j(s)$ in the numerators of equations (3 and 5b) then become polynomials of degree $2n$, while the function $\Delta(s)$ in the denominator gives a polynomial of degree $2n+4$. Hence, the only singularities of the integrand in equation (5*b*) correspond to the $n+2$ pairs of complex conjugate roots of the polynomial $\Delta(s)$. Each of these pairs is associated with a resonant frequency of the system and the attendant damping.

Using this information, the integrals in equation (5b) may be evaluated by standard contour integration techniques (Bielak, 1971). Performing the integration and making use of the residue theorem, equation (5b) becomes

$$h_k(t) = \sum_{l=1}^{n+2} \exp(-\sigma_l t)\,(a_{lk}\cos\beta_l t - b_{lk}\sin\beta_l t); \quad k = 0, \varphi, j; j = 1, \ldots, n \tag{6a}$$

in which the real constants a_{lk}, b_{lk}, σ_l and β_l are defined by

$$a_{lk} + ib_{lk} = 2\,\frac{\Delta_k(s_l)}{\Delta'(s_l)} \tag{6b}$$

$$-\sigma_l + i\beta_l = s_l \tag{6c}$$

where s_l is a root of $\Delta(s)$ and $\Delta'(s_l)$ is the first derivative of $\Delta(s)$ evaluated at s_l. $\Delta'(s_l)$ can be found from the formulas given in the Appendix.

Equation (6a) gives closed-form expressions for the impulse response functions of the system in terms of elementary functions and real constants that can be evaluated explicitly. The time response of the system is then calculable from equations (5a).

Analysis for the Half-Space

Equation (6a) shows that the impulse response functions of a structure supported on a discrete foundation are linear combinations of $n+2$ pairs of terms, corresponding to the $n+2$ pairs of complex conjugate roots of $\Delta(s)$ in equation (5b). Each pair of roots is associated with a resonant frequency and damping.

When the discrete foundation is replaced by the elastic half-space, the impedance functions K_{hh}, K_{hm}, and K_{mm} no longer are linear in s. On physical grounds, however, it is expected that the building-foundation system should still exhibit the same number of significant resonant frequencies. To preserve this feature of the problem, it is assumed that $\Delta(s)$ will again have $n+2$ pairs of nonrepeated complex conjugate roots, one pair to be associated with each significant resonant frequency. It will also be assumed for mathematical purposes that the impedance functions K_{hh}, K_{hm}, and K_{mm} are analytic away from infinity and are such that $\Delta_k(s)/\Delta(s)$ in equation (5b) goes to zero as s approaches infinity.

With these assumptions, the type of contour integration discussed in the preceding section remains valid when the discrete foundation is replaced by an elastic half-space. Hence, the impulse response functions of the corresponding building-foundation system are again given by equation (6a). In this case, however, the quantities a_{lk}, b_{lk}, σ_l, β_l, s_l, $\Delta_k(s_l)$ and $\Delta'(s_l)$ must be obtained by using the impedance functions for the half-space and they will have different values, in general, from those found for the case of the discrete foundation. The assumptions involved here allow what is, in effect, an equivalent linearization of the entire system in the transform domain.

Having determined the impulse response functions of the system, $h_k(t)$, the time histories $v_0(t)$, $\varphi(t)$, and $v_j(t)$ may be obtained by substituting equation (6a) into equation (5a). Upon integration by parts, however, the resulting expressions can be written in a more convenient form subject to a simple physical interpretation.

Defining frequencies and dampings by

$$\tilde{\omega}_l = (\sigma_l^2 + \beta_l^2)^{1/2} \tag{7a}$$

$$\tilde{\eta}_l = \sigma_l/(\sigma_l^2 + \beta_l^2)^{1/2} \tag{7b}$$

and modified excitations by

$$\begin{Bmatrix} \ddot{v}_{l0}{}^e(t) \\ \ddot{v}_{l\varphi}{}^e(t) \\ \ddot{v}_{lj}{}^e(t) \end{Bmatrix} = \begin{Bmatrix} a_{l0} \\ a_{l\varphi} \\ a_{lj} \end{Bmatrix} [\tilde{\omega}_l^2 \dot{v}_g(t) + \tilde{\eta}_l \tilde{\omega}_l \ddot{v}_g(t)] + \begin{Bmatrix} b_{l0} \\ b_{l\varphi} \\ b_{lj} \end{Bmatrix} \tilde{\omega}_l \sqrt{1-\tilde{\eta}_l^2}\, \dot{v}_g(t), \tag{7c}$$

the response of the building-soil system can be written as

$$\begin{Bmatrix} v_0(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = -\sum_{l=1}^{n+2} \frac{1}{\tilde{\omega}_l \sqrt{1-\tilde{\eta}_l^2}} \int_0^t \exp[-\tilde{\eta}_l \tilde{\omega}_l (t-\tau)] \sin \tilde{\omega}_l \sqrt{1-\tilde{\eta}_l^2}(t-\tau) \begin{Bmatrix} \ddot{v}_{l0}{}^e(\tau) \\ \ddot{v}_{l\varphi}{}^e(\tau) \\ \ddot{v}_{lj}{}^e(\tau) \end{Bmatrix} d\tau. \tag{7d}$$

In the derivation of equation (7d) from equations (5a) and (6a), use has been made of the equation

$$\sum_{l=1}^{n+2} \begin{Bmatrix} a_{l0} \\ a_{l\varphi} \\ a_{lj} \end{Bmatrix} = \{0\} \tag{8}$$

which is obtained from the requirement that the velocities $\dot{v}_0(t)$, $\dot{\varphi}(t)$ and $\dot{v}_j(t)$ all vanish at $t = 0$.

The expressions in equation (7d) show that the response of the building-foundation system may be written as a linear combination of the responses of $n+2$, viscously damped, one-degree-of-freedom linear oscillators resting on rigid ground. Each oscillator, described by its undamped natural frequency, $\tilde{\omega}_l$, and fraction of critical damping, $\tilde{\eta}_l$, experiences an acceleration at its base given by $\ddot{v}_{lk}{}^e$. The subscript k takes the values 0, φ and $j (j = 1, \ldots, n)$ corresponding to the displacements $v_0(t)$, $\varphi(t)$ and $v_j(t)$, respectively.

Although equation (7d) resembles the results for normal mode analysis of fixed-base structures, the equation is valid even for building-foundation systems that do not have classical normal modes as no assumption about the existence of such modes was made in the derivation. If the structure-foundation system has normal modes, the coefficients a_{lk} will vanish.

An Alternate Form of the Solution

An alternate expression for the solution may be obtained from equation (4) by selecting the imaginary axis as the Bromwich contour. This is possible because the building-foundation system under investigation is stable, and therefore, any singularities occurring in the transformed space must be either to the left of the imaginary axis, or if on that axis, they can be at most simple poles. Thus, after introducing the variable transformation $s = i\omega$, equation (4) becomes

$$\begin{Bmatrix} v_0(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{\bar{v}}_g(i\omega)}{\Delta(i\omega)} \begin{Bmatrix} \Delta_0(i\omega) \\ \Delta_\varphi(i\omega) \\ \Delta_j(i\omega) \end{Bmatrix} \exp(i\omega t)\, d\omega. \tag{9}$$

The integral operator appearing in equation (9) represents a Fourier integral that may be evaluated by the Fast Fourier Transform (FFT) technique (Bergland, 1969) thereby utilizing the high computational efficiency of the FFT algorithm.

Equation (9) also can be obtained directly by taking the Fourier transform of the equations of motion (1) and using the inverse Fourier theorem. This approach, together with the FFT, was used recently by Liu and Fagel (1971) in a numerical study of a single-story building-foundation system.

It should be noted that equation (9) requires only values of the impedance functions K_{hh}, K_{hm}, and K_{mm} as functions of the real frequency parameter $a_0 = \omega a/V_s$ whereas the analogous equation for the Laplace transform, equation (4), requires knowledge of the impedance functions throughout the complex plane.

Application of Foss's Method to Systems with Discrete Foundations

When the soil is represented by linear, discrete elements, equations (1) reduce to a system of second-order ordinary differential equations with constant coefficients which may be solved by several methods, including the Laplace operational approach used above. Another useful method for the solution of these equations is presented next.

The equations of motion of buildings resting upon linear, discrete idealizations of the half-space can be written as

$$M_0\ddot{\mathbf{X}} + C_0\dot{\mathbf{X}} + K_0\mathbf{X} = -\mathbf{f}\ddot{v}_g(t) \tag{10}$$

where M_0, C_0, and K_0 are $N \times N$ ($N = n+2$) symmetric matrices with K_0 nonsingular, $\mathbf{X}$ is the displacement vector, $\mathbf{f}$ is a known vector, and $\ddot{v}_g(t)$ is the free-field earthquake acceleration.

The classical normal mode method of analysis cannot be used to solve equation (10) as the system does not, in general, possess such modes. Foss, (1958), however, has shown that systems of this type are solvable by modal methods after transforming them to $2N$-space.

Equation (10) is first combined with the identity equation

$$M_0\dot{\mathbf{X}} - M_0\dot{\mathbf{X}} = 0 \tag{11}$$

to obtain a system of first-order differential equations in $2N$ unknowns

$$R\dot{\mathbf{Z}} + S\mathbf{Z} = -\mathbf{F}\ddot{v}_g(t) \tag{12a}$$

in which

$$R = \begin{bmatrix} 0 & M_0 \\ M_0 & C_0 \end{bmatrix}; \quad S = \begin{bmatrix} -M_0 & 0 \\ 0 & K_0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix}; \quad \mathbf{Z} = \begin{bmatrix} \dot{\mathbf{X}} \\ \mathbf{X} \end{bmatrix}. \tag{12b}$$

Equation (12a) may be uncoupled and solved by superposition provided the eigenvalues of $S^{-1}R$ are distinct. Proceeding on this assumption, the matrix

$$U = -S^{-1}R = \begin{bmatrix} 0 & I \\ -K_0^{-1}M_0 & -K_0^{-1}C_0 \end{bmatrix}; \quad \text{a } 2N\times 2N \text{ matrix} \tag{13}$$

may be diagonalized by a similarity transformation Φ, the columns of which are the eigenvectors of R and S. From the fact that R and S are symmetric,

$$\Phi^T R\Phi = \tilde{R}, \text{ a diagonal } 2N\times 2N \text{ matrix} \tag{14a}$$

$$\Phi^T S\Phi = \tilde{S}, \text{ a diagonal } 2N\times 2N \text{ matrix.} \tag{14b}$$

Equations (14) are the orthogonality conditions in $2N$-space and may be expanded in terms of N-space quantities. From the form of equations (12) the ith column, Φ_i, of Φ may be partitioned

$$\Phi_i = \begin{Bmatrix} \alpha_i & \varphi_i \\ & \varphi_i \end{Bmatrix}; \qquad i = 1, 2, \ldots, 2N \tag{15}$$

in which φ_i is an $N\times 1$ column vector and α_i is an eigenvalue of U.

Equations (12a) can be uncoupled by making use of the orthogonality conditions (14). After solving each uncoupled equation, the following solution was obtained by Foss for a system which is initially at rest;

$$\mathbf{Z} = \begin{Bmatrix} \dot{\mathbf{X}} \\ \mathbf{X} \end{Bmatrix} = -\sum_{k=1}^{2N} \left(\frac{G_k}{\tilde{R}_{kk}} \int_0^t \exp\,[\alpha_k(t-\tau)]\ddot{v}_g(\tau)\,d\tau \right) \begin{Bmatrix} \alpha_k & \varphi_k \\ & \varphi_k \end{Bmatrix}. \tag{16}$$

In equation (16), G_k is an element of the $2N\times 1$ column vector, $\mathbf{G} = \Phi^T\mathbf{F}$, and $\tilde{R}_{kk}$ is the kth diagonal element of the matrix $\tilde{R}$.

A more convenient form of the solution for our purposes may be obtained by noting that the eigenvalues α_k occur in complex conjugate pairs as do the corresponding eigenvectors. Thus, the equation for the displacements $\mathbf{X}$ in equation (16) becomes

$$\mathbf{X} = -2\sum_{k=1}^{N} \left(\mathrm{Re}\,\frac{G_k}{\tilde{R}_{kk}}\,\varphi_k \int_0^t \exp\,[\alpha_k(t-\tau)]\ddot{v}_g(\tau)\,d\tau \right). \tag{17}$$

Upon integration by parts, equation (17) may be transformed into

$$\mathbf{X} = -\sum_{k=1}^{N} \frac{1}{\tilde{\theta}_k\sqrt{1-\tilde{\lambda}_k^2}} \int_0^t \exp\,[-\tilde{\lambda}_k\tilde{\theta}_k(t-\tau)]\sin\,[\tilde{\theta}_k\sqrt{1-\tilde{\lambda}_k^2}(t-\tau)]\ddot{v}_k{}^e(\tau)\,d\tau \tag{18a}$$

where the constants $\tilde{\theta}_k$ and $\tilde{\lambda}_k$, and the elements of the equivalent input acceleration vector $\ddot{v}_k{}^e(t)$, are defined by

$$\tilde{\theta}_k = [(\mathrm{Im}\,\alpha_k)^2 + (\mathrm{Re}\,\alpha_k)^2]^{1/2} \tag{18b}$$

$$\tilde{\lambda}_k = \frac{-\mathrm{Re}\,\alpha_k}{[(\mathrm{Im}\,\alpha_k)^2 + \mathrm{Re}\,\alpha_k)^2]^{1/2}} \tag{18c}$$

and

$$\ddot{v}_k{}^e(t) = [\tilde{\theta}_k{}^2 v_g(t) + \tilde{\lambda}_k\tilde{\theta}_k\dot{v}_g(t)]\,\mathrm{Re}\left(-2\frac{G_k}{\tilde{R}_{kk}}\,\varphi_k\right) + \tilde{\theta}_k\sqrt{1-\tilde{\lambda}_k{}^2}\,\dot{v}_g(t)\,\mathrm{Im}\left(-\frac{2G_k}{\tilde{R}_{kk}}\,\varphi_k\right). \tag{18d}$$

Equations (18) are formulas that can be evaluated directly from the eigenvalues and eigenvectors of the matrix U. The equations show that the response of the system may be obtained as a linear combination of the responses of N damped, one-degree-of-freedom oscillators subjected to modified excitations, provided the eigenvalues of the matrix U are distinct. The result holds for discrete, constant, linear, damped systems, irrespective of whether or not the systems possess classical normal modes.

For the particular case of a building supported on a discrete foundation, equations (18) may be shown to be equivalent to the corresponding equations obtained by the Laplace operational method [equations (7)]. The essential difference between the two formulations lies in the method of determining the natural frequencies, damping ratios, and base accelerations of the equivalent linear oscillators. In some instances, it may be more convenient to use equations (18) for numerical calculations, as it is often easier to solve an eigenvalue problem than to obtain directly the roots of the corresponding frequency equation.

Although developed for discrete, constant, linear systems, this technique can also be applied to structures founded on an elastic half-space by iterating between successive discrete approximations to the half-space.

APPLICATIONS

The earthquake response and the steady-state response to sinusoidal excitation of some idealized building-foundation systems will next be examined to illustrate the use of the methods developed above. In addition to numerical examples, formulas will be presented for some of the more important parameters of response for single-story structure-foundation systems and for the fundamental mode response of multi-degree-of-freedom systems.

Dynamic soil coefficients. To apply the analysis, it is necessary to know the impedance functions K_{hh}, K_{hm}, and K_{mm}, which relate the stress resultants of the contact area to the displacements experienced by a rigid disc oscillating on the surface of the elastic half-space. These impedance functions are found from the solution of a mixed boundary-value problem, referred to as the complete problem, in which a horizontal uniform translation and a rigid rotation about an axis parallel to the plane of the disc are prescribed under the disc, and the stresses are specified to be zero over the remainder of the surface of the half space. Although a formal solution has been derived for this problem (Bielak, 1971), no numerical results have been obtained to date other than for the corresponding static problem. Gladwell (1968), Luco and Westmann (1971), Veletsos and Wei (1971), and Bycroft (1956) in an approximate solution, however, have presented numerical results for a related simpler mixed boundary-value problem, the relaxed problem, for the case of steady-state harmonic oscillations of the disc. In the relaxed problem, simplifying assumptions are made regarding the conditions of contact between the disc and the supporting medium.

As a result of these assumptions, the rotation of the disc induced by the horizontal force and the horizontal displacement induced by the moment can only be determined approximately. Another consequence of the assumptions is that neither the horizontal displacement caused by the horizontal force, nor the rotation of the disc caused by the overturning moment coincide exactly with the corresponding values for the complete problem.

Fortunately, however, the errors introduced by the simplified boundary conditions under the disc seem to be small. The analogous complete and relaxed problems for the infinite, rigid strip have been analyzed by Luco (1969), who showed that differences in the impedance functions for the two problems are significant only for large values of the frequency parameter and small values of Poisson's ratio. Moreover, Luco (1969) and Veletsos and Wei (1971) have shown that the off-diagonal elements K_{hm} and K_{mh} in the impedance matrix of equation (2) are generally small compared to K_{hh} and K_{mm}.

To investigate further the effect of these assumptions on the response of building-foundation systems, the fundamental frequency of a single-story building resting on the half-space was calculated for a wide range of the system parameters (Bielak, 1971) using static values of the impedance matrices. Two types of bond between the base of the building and its foundation were considered, corresponding to the complete and the relaxed boundary-value problems discussed above. The difference in the values of the fundamental frequency was in no case greater than 5 per cent.

In view of these results, and in order to use the numerical values of the impedance functions now available, it is assumed in the calculations which follow that the values of K_{hh} and K_{mm} found from the solution of the relaxed mixed boundary-value problem are sufficiently accurate, and further that the off-diagonal terms of the impedance matrix can be neglected. It is thought that these assumptions do not affect the dynamics of the problem significantly. The results in this section can be updated, of course, as more complete numerical results become available.

Numerical Evaluation of the Soil Coefficients

Bycroft (1956) and Gladwell (1968) have shown that for steady-state harmonic vibration of the disc, the functions K_{hh} and K_{mm} can be expressed formally as

$$K_{hh}(ia_0) = k_{hh}(a_0)+ia_0c_{hh}(a_0) \tag{19a}$$

$$K_{mm}(ia_0) = k_{mm}(a_0)+ia_0c_{mm}(a_0) \tag{19b}$$

in which the functions k_{hh}, c_{hh}, k_{mm} and c_{mm} are real, and $a_0 = \omega a/V_s$. Hsieh (1962) has shown that k_{hh} and k_{mm} can be interpreted as the stiffnesses of frequency-dependent linear springs, whereas c_{hh} and c_{mm} are associated with viscous dampers, also functions of frequency. These stiffnesses k_h and k_m and damping coefficients c_h and c_m are

$$k_h = \mu a k_{hh}(a_0, \sigma), \qquad c_h = \frac{\mu a^2}{V_s} c_{hh}(a_0, \sigma) \tag{20a}$$

$$k_m = \mu a^3 k_{mm}(a_0, \sigma), \qquad c_m = \frac{\mu a^4}{V_s} c_{mm}(a_0, \sigma). \tag{20b}$$

The functions k_{hh}, k_{mm}, c_{hh} and c_{mm} may be calculated from available numerical results for values of the frequency parameter, a_0, up to 10 and for several values of Poisson's ratio, σ. It is convenient for later work to express these functions in the form

$$k_{hh}(a_0, \sigma) = \frac{8}{2-\sigma}\beta_h(a_0, \sigma) = \sigma_h\beta_h(a_0, \sigma) \tag{21a}$$

$$c_{hh}(a_0, \sigma) = \zeta_h(a_0, \sigma)k_{hh}(a_0, \sigma) \tag{21b}$$

$$k_{mm}(a_0, \sigma) = \frac{8}{3(1-\sigma)}\beta_m(a_0, \sigma) = \sigma_m\beta_m(a_0, \sigma) \tag{21c}$$

$$c_{mm}(a_0, \sigma) = \zeta_m(a_0, \sigma)k_{mm}(a_0, \sigma) \tag{21d}$$

in which the constants σ_h and σ_m are the static values of the stiffnesses k_{hh} and k_{mm}, respectively, and the functions β_h and β_m measure the ratio of the dynamic stiffnesses to their static values. ζ_h and ζ_m are related to the energy radiated into the elastic half-space by horizontal translation and rotation, respectively; however, these parameters are not ratios of critical damping.

The functions β_h, β_m, ζ_h, and ζ_m, calculated from the results of Luco and Westmann (1971) are shown in Figure 3 for three values of Poisson's ratio and values of a_0 from 0 to 2. This range of a_0 is sufficient for most practical applications.

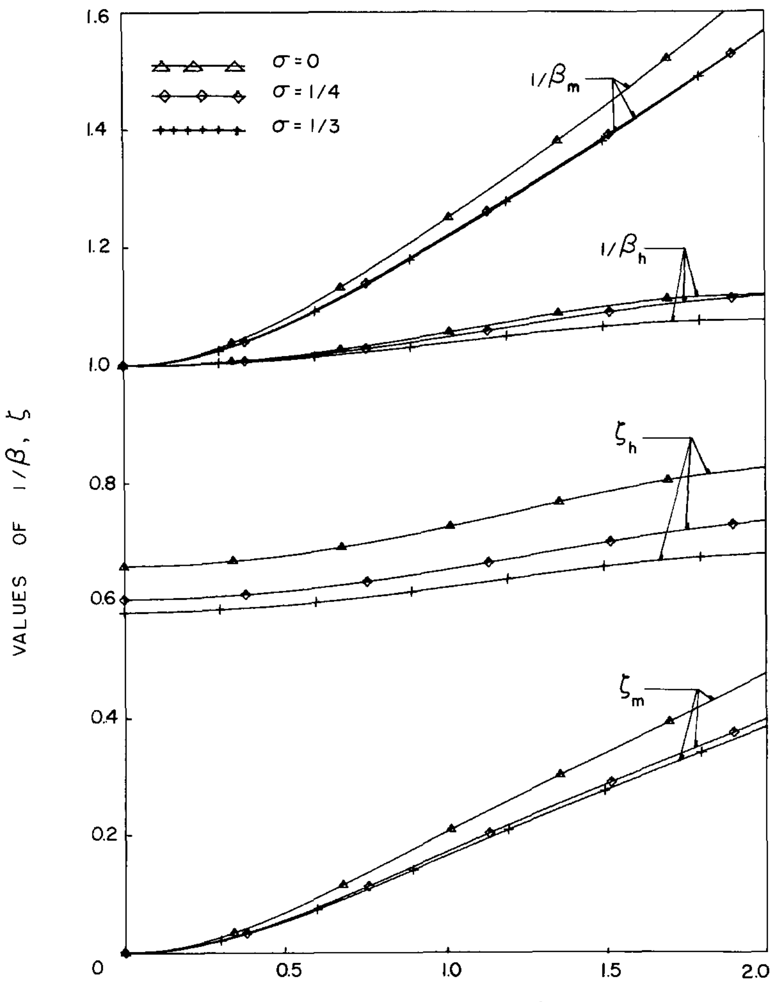


FIG. 3. Dynamic soil coefficients.

Impedance Functions for Transient Vibrations

To study the earthquake response of building-foundation systems by means of equations (5), it is necessary to know K_{hh} and K_{mm} as functions of s_0, a complex number, rather than ia_0. Although numerical values throughout the complex plane are not yet available, it is possible to obtain $K_{hh}(s_0)$ and $K_{mm}(s_0)$ by analytic continuation from the known solutions $K_{hh}(ia_0)$ and $K_{mm}(ia_0)$. For the small values of damping coefficients that typify most applications, a particularly convenient approximation is (Bielak, 1971)

$$K_{hh}(s_0) = k_{hh}(\mathrm{Im}\ s_0)+s_0 c_{hh}(\mathrm{Im}\ s_0) \tag{22a}$$

$$K_{mm}(s_0) = k_{mm}(\mathrm{Im}\ s_0)+s_0 c_{mm}(\mathrm{Im}\ s_0). \tag{22b}$$

This representation, like equation (19), has the advantage of involving only the real functions $k_{hh}(a_0)$, $c_{hh}(a_0)$, $k_{mm}(a_0)$ and $c_{mm}(a_0)$ of equation (21).

SINGLE-STORY STRUCTURE ON AN ELASTIC HALF-SPACE

For the first example, an idealized single-story interacting system will be considered, as shown in Figure 4. The single-story structure of height h_1 is linear, viscously damped and has a base mass resting on the surface of the half-space. For fixed-base response, the structure has a stiffness k_1, mass m_1, undamped natural frequency $\omega_1 = (k_1/m_1)^{1/2}$, damping coefficient c_1, and fraction of critical damping η_1.

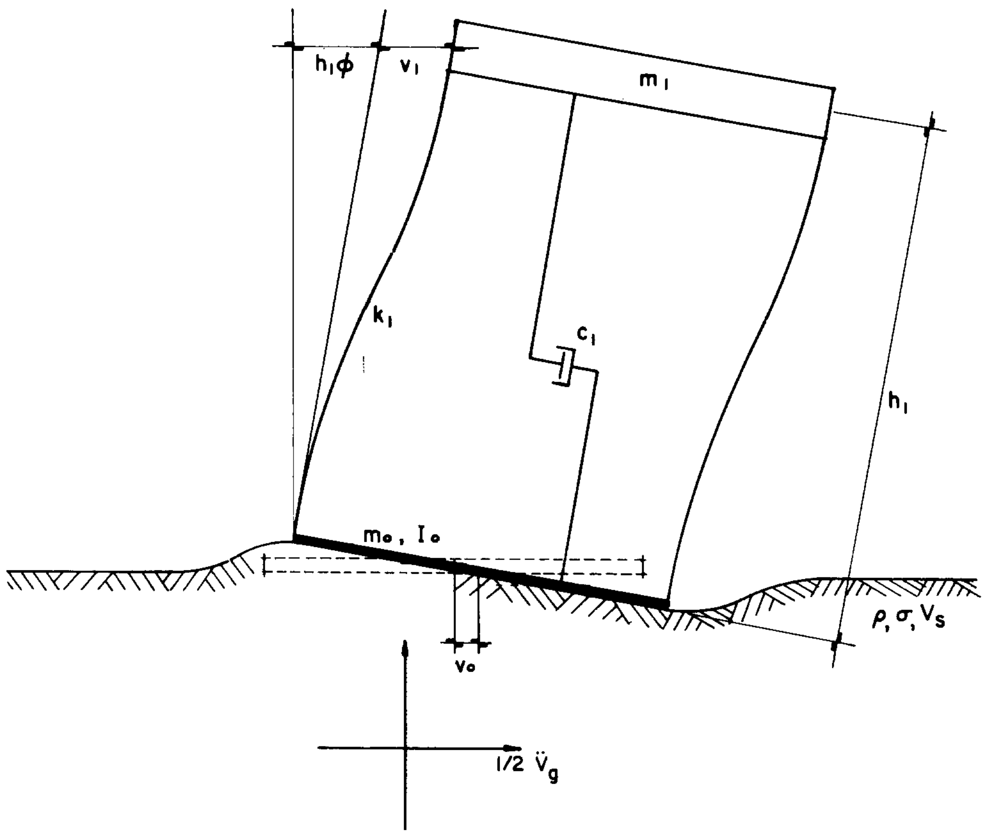


FIG. 4. Single-story building-foundation system.

Dimensionless expressions for the Laplace transforms of the horizontal translation, v_0, of the base mass relative to the free-field motion; for the total translation of the base mass, $y_0 = v_0+v_g$; for the rotation, φ, of the system; and for the relative displacement, v_1, of the top mass relative to the base, excluding rotation, can be obtained from equations (3) and (22), and the results in the Appendix. For this example, n is unity and K_{hm} and K_{mh} vanish.

$$\frac{\bar{v}_0}{\bar{v}_g} = -\frac{s_0^2}{\Delta}\left\{\left(1+\frac{s_0^2 b_m}{\beta_m\sigma_m}+\zeta_m s_0\right)\left[\left(1+2\eta_1\frac{s_0}{a_1}\right)(b_1+b_h)\frac{1}{\beta_h\sigma_h}+\frac{s_0^2 b_h}{a_1^2\beta_h\sigma_h}\right] + \frac{b_1 b_h \alpha_1^2 s_0^2}{\beta_h\beta_m\sigma_h\sigma_m}\left(1+2\eta_1\frac{s_0}{a_1}\right)\right\} \tag{23a}$$

$$\frac{\bar{y}_0}{\bar{v}_g} = \frac{1+s_0\zeta_h}{\Delta}\left[\left(1+2\eta_1\frac{s_0}{a_1}\right)\frac{s_0{}^2 b_1 \alpha_1{}^2}{\beta_m\sigma_m} + \left(1+2\eta_1\frac{s_0}{a_1}+\frac{s_0{}^2}{a_1{}^2}\right)\left(1+\frac{s_0{}^2 b_m}{\beta_m\sigma_m}+s_0\zeta_m\right)\right] \tag{23b}$$

$$\frac{\bar{\varphi}h_1}{\bar{v}_g} = -\frac{s_0{}^2}{\Delta}(1+s_0\zeta_h)\left(1+2\eta_1\frac{s_0}{a_1}\right)\frac{b_1\alpha_1{}^2}{\beta_m\sigma_m} \tag{23c}$$

$$\frac{\bar{v}_1}{\bar{v}_g} = -\frac{s_0{}^2}{a_1{}^2\Delta}(1+s_0\zeta_h)\left(1+\frac{s_0{}^2 b_m}{\beta_m\sigma_m}+s_0\zeta_m\right) \tag{23d}$$

in which

$$\begin{aligned}\Delta = {} & s_0{}^2 b_1\left(1+2\eta_1\frac{s_0}{a_1}\right)\left[\frac{1}{\beta_h\sigma_h}\left(1+\frac{s_0{}^2 b_m}{\beta_m\sigma_m}+s_0\zeta_m\right) + \frac{\alpha_1{}^2}{\beta_m\sigma_m}\left(1+\frac{s_0{}^2 b_h}{\beta_h\sigma_h}+s_0\zeta_h\right)\right] \\ & + \left(1+2\eta_1\frac{s_0}{a_1}+\frac{s_0{}^2}{a_1{}^2}\right)\left[\frac{s_0{}^4 b_h b_m}{\beta_h\beta_m\sigma_h\sigma_m}+\frac{s_0{}^2 b_h}{\beta_h\sigma_h}(1+s_0\zeta_m) \right. \\ & \left. + \frac{s_0{}^2 b_m}{\beta_m\sigma_m}(1+s_0\zeta_h)+1+s_0(\zeta_m+\zeta_h)+s_0{}^2\zeta_h\zeta_m\right].\end{aligned} \tag{23e}$$

Equations (23), with equations (21), give explicitly the transfer functions of the single-story building foundation system in terms of Poisson's ratio and the dimensionless parameters a_1, α_1, η_1, b_1, b_h and b_m, and the transform parameter s_0.

The system parameters are given by

$$a_1 = \frac{\omega_1 a}{V_s} \tag{24a}$$

$$\alpha_1 = \frac{h_1}{a} \tag{24b}$$

$$\eta_1 = \frac{c_1}{2m_1\omega_1} \tag{24c}$$

$$b_1 = \frac{m_1}{\rho a^3} \tag{24d}$$

$$b_h = \frac{m_0}{\rho a^3} \tag{24e}$$

$$b_m = \frac{I_t}{\rho a^5}. \tag{24f}$$

Of these parameters, only a_1 is a function of the soil stiffness. In fact, the rigidity of the soil, as measured by its shear-wave velocity, V_s, only enters the problem in conjunction with ω_1. Therefore, the dynamic coupling between a building of this type and the surrounding ground will depend on the relative stiffnesses of the superstructure and its foundation, and not on the rigidity of the soil *per se*.

The transfer functions given by equations (23) were obtained for a building-foundation system whose base mass can both rotate and translate with respect to the free-field displacement. Constraints, however, may be imposed on the base which preclude one of these motions, for example, some systems founded on piles might not allow rocking of the base. The dynamic behavior of constrained systems can still be described by equations (23); it is only necessary to set $1/\beta_m$ and ζ_m equal to zero in these equations if the base is not allowed to rotate, and to eliminate the terms containing $1/\beta_h$ and ζ_h if the base cannot move horizontally with respect to the free-field displacement.

Steady-State Response

The steady-state harmonic response of the building-foundation system shown in Figure 4 may be obtained readily from the foregoing results. Assuming a free-field surface motion $v_g(t) = \bar{v}_g \operatorname{Re}(\exp i\omega t)$ where $\bar{v}_g$ is the amplitude of the motion and ω the

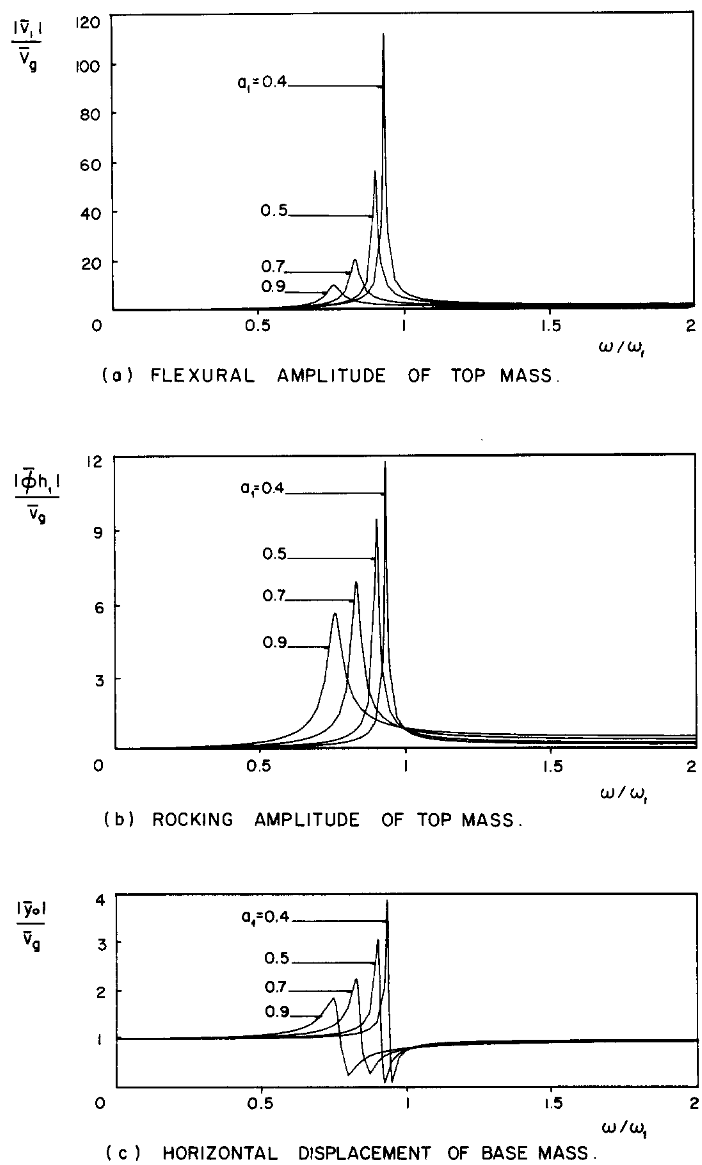


FIG. 5. Amplification ratios for the frequency response of a single-story interaction system ($\alpha_1 = 1.5$, $b_1 = 1, \sigma = \frac{1}{4}, b_h = b_m = \eta_1 = 0$).

frequency of oscillation, the harmonic displacements $v_0(t)$, $y_0(t)$, $\varphi(t)$ and $v_1(t)$ are given by

$$\begin{Bmatrix} v_0(t) \\ y_0(t) \\ \varphi(t) \\ v_1(t) \end{Bmatrix} = \mathrm{Re} \begin{Bmatrix} \bar{v}_0 \\ \bar{y}_0 \\ \bar{\varphi} \\ \bar{v}_1 \end{Bmatrix} \exp(i\omega t) \tag{25}$$

where the complex quantities $\bar{v}_0$, $\bar{y}_0$, $\bar{\varphi}$ and $\bar{v}_1$ are found from equations (23) with s_0 replaced by ia_0.

The structural damping ratio, η_1, is taken to be zero for all numerical calculations in the two following subsections. Because of this, all of the energy dissipated will be by wave radiation into the elastic half-space. Also, calculations will be presented only for one value of Poisson's ratio ($\sigma = 1/4$) because similar results are expected for other values. For purposes of clarity, numerical evaluation of the steady-state response of the interaction system will be obtained first for the limiting case in which both b_h and b_m vanish, i.e., a system with negligible base mass and negligible total moment of inertia, I_t. Systems with b_h, b_m different from zero will be examined below.

TABLE 1

Resonant Frequencies and Amplication Factors of Single-Story Interaction System

($b_h = b_m = \eta_1 = 0$, $\sigma = 1/4$)

a_1	$\tilde{\omega}_1/\omega_1$		$\lvert\bar{v}_1\rvert/\bar{v}_g$		$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$		$\lvert\bar{v}_0\rvert/\bar{v}_g$	
(1)	Exact (2)	Approx. (3)	Exact (4)	Approx. (5)	Exact (6)	Approx. (7)	Exact (8)	Approx. (9)
.4	.937	.936	111.7	107.0	11.75	11.26	3.84	3.77
.5	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
.7	.835	.832	20.25	18.64	6.83	6.29	2.08	2.03
.9	.761	.755	9.70	8.79	5.54	5.02	1.62	1.59

(a) $b_1 = 1.0$, $\alpha_1 = 1.5$

b_1	$\tilde{\omega}_1/\omega_1$		$\lvert\bar{v}_1\rvert/\bar{v}_g$		$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$		$\lvert\bar{v}_0\rvert/\bar{v}_g$	
(1)	Exact (2)	Approx. (3)	Exact (4)	Approx. (5)	Exact (6)	Approx. (7)	Exact (8)	Approx. (9)
.5	.949	.949	105.9	99.25	8.83	8.33	2.79	2.75
1.0	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
1.5	.868	.866	39.64	37.59	9.90	9.38	3.19	3.11

(b) $a_1 = 0.5$, $\alpha_1 = 1.5$

α_1	$\tilde{\omega}_1/\omega_1$		$\lvert\bar{v}_1\rvert/\bar{v}_g$		$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$		$\lvert\bar{v}_0\rvert/\bar{v}_g$	
(1)	Exact (2)	Approx. (3)	Exact (4)	Approx. (5)	Exact (6)	Approx. (7)	Exact (8)	Approx. (9)
1.0	.943	.941	60.35	56.41	4.50	4.21	3.20	3.12
1.5	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
2.0	.861	.860	52.40	50.17	15.49	14.83	2.80	2.77

(c) $a_1 = 0.5$, $b_1 = 1.0$

1. Systems with negligible base masses. Having fixed the values of b_h, b_m, η_1, and σ, the frequency response of the system will depend solely on the frequency ratio a_1, the height parameter α_1 and the mass ratio b_1, defined by equations (24). Calculations have been carried out for several combinations of these parameters to assess their influence on the steady-state response of the system. Values used are: (1) $a_1 = 0.4, 0.5, 0.7, 0.9$; (2) $\alpha_1 = 1.0, 1.5, 2.0$; and (3) $b_1 = 0.5, 1.0, 1.5$. The values chosen are intended to approximate real structures. For instance, $a_1 = 0.7$ might correspond to a reinforced concrete nuclear reactor containment vessel of radius $a = 60$ ft and natural frequency $f_1 = 4$ cps, founded on a soil with a shear-wave velocity, V_s, of 2150 ft/sec.

The results of the calculations are presented in Figure 5 in the form of frequency-response curves. Three sets of curves are given, each illustrating the variation of an amplitude magnification factor, $|\bar{v}_1|/\bar{v}_g$, $|\varphi h_1|/\bar{v}_g$ or $|\bar{y}_0|/\bar{v}_g$, obtained from equations (23), as functions of the frequency ratio ω/ω_1. In Table 1, peak values of the amplification factors $|\bar{v}_1|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$, and $|\bar{v}_0|/\bar{v}_g$ and the resonant frequency ratio, $\tilde{\omega}_1/\omega_1$, are presented for other combinations of the parameters a_1, b_1 and α_1. Table 1 also includes approximate values of these results, calculated from

$$\frac{\tilde{\omega}_1}{\omega_1} = \frac{1}{\left[1+a_1{}^2 b_1\left(\dfrac{1}{\beta_h\sigma_h}+\dfrac{\alpha_1{}^2}{\beta_m\sigma_m}\right)\right]^{1/2}} \tag{26a}$$

$$\max\frac{|\bar{v}_1|}{\bar{v}_g} = \frac{\left[1+a_1{}^2 b_1\left(\dfrac{1}{\beta_h\sigma_h}+\dfrac{\alpha_1{}^2}{\beta_m\sigma_m}\right)\right]^{1/2}}{2\eta_1+a_1{}^3 b_1\left(\dfrac{\zeta_h}{\beta_h\sigma_h}+\dfrac{\zeta_m\alpha_1{}^2}{\beta_m\sigma_m}\right)} \tag{26b}$$

$$\max\frac{|\bar{\varphi}h_1|}{\dot{\bar{v}}_g} = \frac{a_1{}^2 b_1\alpha_1{}^2}{\beta_m\sigma_m}\max\frac{|\bar{v}_1|}{\bar{v}_g} \tag{26c}$$

$$\max\frac{|\bar{v}_0|}{\bar{v}_g} = \frac{a_1{}^2 b_1}{\beta_h\sigma_h}\max\frac{|\bar{v}_1|}{\bar{v}_g}. \tag{26d}$$

Equation (26a) approximates the natural frequency of the system under study by neglecting the effect of damping on $\tilde{\omega}_1$. Recalling that $b_h = b_m = 0$, equation (26a) was obtained by solving the frequency equation

$$[\Delta_e(ia_1\tilde{\omega}_1/\omega_1]_{b_h=b_m=0} = 0 \tag{27a}$$

in which Δ_e is defined by

$$\Delta_e(s_0) = [\Delta(s_0)]_{\eta_1=\zeta_h=\zeta_m=0}. \tag{27b}$$

The function $\Delta(s_0)$ is given by equation (23e). Successive approximations are required to obtain $\tilde{\omega}_1/\omega_1$ from equation (26a) because β_h and β_m (Figure 3) must be evaluated at $a_0 = \tilde{a}_1(\tilde{a}_1 = \tilde{\omega}_1 a/V_s)$.

With the resonant frequency established, the approximate peak values of the amplification factors $|\bar{v}_1|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$, and $|\bar{v}_0|/\bar{v}_g$ were obtained by evaluating equations (23) at $s_0 = i\tilde{a}_1$ and retaining only first-order terms in η_1, ζ_h, and ζ_m.

As shown in Table 1, estimates for the resonant frequency calculated from equation (26a) fall within 1 per cent of the values computed from equations (23), whereas an error of ten per cent or less is associated with estimates of peak values of response. This difference in accuracy is to be expected as the peak values are more sensitive than the resonant frequency to the damping coefficients η_1, ζ_h, and ζ_m, whose higher powers have been neglected in the derivation of equations (26).

Equations (26) provide relatively simple expressions that show explicitly the effects of the individual parameters on the resonance frequency and peak response. It is seen, for example, that the resonant frequency $\tilde{\omega}_1/\omega$ is always less than unity, decreasing monotonically with increasing values of α_1 and $a_1{}^2 b_1 = k_1/\mu a$.

Equations (26), along with Figure 5 and Table 1, show the effect of radiation damping in limiting the response. With η_1 equal to zero, the peak value of $|\bar{v}_1|/\bar{v}_g$ is inversely proportional to $a_1{}^3$ for small values of a_1 (hard soils), whereas it approaches zero as $1/a_1{}^2$ for a_1 large (soft soils). When η_1 is not zero, interaction does not necessarily decrease the response and the peak value of $|\bar{v}_1|/\bar{v}_g$ can be seen from equation (26b) to be smaller or larger than the value $(1/2\eta_1)$ for the system on rigid soil, depending on the values of a_1, b_1, and α_1.

The two amplification ratios $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_0|/\bar{v}_g$ must vanish and $|\bar{y}_0|/\bar{v}_g$ must equal unity for rigid soils ($V_s = \infty$, $a_1 = 0$). Equations (26c) and (26d) show, however, that when $\eta_1 = 0$, the peak values of $\bar{v}_0$ and $\bar{\varphi}h_1$ become proportional to $1/a_1$ for small values of a_1. Thus, as V_s approaches infinity and a_1 approaches zero, there are singularities in the response. The approach of the amplification ratios to their asymptotic values and the nature of the singularities are indicated by the curves in Figure 5. At the other extreme, when V_s becomes small and a_1 large, the system becomes, in effect, a rigid body vibrating on the elastic half-space. The corresponding resonant frequency and peak values of the amplification ratios $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_0|/\bar{v}_g$ can readily be obtained from equations (26).

The base displacement $|\bar{y}_0|/\bar{v}_g$ (Figure 5c) shows both a maximum and a minimum. It can be shown that the peak occurs near $\tilde{\omega}_1/\omega_1$ and the minimum near the frequency the system would have if it could rotate, but not translate, with respect to the free-field motion. The minimum will be zero, and will occur at ω_1, if the system can translate but is constrained against rotation. This behavior, characteristic of vibration absorbers, also occurs when the mass ratios b_h and b_m are nonzero (Bielak, 1971). The effect will diminish as η_1 increases from zero.

2. *Systems with base masses.* The response of a single-story building-foundation system with a base mass will differ somewhat from that of the same system with negligible values of b_h and b_m. For example, the system shown in Figure 4 will exhibit three distinct resonant frequencies in contrast to the single resonant frequency, $\tilde{\omega}_1$, observed when the base mass and total moment of inertia were neglected. It is clearly of practical interest to know how $\tilde{\omega}_1$ is modified and how the additional frequencies appear in the response as b_h and b_m increase from zero.

The resonant frequencies of the system shown in Figure 4 can be approximated by the zeros of

$$\Delta_e(ia_1\omega/\omega_1) = 0 \tag{28}$$

if it is assumed, as before, that the coefficients of damping η_1, ζ_h, and ζ_m may be neglected without affecting significantly the values of these frequencies. The function $\Delta_e(s_0)$ is defined by equations (27b) and (23e). A useful approximation for the fundamental frequency, $\tilde{\omega}_1$, is found by retaining only first-order terms in b_h and b_m,

$$\frac{\tilde{\omega}_1}{\omega_1} = \frac{1}{\left[1+a_1{}^2 b_1\left(\dfrac{1}{\beta_h\sigma_h}+\dfrac{\alpha_1{}^2}{\beta_m\sigma_m}\right)\right]^{1/2}}\left\{1-\frac{1}{2}\cdot\frac{a_1{}^4 b_1\left(\dfrac{b_h}{\beta_h{}^2\sigma_h{}^2}+\dfrac{b_m\alpha_1{}^2}{\beta_m{}^2\sigma_m{}^2}\right)}{\left[1+a_1{}^2 b_1\left(\dfrac{1}{\beta_h\sigma_h}+\dfrac{\alpha_1{}^2}{\beta_m\sigma_m}\right)\right]^2}\right\}. \tag{29}$$

Equation (29) shows that the fundamental resonant frequency of the system decreases as b_h and b_m increase from zero. This reduction may be small, however, even for relatively large values of b_h and b_m. For example, with $a_1 = 0.5, b_1 = 1.5, \alpha_1 = 1.5$, and $\sigma = 0.25$, $\tilde{\omega}_1/\omega_1$ is equal to 0.866 when b_h and b_m are both zero. A further reduction in $\tilde{\omega}_1/\omega_1$ of only 2.5 per cent occurs when $b_h = 3$ and $b_m = 4$, values which typically indicate a base mass twice as large as the top mass.

To study the remaining resonant frequencies, it is convenient to write the frequency equation (28) in the inverted form

$$b_h = \frac{\beta_h \sigma_h}{a_1{}^2(\tilde{\omega}/\omega_1)^2} \frac{[\Delta_e]_{b_h=0}}{[\Delta_e]_{b_h=0,1/\beta_h=0}}. \tag{30}$$

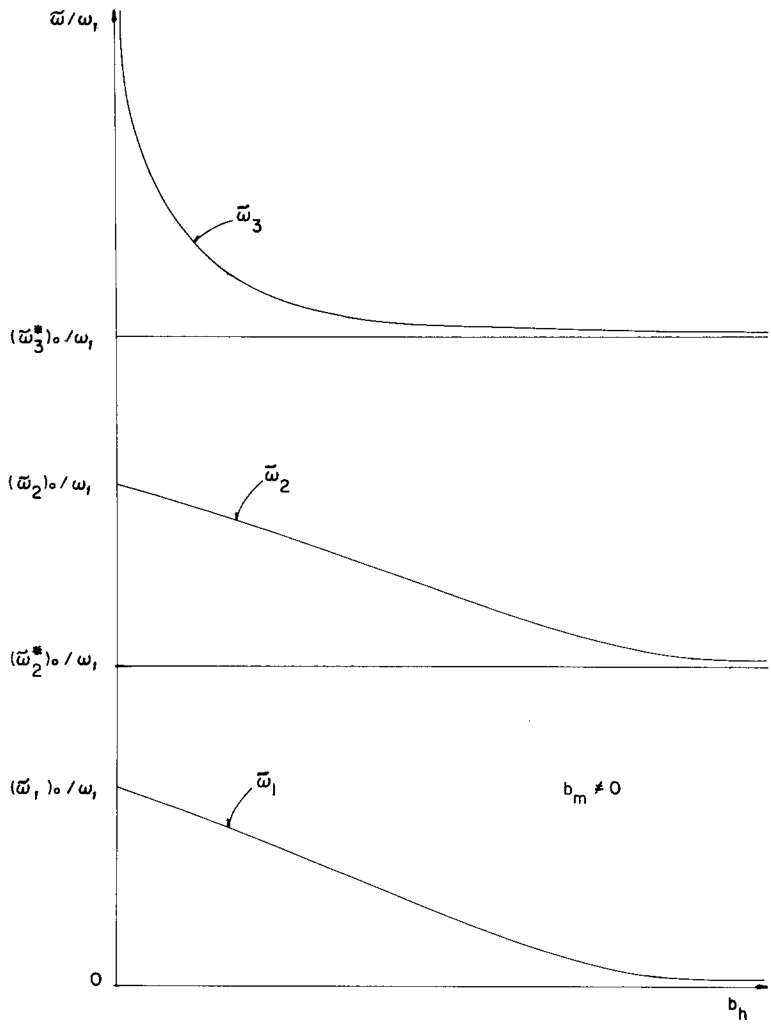


FIG. 6. Resonant frequencies of a single-story interaction system with base mass. Variation with b_h for fixed value of b_m.

Equation (30) defines three distinct branches, each describing the behavior of one resonant frequency. Figure 6 shows the variation in the frequencies with b_h for a fixed value of b_m. It is seen that the resonant frequencies decrease monotonically for increasing values of b_h, each becoming asymptotic to a horizontal line. The initial values of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are the roots of the frequency equation

$$[\Delta_e(ia_1\tilde{\omega}/\omega_1)]_{b_h=0} = 0 \tag{31a}$$

whereas the two nonzero asymptotes come from the roots of

$$[\Delta_e(ia_1\tilde{\omega}/\omega_1)]_{b_h=0,\ 1/\beta_h=0} = 0. \tag{31b}$$

The two roots of equation (31a) are the natural frequencies of a system for which b_h vanishes, whereas equation (31b) represents the frequency equation of a similar system, except that the base mass can only rotate.

It can be shown that equation (30) applies also for multistory building-foundation systems, provided equation (28) is interpreted as the frequency equation of the multistory system. This result can be used to show that all the $n+2$ resonant frequencies of an n-story building-foundation system decrease (or remain constant) for increasing values of b_m and b_h.

In addition to modifying the resonant frequencies of a single-story interaction system, the parameters b_m and b_h also affect the peak values of the response. This is illustrated in Figure 7, which gives the frequency response for several values of b_m of a particular building-foundation system with $b_h = 0$. In the figure, the amplification factors $|\bar{v}_1|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$, and $|\bar{y}_0|/\bar{v}_g$ obtained from equations (23) are again plotted in terms of the frequency ratio ω/ω_1.

Two distinct resonant frequencies can be recognized in Figure 7 for each nonzero value of b_m; the third resonant frequency is at infinity because b_h is zero in this example. For the values of b_m shown in the figure, the peak value of the response at the second resonant frequency is much smaller than that of the fundamental mode, indicating that a larger amount of effective damping is associated with the second mode of vibration than with the first.

It may be noted also that values of b_m up to about 3 do not affect significantly the maximum response of the fundamental mode nor the location of the fundamental frequency. These observations suggest that the influence of b_h and b_m on the response of the fundamental mode of a building-foundation system may be negligible over a significant range of these parameters. As indicated by equation (29), the range of validity of this simplification depends on the values of a_1, b_1 and α_1.

Earthquake Response

The response of the single-story interaction system to the earthquake motion $\ddot{v}_g(t)$ may be computed by inverting the Laplace transformations of the response. Thus, proceeding as with equations (3), expressions corresponding to (7d) for the displacements $v_0, y_0, \varphi h_1$ and v_1 may be obtained from equations (23) for given values of the parameters $a_1, b_1, \alpha_1, \eta_1, b_h, b_m$ and σ. The response of the system is then reduced to the sum of the responses of three viscously damped, one-degree-of-freedom linear oscillators resting on rigid ground.

It is not generally practicable to obtain formulas for the earthquake response in terms of the system parameters because of the difficulty in finding explicit solutions for the roots of the function Δ (equation 23e). It is, however, possible to solve explicitly for these roots when b_h and b_m vanish. The results of the previous section indicate that this may be an approach of practical interest, because the results may be used to estimate the response of systems for which b_h and b_m depart appreciably from zero.

When b_h and b_m are small, it seems reasonable to assume that the system has only one significant frequency; i.e., only one of the pairs of complex conjugate roots of the equation $\Delta = 0$ need be taken into consideration as the other two pairs of roots are associated with high frequencies and large amounts of damping, and, therefore, do not affect materially

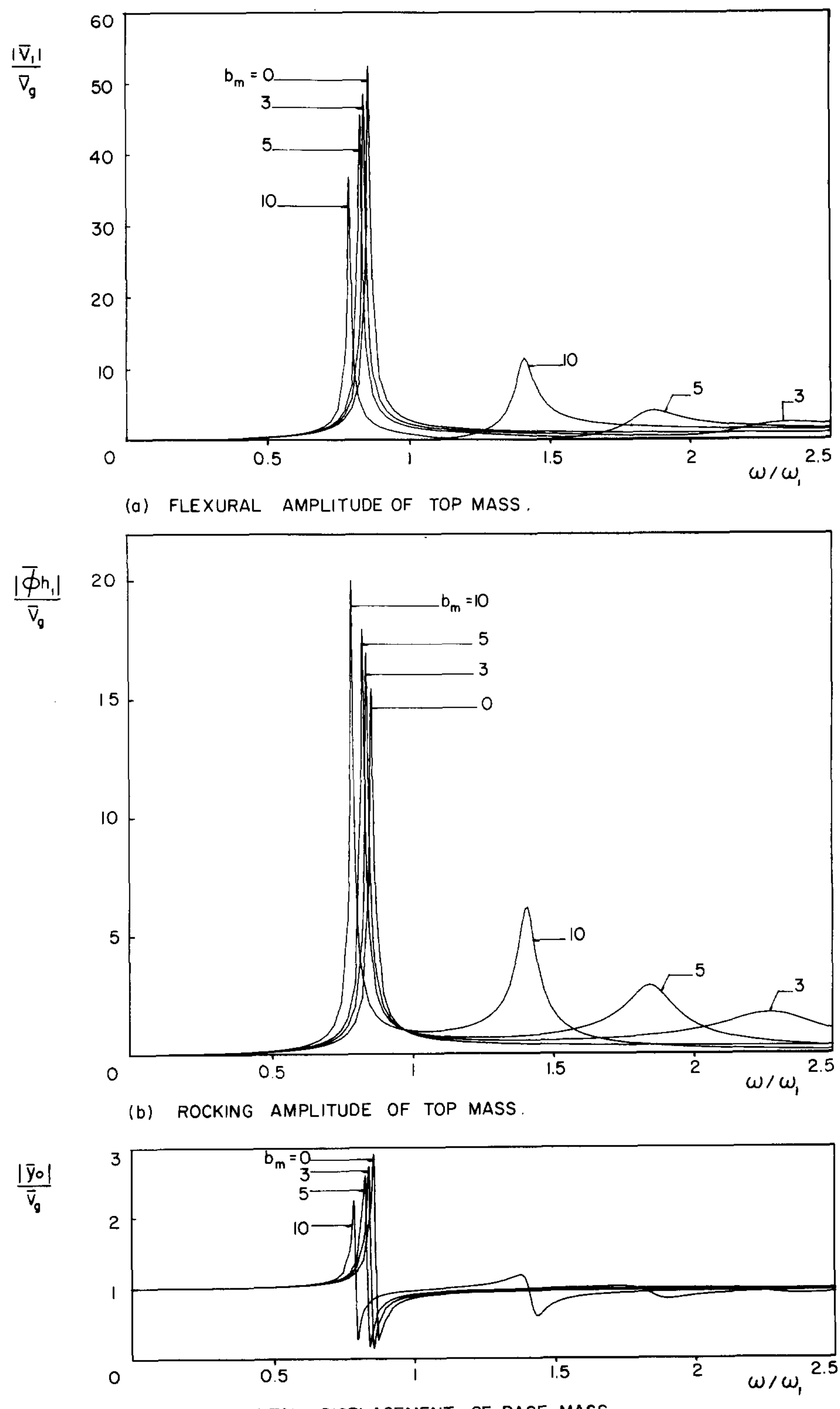


FIG. 7. Amplification ratios for the frequency response of a single-story interaction system. ($a_1 = 0.5$, $b_1 = 1, \alpha_1 = 2, \sigma = \frac{1}{4}, b_h = \eta_1 = 0$).

the response of the system. When b_h and b_m vanish, the significant root, s_1, can be obtained from equation (23e) by retaining only first-order terms in η_1, ζ_h, and ζ_m,

$$\frac{s_1}{\omega_1} = -\frac{\eta_1 + \dfrac{a_1{}^3 b_1}{2}\left(\dfrac{\zeta_h}{\beta_h \sigma_h} + \dfrac{\zeta_m \alpha_1{}^2}{\beta_m \sigma_m}\right)}{\left[1 + a_1{}^2 b_1\left(\dfrac{1}{\beta_h \sigma_h} + \dfrac{\alpha_1{}^2}{\beta_m \sigma_m}\right)\right]^2} + \frac{i}{\left[1 + a_1{}^2 b_1\left(\dfrac{1}{\beta_h \sigma_h} + \dfrac{\alpha_1{}^2}{\beta_m \sigma_m}\right)\right]^{1/2}}. \tag{33}$$

With the root s_1 determined, the impulse response functions of the system can be found from equations (6). The expressions for the displacements v_0, $h_1\varphi$, and v_1 may then be written in the form of equation (5a). Thus

$$\begin{Bmatrix} v_0(t) \\ h_1\varphi(t) \\ v_1(t) \end{Bmatrix} = -\frac{1}{\tilde{\omega}_1}\begin{Bmatrix} \dfrac{a_1{}^2 b_1}{\beta_h \sigma_h} \\ \dfrac{a_1{}^2 b_1 \alpha_1{}^2}{\beta_m \sigma_m} \\ 1 \end{Bmatrix} \int_0^t \exp\left[-\tilde{\eta}_1 \tilde{\omega}_1 (t-\tau)\right] \sin \tilde{\omega}_1 (t-\tau) \ddot{v}_g{}^e(\tau)\, d\tau \tag{34a}$$

where $\tilde{\omega}_1$ is again given by equation (26a), and

$$\tilde{\eta}_1 = \frac{\eta_1 + \dfrac{a_1{}^3 b_1}{2}\left(\dfrac{\zeta_h}{\beta_h \sigma_h} + \dfrac{\zeta_m \alpha_1{}^2}{\beta_m \sigma_m}\right)}{\left[1 + a_1{}^2 b_1\left(\dfrac{1}{\beta_h \sigma_h} + \dfrac{\alpha_1{}^2}{\beta_m \sigma_m}\right)\right]^{3/2}} \tag{34b}$$

$$\ddot{v}_g{}^e(t) = \left(\frac{\tilde{\omega}_1}{\omega_1}\right)^2 \ddot{v}_g(t). \tag{34c}$$

Equation (34a) implies that to first-order in $\tilde{\eta}_1$, the earthquake response of the single-story building-foundation system shown in Figure 4 is equivalent to the response of a one-degree-of-freedom damped oscillator resting upon rigid ground. This equivalent oscillator, defined by its undamped natural frequency $\tilde{\omega}_1$ (26a) and critical damping ratio $\tilde{\eta}_1$ (34b), is subjected to the effective acceleration $\ddot{v}_g{}^e(t)$ (34c), identical to $\ddot{v}_g(t)$ but scaled by the factor $(\tilde{\omega}_1/\omega_1)^2$. The deformation of the equivalent oscillator equals the relative story displacement, v_1, of the original system and the displacements $v_0(t)$ and $h_1\varphi(t)$ are related to $v_1(t)$ by

$$v_0(t) = \frac{a_1{}^2 b_1}{\beta_h \sigma_h} v_1(t) \tag{35a}$$

$$h_1\varphi(t) = \frac{a_1{}^2 b_1 \alpha_1{}^2}{\beta_m \sigma_m} v_1(t). \tag{35b}$$

One of the practical implications of the foregoing is that the earthquake response of single-story building-foundation systems may be obtained from standard response spectra. For example, to compute the maximum relative displacement, v_1, of such a system to a specified free-field earthquake motion, $\ddot{v}_g(t)$, it is only necessary to evaluate the modified natural frequency $\tilde{\omega}_1$ and the modified critical damping ratio $\tilde{\eta}_1$. The desired displacement is then obtained by multiplying the spectral displacement for this frequency and damping by the frequency ratio $(\tilde{\omega}_1/\omega_1)^2$. Rainer (1971) also has suggested this approach for calculating v_1 from examination of frequency response curves of interacting systems.

The equivalent oscillator technique can be used to obtain approximately the transient response of single-story interaction systems even in the absence of explicit formulas for $\tilde{\omega}_1$ and $\tilde{\eta}_1$, if these quantities may be evaluated, for example, from the steady-state harmonic response of the system. In this case, $\tilde{\omega}_1$ is the fundamental frequency of the system and $\tilde{\eta}_1$ is found by using the analogy between the interaction system and the equivalent single-degree-of-freedom oscillator.

$$\tilde{\eta}_1 \approx \frac{1}{2}\left(\frac{\tilde{\omega}_1}{\omega_1}\right)^2 \frac{1}{\max \dfrac{|\bar{v}_1|}{\bar{v}_g}} \tag{36}$$

where $\max |\bar{v}_1|/\bar{v}_g$ is the peak amplitude ratio of the interaction system under harmonic excitation. This approximate technique also may be used when b_h and b_m are large enough that their influence upon $\tilde{\omega}_1$ and $\tilde{\eta}_1$ must be taken into consideration, provided that the effect on the response of the frequencies $\tilde{\omega}_2$ and $\tilde{\omega}_3$, can still be neglected.

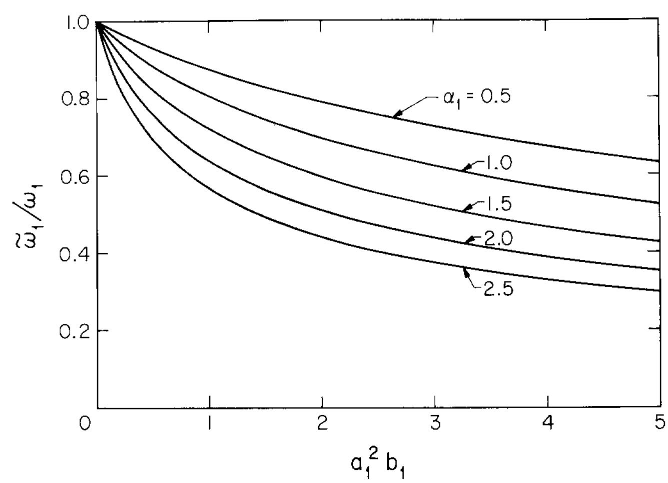


FIG. 8. Resonant frequency of single-story interaction system, equation (26a).

To illustrate the effect of the system parameters on the resonant frequency and fraction of critical damping of the single-story interaction system, equations (26a) and (34b) have been computed for several values of a_1, b_1, α_1 and η_1 with $\sigma = 0.25$. The results, given in Figures 8 and 9, show that a significant reduction in the resonant frequency of the system relative to the fixed-base frequency is obtained for slender buildings and for buildings which are stiff with respect to the soil. The effective damping in the system is seen always to increase for increasing values of a_1 and α_1 if there is no damping in the superstructure. If η_1 is not zero, however, the damping ratio $\tilde{\eta}_1$ can be less than or greater than η_1, depending on the values of the parameters η_1, a_1, α_1, b_1 and σ.

As a result of the changes in the natural frequency and ratio of critical damping of the interaction system, the earthquake response of the building on a flexible foundation will differ from that of the building on rigid ground. Whether there will be an increase or decrease of the response will depend upon the values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$, and upon the detailed time history of the particular earthquake under consideration. An estimate of the effect of soil-structure interaction on the average earthquake response of single-story systems can be obtained by using the average velocity spectra formula derived by Housner and

Jennings (1964). Let R_1 be the ratio of the average maximum value of the velocity, $\dot{v}_1$, of the interaction system to the average maximum velocity of the corresponding fixed-base structure. Hence, a reduction in the average response from the effects of interaction is indicated by $R_1 < 1$.

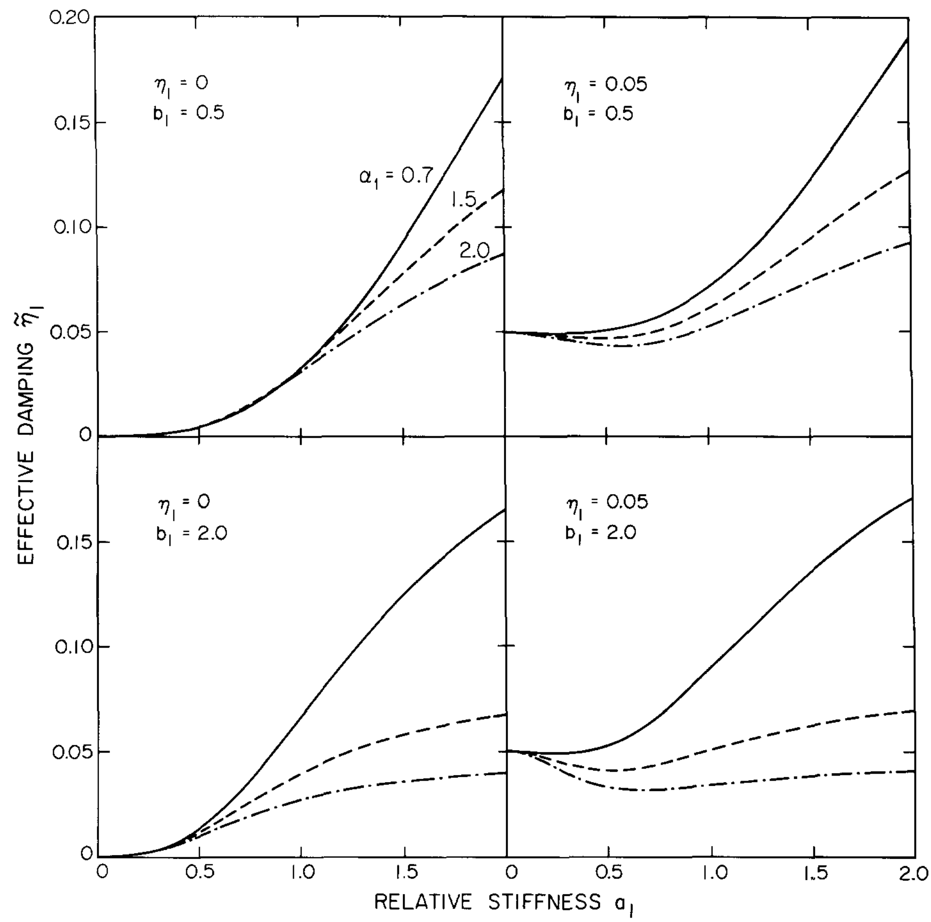


FIG. 9. Damping factor of single-degree-of-freedom oscillator equivalent to a single-story interaction system, equation (34b).

The ratio R_1 can be evaluated from equation (34),

$$R_1 = \frac{\left(\dfrac{\tilde{\omega}_1}{\omega_1}\right)^2 \langle S_v(\tilde{\omega}_1, \tilde{\eta}_1)\rangle}{\langle S_v(\omega_1, \eta_1)\rangle} \tag{37a}$$

where $\langle S_v(\omega, \eta)\rangle$ is the average of normalized velocity spectra of a simple oscillator with natural frequency ω and fraction of critical damping η. Housner and Jennings (1964) approximate $\langle S_v(\omega, \eta)\rangle$ by

$$\langle S_v(\omega, \eta)\rangle = 1.796\left[\frac{\pi G(\omega)}{2\eta\omega}(1-e^{-15.2\eta\omega})\right]^{1/2}. \tag{37b}$$

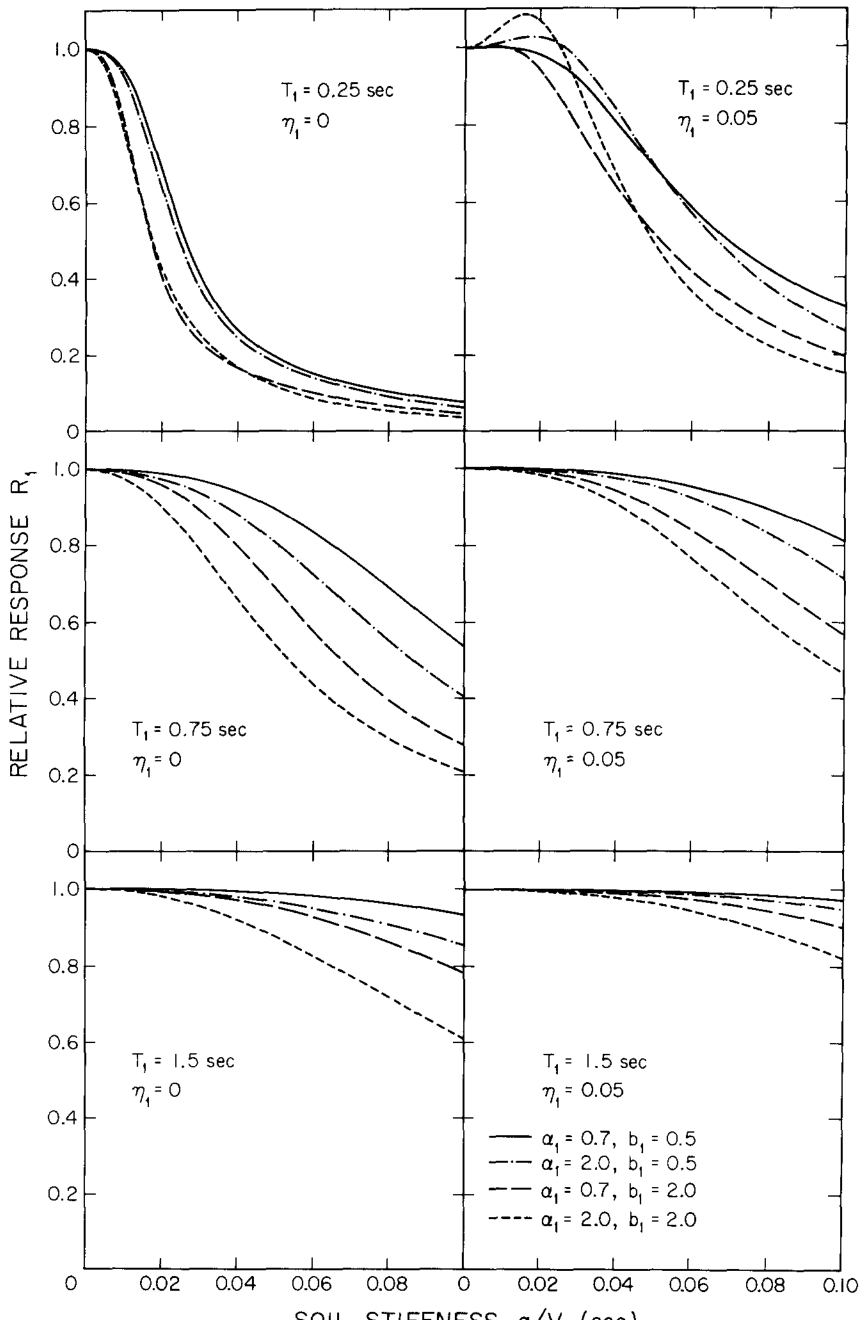


FIG. 10. Effects of interaction on earthquake response of single-story systems as measured by ratio of velocity spectra for elastic and rigid foundations.

In equation (37b) the power spectral density $G(\omega)$ is given by

$$G(\omega) = \frac{0.01238\left(1+\dfrac{\omega^2}{147.8}\right)}{\left(1-\dfrac{\omega^2}{242}\right)^2+\dfrac{\omega^2}{147.8}} \, . \tag{37c}$$

Equation (37a) has been evaluated as a function of the soil rigidity, measured by the parameter a/V_s for several values of the parameters η_1, α_1, b_1 and $T_1(T_1 = 2\pi/\omega_1)$ with $\sigma = 0.25$. The results, shown in Figure 10, indicate that interaction leads to a reduction in the average response if the superstructure is undamped. An increase in the average response, however, may occur for larger values of η_1. For a given soil, the effect of interaction decreases as the building becomes more flexible, and for a given structure, R_1 generally decreases as the soil becomes softer. Figure 10 also shows that increasing values of α_1 and b_1 tend to accentuate the effect of interaction, whereas R_1 becomes closer to unity as η_1 increases.

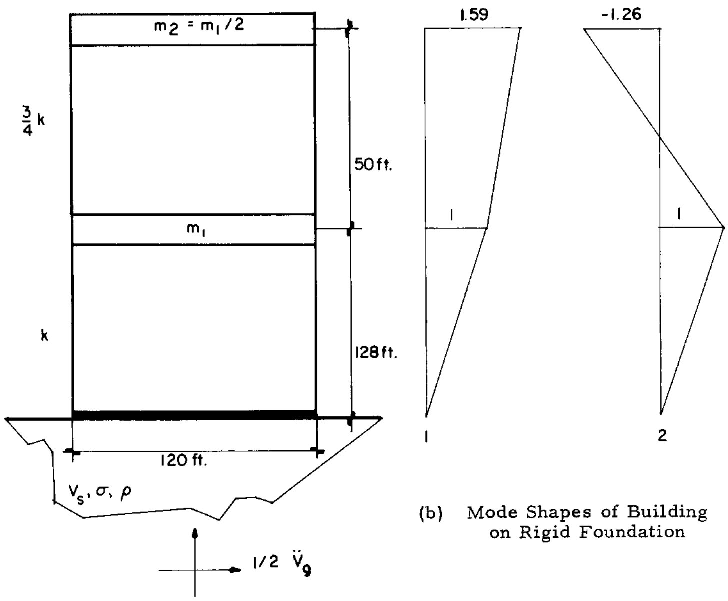
Two-Story Building-Foundation System

The earthquake response of the idealized two-story system shown in Figure 11 will be used to illustrate the application of the analysis to multistory building-foundation systems. The values of the system parameters have been selected to approximate a nuclear reactor containment vessel (Scavuzzo *et al.*, 1971). Qualitatively, the example is a rather extreme example of interaction.

Four mode solution. The response of the idealized system will first be obtained from equation (7) as a linear combination of the individual responses of four simple linear oscillators resting upon a rigid ground. The natural frequencies $\tilde{\omega}_l$ and critical damping ratios $\tilde{\eta}_l$ of the four equivalent oscillators are determined by equations (7a), (7b) and (6c). The results are presented in Table 2 for several values of the shear-wave velocity of the elastic medium. It is seen that the fundamental frequency, $\tilde{\omega}_1$, is reduced considerably as the soil becomes soft, but that $\tilde{\omega}_2$ remains almost constant for all values of V_s. The frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$, which arise from the introduction of rocking and relative lateral motion of the base, decrease monotonically from infinity for decreasing values of V_s. For softer soils, $\tilde{\omega}_3$ so defined becomes less than $\tilde{\omega}_2$. Table 2 also shows that the damping associated with the frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ is negligible for hard soils but increases as the soil becomes softer. In contrast with $\tilde{\eta}_1$ and $\tilde{\eta}_2$, the critical damping ratios $\tilde{\eta}_3$ and $\tilde{\eta}_4$ are always large, even for hard soils. It should be realized that these large values of damping do not necessarily imply large amounts of energy dissipation by these modes. The dissipated energy will be small if the modal amplitudes are small, even if the damping coefficients are large.

The participation factors a_{lk} and b_{lk} appearing in equation (7c) can be computed from equation (6b). With $\tilde{\omega}_l$, $\tilde{\eta}_l$ and the modified excitations determined, the earthquake response of the system may be obtained from equation (7d) by means of standard numerical techniques for evaluating the transient response of single-degree-of-freedom linear oscillators.

As an example, consider the system subjected to the free-field acceleration $\ddot{v}_g(t)$ depicted in Figure 12, the corrected version of the N33°E component of the earthquake motion (first event) recorded at the nuclear power plant at San Onofre, California on April 9, 1968 (Cloud and Hudson, 1968). In the calculations the shear-wave velocity of the soil is 1500 ft/sec, but for purposes of comparison, the response also was obtained for a rigid soil.



(a) Model

(b) Mode Shapes of Building on Rigid Foundation

$\omega_1 = 25.15$ rad/sec ($T_1 = 0.25$ sec)

$\omega_2 = 55.30$ rad/sec

$m_1 = .475 \times 10^6$ lb-sec^2/ft

$m_2 = .2375 \times 10^6$ lb-sec^2/ft

$m_0 = 1.07 \times 10^6$ lb-sec^2/ft

$I_t = 16.48 \times 10^8$ lb-sec^2/ft

$r = 60$ ft.

$\rho = 3.73$ lb-sec^2/ft^4 ($\gamma_s = 120$ lb/ft^3)

$\sigma = 1/4$

Undamped Building

(c) Values of the Parameters

FIG. 11. Idealized two-story building-foundation system approximating a nuclear reactor containment vessel.

TABLE 2

RESONANT FREQUENCIES AND CRITICAL DAMPING RATIOS OF TWO-STORY SYSTEM

V_s (ft/sec)	$\tilde{\omega}_l$ (rad/sec)				$\tilde{\eta}_l$ (%)			
	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 1$	$l = 2$	$l = 3$	$l = 4$
800	8.87	54.84	23.86	87.73	4.26	0.88	58.69	16.14
1,200	12.47	54.86	35.86	90.79	3.27	1.05	58.74	23.55
1,500	14.70	54.86	44.93	93.66	2.57	1.06	58.86	27.61
2,700	20.12	54.86	81.29	117.30	0.87	0.53	59.93	34.47
∞	25.15	55.30	∞	∞	0	0	—	—

Figure 13 gives time histories for $v_1(t)$, $v_2(t)$, $h_1\varphi(t)$ and $v_0(t)$. To study the effect of the terms associated with the frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$, two families of curves have been included in Figure 13, one obtained by omitting the terms on the right-hand side of equation (7d) which contain the frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$, and the other which includes all four terms. As the figure shows, the difference is only important for the horizontal displacement of the base. Figure 13 also shows that the displacement of the first story caused by rocking is nearly twice as large as the corresponding flexural displacement, and that the displacements are determined primarily by the fundamental mode of the system.

The corresponding relative displacements $v_1(t)$ and $v_2(t)$ for the structure founded on rigid soil are shown in Figure 14. The structure is undamped in this condition. Comparing Figures 13 and 14, it may be seen that the deformable foundation reduces both the dominant frequency of vibration and the maximum amplitude of the flexural displacements. Furthermore, the displacements $v_0(t)$ and $h_1\varphi(t)$, which are significant for the flexible foundation, vanish identically for a rigid soil.

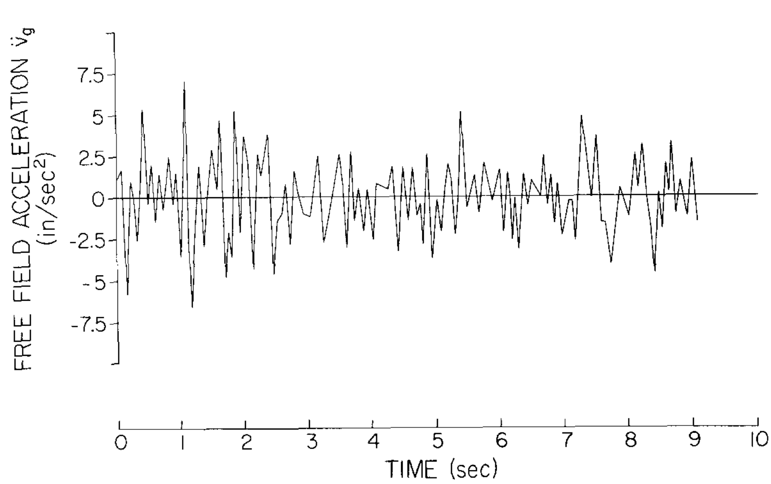


FIG. 12. N33°E component of ground motion recorded at the nuclear power plant at San Onofre, California on April 9, 1968.

Figure 15 shows the lateral acceleration of the base of the building, $\ddot{y}_0$, obtained by adding $\ddot{v}_0$ and $\ddot{v}_g$. The difference between the base acceleration in Figure 15 and the free-field acceleration (Figure 12) is caused by the dynamic coupling between the building and the surrounding soil.

Being in buildings, the majority of strong-motion accelerometers record the motion of the base of the structure rather than the free-field motion. Thus, some records may not portray the free-field motion but might, in some instances, be influenced by the properties of the building in which they were obtained. Any significant effects of this nature would be best measured by differences in response spectra of single-degree-of-freedom oscillators. It is, therefore, of interest to compute response spectra for the base motion $\ddot{y}_0$ (Figure 15) and to compare them with the spectra obtained from $\ddot{v}_g$ (Figure 12). Two pairs of such velocity spectra are shown in Figure 16, one for undamped oscillators and the other for 5 per cent damping. Even though $\ddot{y}_0(t)$ and $\ddot{v}_g(t)$ appear quite different in this somewhat severe example of interaction, significant differences in the corresponding spectra occur mainly in the neighborhood of the resonant frequencies of the two-story building-foundation system and, in general, the spectra are quite similar. This similarity may not be general, however, and this point needs further investigation.

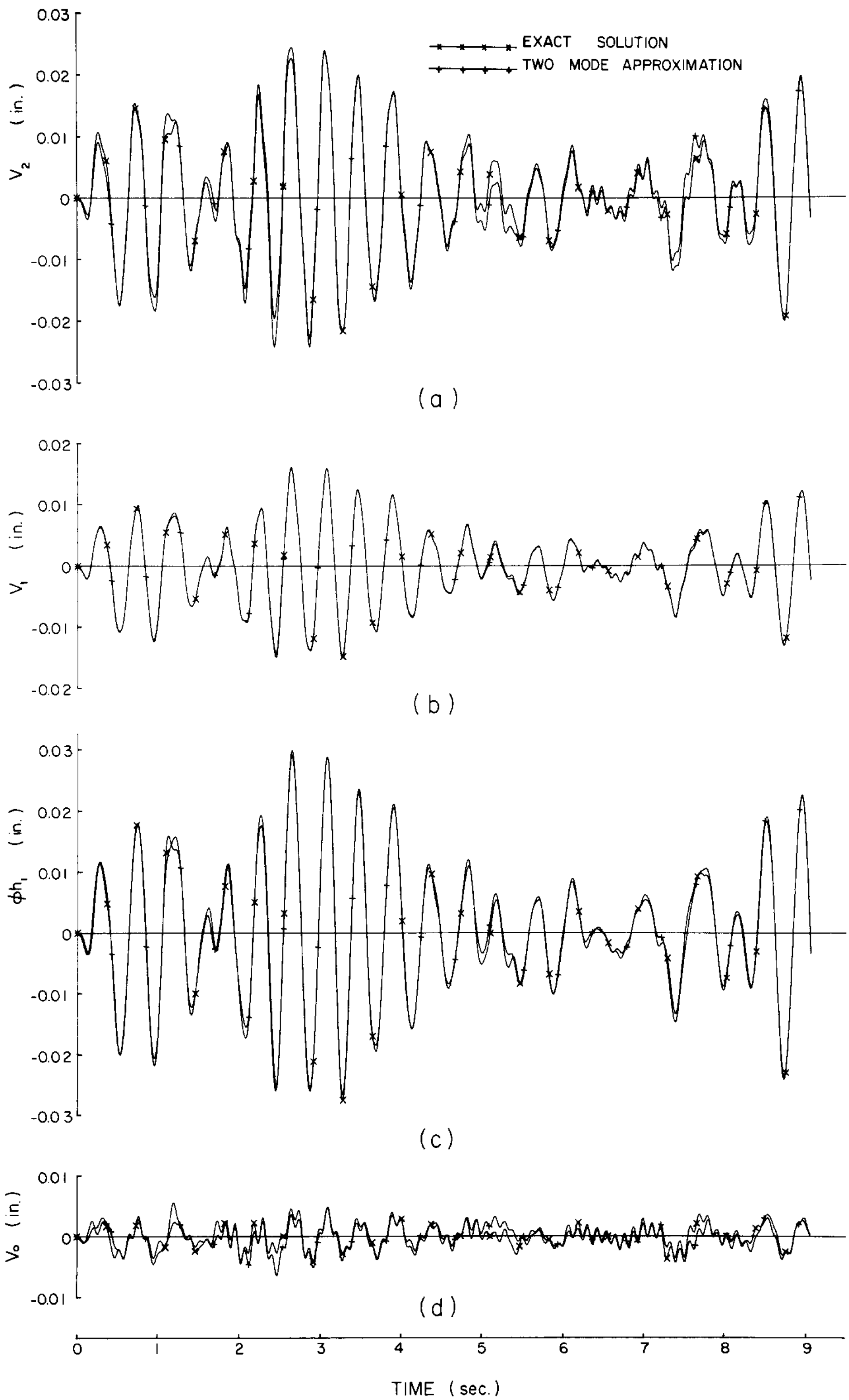


FIG. 13. Response of the two-story interaction system to the San Onofre accelerogram, including complete (4 mode) and approximate (2 mode) solutions.

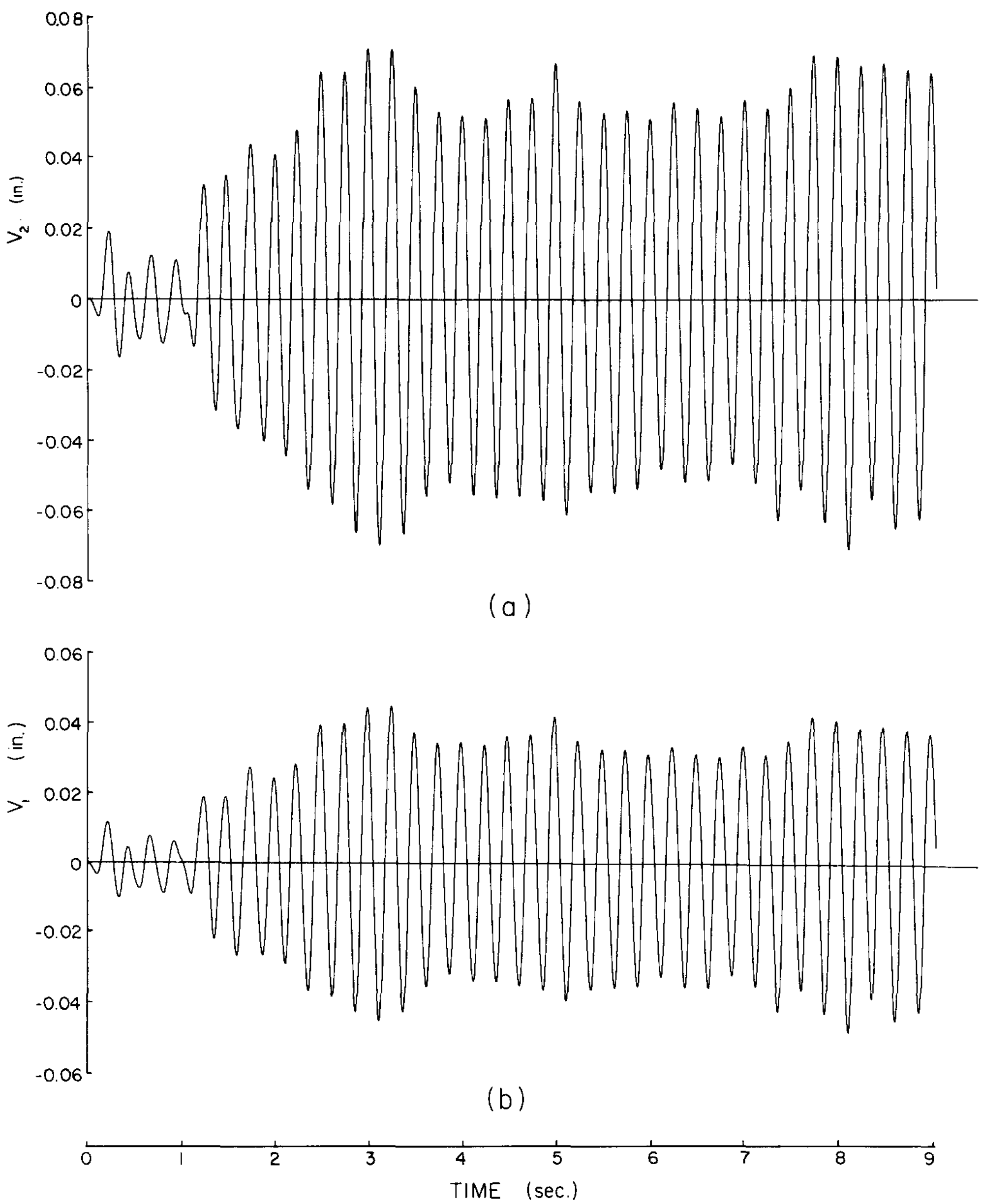


FIG. 14. Response of the two-story interaction system with rigid soil to the San Onofre accelerogram.

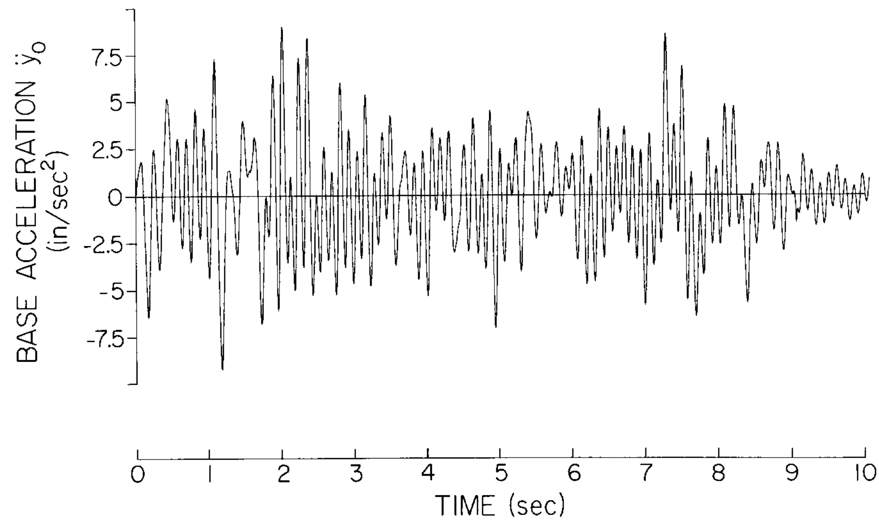


FIG. 15. Lateral acceleration of the base of the two-story interaction system. To be compared with the free-field motion in Figure 12.

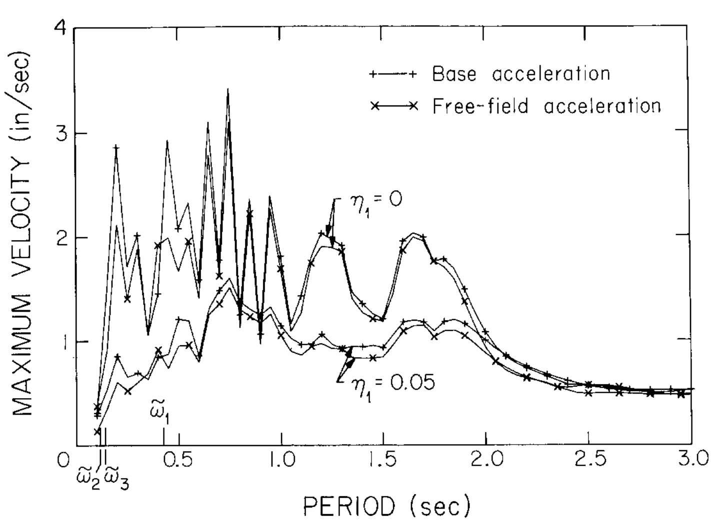


FIG. 16. Comparison of velocity spectra of free-field acceleration and base acceleration for the two-story interaction system.

Approximate Properties of Fundamental Modes

The fundamental mode dominates many important response variables and is a major contributor to nearly every response parameter of interest, especially for tall buildings (Jennings, 1969). In addition, the effects of interaction are most pronounced in the fundamental mode, as was the case, for example, in the two-story structure examined above. Therefore, it is useful to know approximately what the interaction effects will be for the fundamental mode, without simultaneously evaluating the effects on all modes of response. In many instances, an analysis that considers interaction effects only on the fundamental mode may be sufficiently accurate for purposes of design.

Approximate values of the fundamental frequency $\tilde{\omega}_1$, and damping factor $\tilde{\eta}_1$ can be obtained from equations (26a) and (34b), provided the parameters b_1 and α_1 are defined by

$$b_1 = \frac{M_1}{\rho a^3} \tag{38a}$$

$$\alpha_1 = \frac{H_1}{a}, \tag{38b}$$

with a_1 again given by equation (24a). Expressions for the modal parameters M_1 and H_1, in terms of the fixed-base fundamental mode shape, are included in the Appendix. It is noted again that β_h and β_m in equations (26a) and (34b) are to be evaluated at $a_0 = \tilde{\omega}_1 a/V_s$.

For a first approximation of the fundamental mode response, the coefficients $a_{1k}(k = 0, \varphi, j; j = 1, \ldots, n)$ in equation (7c) can be set equal to zero, as they would vanish identically if the structure possessed classical normal modes. The coefficients b_{1k} in equation (7c) can be determined from the first mode, X_{j1}, of the building on a rigid foundation. By comparing equations (7) with the approximate solution, equations (34), for the ingle-story building-foundation system, it can be found that

$$\begin{Bmatrix} b_{10} \\ \\ b_{1\varphi} \\ \\ b_{1j} \end{Bmatrix} = \frac{\tilde{\omega}_1}{\omega_1^{\,2}} \cdot \frac{\sum_i m_i X_{i1}}{\sum_i m_i X_{i1}^2} \begin{Bmatrix} \dfrac{a_1^{\,2} b_1}{\beta_h \sigma_h} X_{11} \\ \\ \dfrac{a_1^{\,2} b_1 \alpha_1^{\,2}}{\beta_m \sigma_m H_1} X_{11} \\ X_{j1} \end{Bmatrix}. \tag{39}$$

With $\tilde{\omega}_1$, $\tilde{\eta}_1$, a_{1k} and b_{1k} determined, the approximate response of the fundamental mode can be calculated from equations (7c) and the first term of (7d).

To verify the accuracy of the approximate formulas, the fundamental frequency and the corresponding damping ratio have been calculated for the two-story building-foundation system considered above. The exact values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$, reproduced from Table 2, are given in Table 3 together with the approximate values calculated from equations (26a), (34b) and (38). Good agreement is obtained between the exact and the approximate values for the sample structure in spite of the fact that equations (26a) and (34b) were derived for structures with no base masses. The two-story system has a base mass which is larger than the sum of the two top masses.

TABLE 3

FUNDAMENTAL RESONANT FREQUENCY AND CRITICAL DAMPING RATIO OF TWO-STORY SYSTEM

V_s (ft/sec)	$\tilde{\omega}_1$ (rad/sec)		$\tilde{\eta}_1$ (%)	
	Exact	Approximate	Exact	Approximate
800	8.87	9.16	4.26	4.47
1,200	12.47	12.77	3.27	3.33
1,500	14.70	14.97	2.57	2.57
2,700	20.12	20.23	0.87	0.84
10,000	24.68	24.68	0.02	0.02
∞	25.15	25.15	0	0

The exact (6b) and the approximate (39) values of the participation factors a_{1k} and b_{1k} are given in Table 4 for the two-story example, with $V_s = 1500$ ft/sec. It is seen that the exact coefficients a_{1k} are small compared to the corresponding coefficients b_{1k}, and, hence, it is justified to neglect them completely, with the possible exception of a_{10}. The error in the approximations for b_{11} and b_{12} is less than 10 per cent, but the error in b_{10} and $b_{1\varphi}$ is about 20 per cent. The comparison suggests that the approximate approach will yield good results for the effects of interaction on the relative deflections of the superstructure in the fundamental mode, with a less satisfactory approximation for the movement of the base mass. Even this result, however, should indicate correctly the magnitude of the interaction effect.

TABLE 4

COMPARISON OF PARTICIPATION FACTORS FOR FUNDAMENTAL MODE RESPONSE

Values	a_{10}	$a_{1\varphi}$	a_{11}	a_{12}	b_{10}	$b_{1\varphi}$	b_{11}	b_{12}
Exact	0.00186	0.00000665	−0.00039	−0.00066	0.00464	0.000289	0.0200	0.0311
Approximate	0	0	0	0	0.00354	0.000235	0.0188	0.0298

MULTISTORY BUILDING-FOUNDATION SYSTEMS

In this section the resonant frequencies of an idealized ten-story building resting upon an elastic half-space are calculated and general multistory systems are discussed briefly.

The natural frequencies and mode shapes of the ten-story building are the values calculated by Housner and Brady (1963) for case 10c, a two-bay steel frame with infinitely rigid floor girders, a story height of 12 ft, a bay width of 20 ft, tributary floor area of 40 by 40 ft, and a lumped weight per floor of 160,000 lb. The fundamental mode of the rigidly founded building is illustrated in Figure 17. To conform with the present analysis, it is assumed that the ten-story building has a circular, massless base with the same area as that of the actual structure. The centroidal moments of inertia of the floors are neglected compared to those about the base of the building, that is, I_t is taken to be zero. Under these assumptions the system possesses ten natural frequencies.

The foundation medium is taken to have a unit weight of 120 lb/ft^3 and a Poisson's ratio of 0.25. Several shear-wave velocities are considered, ranging from 300 ft/sec to the limiting rigid condition.

With the system specified, the resonant frequencies and critical damping ratios can be obtained from the roots of the frequency function, equation (A5) in the appendix. The resonant frequencies have been calculated for rocking and horizontal translation of the base, and for rocking only. Results for the fundamental frequency are presented in Table 5 for several values of V_s. Also shown in Table 5 are approximate values of the fundamental frequency calculated from equation (26a). To use this equation for multistory buildings, the parameters a_1, b_1 and α_1 have been defined by equations (24a), (38a) and (38b), respectively. Table 5 indicates excellent agreement between the exact and approximate values of the fundamental frequency of the system.

Table 5 shows that for soft soil an appreciable reduction in the fundamental frequency takes place when rocking of the base is permitted. The additional reduction resulting from inclusion of horizontal translation of the base is small, however. This result is also indicated by equation (26a). Used as an approximation for the fundamental frequency of the system, equation (26a) shows that the effect of translation of the base will not be

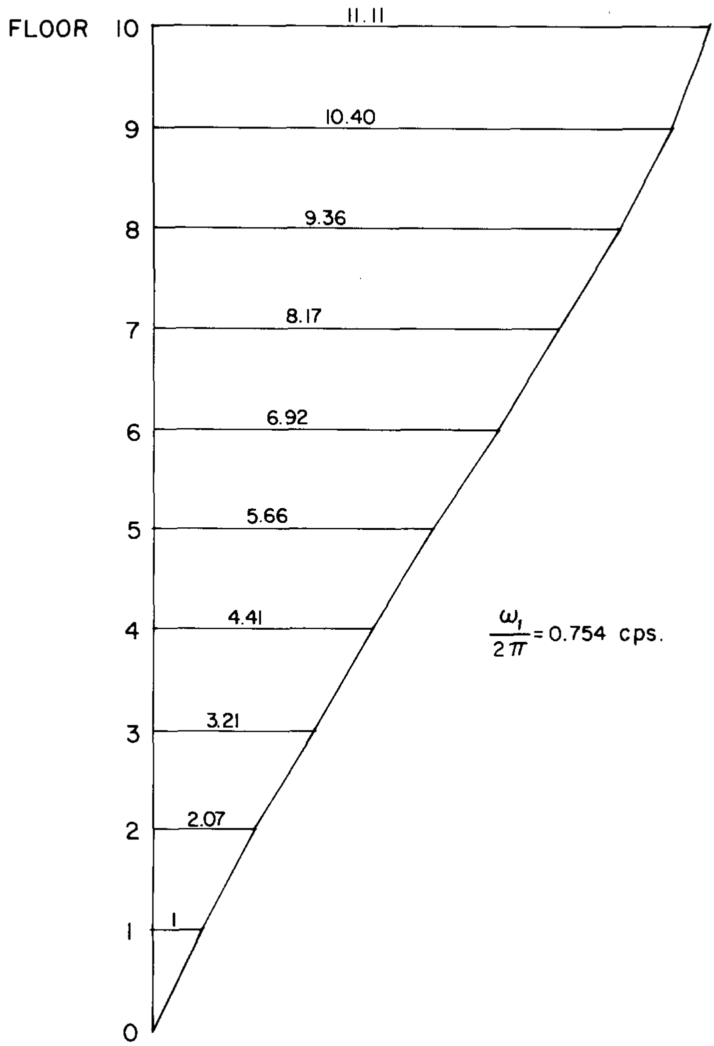


FIG. 17. Fundamental fixed-based mode of a ten-story building.

TABLE 5

FUNDAMENTAL RESONANT FREQUENCY OF TEN-STORY BUILDING ON THE ELASTIC HALF-SPACE

V_s (ft/sec)	Rocking and Hor. Transl. (Exact) (cps)	Rocking Only (Exact) (cps)	Rocking and Hor. Transl. (Approximate) (cps)
∞	0.754	0.754	0.754
5,000	0.753	0.753	0.753
2,000	0.750	0.750	0.750
1,500	0.747	0.747	0.747
1,000	0.738	0.739	0.738
700	0.722	0.724	0.723
500	0.695	0.698	0.696
400	0.667	0.671	0.668
300	0.615	0.620	0.618

significant for systems in which $(H_1/a)^2$ is large. Thus, for sufficiently tall buildings, the effect of translation on the fundamental frequency will be negligible compared to that of rocking.

The tendency for softer soils to reduce the resonant frequency was not observed for the higher modes. For all values of V_s, the second and higher resonant frequencies did not differ by more than 1 per cent from the corresponding frequencies of the rigidly founded building. As the second resonant frequency of the two-story example studied in the preceding section also was essentially independent of the stiffness of the soil, it becomes of interest to determine whether more general systems show this behavior.

Vibration tests have shown that many tall buildings have fundamental modes resembling straight lines (Jennings, 1969). If the superstructure possesses classical normal modes with the fundamental mode shape given by a straight line, and if only rocking of the base is permitted, then it has been shown that the contributions of the second and higher modes to the overturning moment at the base of the building vanish identically (Bielak, 1969). Because only the fundamental mode has a nonvanishing base moment, and, therefore, a tendency to rotate, the second and higher natural frequencies will remain unchanged regardless of the stiffness of the soil. That the rotation of the base does not occur in the higher modes under these circumstances also has been observed by Tajimi (1967).

In general, the inclusion of rocking and the centroidal moments of inertia of the base and upper masses introduces an additional degree of freedom and natural frequency into the system. For the case of a straight line fundamental mode, this frequency is associated with an out-of-phase combination of rocking and interstory deflection. This frequency is typically higher than the range spanned by the other natural frequencies and is associated with high values of effective damping. Therefore, it does not contribute significantly to the earthquake response.

These results suggest that the effect of an elastic foundation, as measured by changes in the natural frequencies, dampings, and mode shapes of a structure as the underlying soil becomes softer, is often negligible for modes higher than the first for many types of structures, including tall buildings. However, the fundamental frequency of a structure can decrease significantly as the soil becomes softer and, except for short buildings, the reduction in the fundamental frequency is primarily caused by rocking rather than by horizontal translation of the base. There will also be attendant changes in the mode shape and damping.

Conclusions

A method has been presented for calculating the earthquake response of multistory structures founded on an elastic half-space. Under the assumption that an n-story structure-foundation system has, in general, $n+2$ significant natural frequencies, it was shown that the response can be written as the sum of the responses to modified excitation of $n+2$ one-degree-of-freedom, viscously damped, linear oscillators resting on rigid ground. The result holds even though the system does not necessarily possess classical normal modes and has the advantage of simplifying the calculations and giving physical insight into the dynamics of building-foundation systems.

For the case of a single-story building on a flexible foundation, approximate formulas were obtained for the natural frequency, $\tilde{\omega}_1$, critical damping ratio, $\tilde{\eta}_1$, and the amplitude of the modified excitation. It was found from these results that the fundamental natural frequency of the single-story building, as well as the amplitude of the equivalent input acceleration, always decreases as a result of the dynamic coupling between the building and the soil, but that the effective damping in the system can be increased or decreased by soil-structure interaction, depending on the parameters of the system. When these effects were combined into estimates of the average earthquake response, it was found that the average maximum response of an undamped one-story building was always decreased by the effects of interaction. For a damped structure, however, either a decrease or an increase in response was obtained, depending on the parameters of the system. Thus, the effects of interaction are not always conservative.

The results for the one-story structure showed also that the effect of the base mass on the dynamic properties (Figure 4) are small and that simple approximations for the natural frequency and damping based on a negligible base mass may suffice for many purposes. These approximate results, equations (26a) and (34) and Figures 8 through 10, are suitable for use in estimating the earthquake response of single-degree-of-freedom interacting systems, such as large electrical transmission equipment on separate footings or similarly founded mechanical equipment, as well as one-story building structures. The soil properties required for such applications are the shear-wave velocity, V_s, and Poisson's ratio, σ, which the analysis assumes to be constant. For approximate application of the results to sites with soil properties varying with depth, it is recommended that average properties within about three foundation diameters (or equivalent) be used.

The effect of soil-structure interaction on the earthquake response of multistory structures seems to occur predominantly in the fundamental mode and for many structures, such as tall buildings, the effects of interaction may be negligible for higher modes if the fundamental mode of the fixed-base structure is approximately a straight line. Although all of the resonant frequencies decrease as a result of interaction, typically, only the fundamental frequency decreases significantly as the soil becomes softer. Except for short structures, the reduction is due essentially to rocking of the structure, rather than to translation of the base mass. Approximate formulas for the effect of interaction on the fundamental mode response were developed in the text.

Many approximate calculations of earthquake response of interacting systems are based on mathematical models in which springs and dashpots are used to model the elastic foundation of the structure. To model an elastic half-space exactly, the springs and dashpots must be frequency-dependent, which is awkward for calculations. The simplification that is usually made is to replace the frequency dependence by the static (zero frequency) values, average values, or some other representative values. The results of the present study indicate that the most representative frequency for determining approximate, constant values of these discrete elements would be the fundamental frequency of the interacting system.

Acknowledgment

A large part of the results presented in this paper are from the second author's recent Doctoral Thesis (Bielak, 1971). The financial support of the National Science Foundation during the thesis work is gratefully acknowledged.

Appendix

Transfer functions for the response of the building-foundation system (equations, 3), (Bielak, 1971).

$$\begin{aligned}\Delta_0(s) = & -[s^2 \sum_{\substack{j,k \\ j\neq k}} (M_j I_k - Z_j Z_k)\hat{F}_{jk}(s) + (s^2 I_t + \mu a^3 K_{mm})(\Sigma_j \hat{F}_j(s) M_j) \\ & + s^2 m_0(\Sigma_j \hat{F}_j(s) I_j) - \mu a^2 K_{hm}(\sum_j \hat{F}_j(s) Z_j) \\ & + m_0(s^2 I_t + \mu a^3 K_{mm}) \prod_{k=1}^{n} (s^2 + 2\eta_k \omega_k s + \omega_k{}^2)]\end{aligned} \quad \text{(A1)}$$

$$\Delta_\varphi(s) = -\{\mu a K_{hh}(\Sigma_j \hat{F}_j(s) Z_j) - \mu a^2 K_{hm}[(\Sigma_j \hat{F}_j(s) M_j) + m_0 \prod_{k=1}^{n} (s^2 + 2\eta_k \omega_k s + \omega_k{}^2)]\} \tag{A2}$$

$$\Delta_j(s) = -\sum_l \xi_l(s) X_{jl}; \quad j = 1, 2, \ldots, n \tag{A3}$$

in which

$$\begin{aligned} \xi_l(s) = & -\mu a^2 s^2 K_{hm} \sum_{\substack{k \\ k \neq l}} (\psi_l Z_k - \gamma_l M_k) \hat{\Pi}_{kl}(s) \\ & + \mu a s^2 K_{hh} \sum_{\substack{k \\ k \neq l}} (\psi_l I_k - \gamma_l Z_k) \hat{\Pi}_{kl}(s) + (\mu a s^2 K_{hh} I_t \\ & + \mu^2 a^4 K_{hh} K_{mm}) \hat{\Pi}_l(s) \psi_l + \mu a^2 s^2 m_0 K_{hm} \hat{\Pi}_l(s) \gamma_l \end{aligned} \tag{A4}$$

$$\begin{aligned} \Delta = & s^4 \sum_{\substack{j,k \\ j \neq k}} (M_j I_k - Z_j Z_k) \hat{F}_{jk}(s) + s^2(s^2 I_t + \mu a^3 K_{mm})(\Sigma_j \hat{F}_j(s) M_j) \\ & + s^2(s^2 m_0 + \mu a K_{hh})(\Sigma_j \hat{F}_j(s) I_j) - 2s^2 \mu a^2 K_{hm}(\Sigma_j \hat{F}_j(s) Z_j) \\ & + \prod_{k=1}^{n} (s^2 + 2\eta_k \omega_k s + \omega_k{}^2)[s^4 m_0 I_t + \mu a^3 s^2 m_0 K_{mm} \\ & + \mu a s^2 K_{hh} I_t + \mu^2 a^4 (K_{hh} K_{mm} + K_{hm}^2)]. \end{aligned} \tag{A5}$$

In these equations, X_{jk} = jth component of $\mathbf{X}_k$ = modal displacement of the jth mass in the kth mode of the superstructure, if it were supported on a rigid foundation.

$$\psi_k = \frac{\mathbf{X}_k{}^T M\{\mathbf{1}\}}{\mathbf{X}_k{}^T M \mathbf{X}_k}; \tag{A6}$$

$$\gamma_k = \frac{\mathbf{X}_k{}^T M \mathbf{h}}{\mathbf{X}_k{}^T M \mathbf{X}_k}; \qquad \mathbf{h} = \{h_j\} \tag{A7}$$

ω_k = undamped natural frequency of the kth mode of the superstructure, given by

$$\omega_k{}^2 = \frac{\mathbf{X}_k{}^T K \mathbf{X}_k}{\mathbf{X}_k{}^T M \mathbf{X}_k} \tag{A8}$$

η_k = critical damping ratio of the kth mode of the superstructure, defined by

$$2\eta_k \omega_k = \frac{\mathbf{X}_k{}^T C \mathbf{X}_k}{\mathbf{X}_k{}^T M \mathbf{X}_k} \tag{A9}$$

$$M_j = \frac{(\Sigma_i m_i X_{ij})^2}{\Sigma_i m_i X_{ij}^2} \tag{A10}$$

$$Z_j = M_j H_j = \frac{(\Sigma_i m_i X_{ij})(\Sigma_i m_i h_i X_{ij})}{\Sigma_i m_i X_{ij}^2} \tag{A11}$$

$$I_j = \frac{(\Sigma_i m_i h_i X_{ij})^2}{\Sigma_i m_i X_{ij}^2}. \tag{A12}$$

The functions $\hat{F}_j(s)$, $\hat{F}_{jk}(s)$, $\hat{\Pi}_l(s)$ and $\hat{\Pi}_{kl}(s)$ are defined, respectively, by

$$\hat{F}_j(s) = (\omega_j^2 + 2\eta_j\omega_j s) \prod_{\substack{k=1\\k\neq j}}^{n} (s^2 + 2\eta_k\omega_k s + \omega_k^2) \tag{A13}$$

$$\hat{F}_{jk}(s) = (\omega_j^2 + 2\eta_j\omega_j s)(\omega_k^2 + 2\eta_k\omega_k s) \prod_{\substack{l=1\\l\neq j, l\neq k}}^{n} (s^2 + 2\eta_l\omega_l s + \omega_l^2) \tag{A14}$$

$$\hat{\Pi}_{kl}(s) = (\omega_k^2 + 2\eta_k\omega_k s) \prod_{\substack{m=1\\m\neq l, m\neq k}}^{n} (s^2 + 2\eta_m\omega_m s + \omega_m^2) \tag{A15}$$

$$\hat{\Pi}_l(s) = \prod_{\substack{m=1\\m\neq l}}^{n} (s^2 + 2\eta_m\omega_m s + \omega_m^2). \tag{A16}$$

References

Bergland, G. D. (1969). A guided tour of the Fast Fourier Transform, *IEEE Spectrum* **6,** 41–52.

Bielak, J. (1969). Base moment for a class of linear systems, *J. Eng. Mech. Div., ASCE* **95,** 1053–1062.

Bielak, J. (1971). Earthquake Response of Building-Foundation Systems, *Earthquake Engineering Research Laboratory Report EERL* 71–04, California Institute of Technology.

Bycroft, G. N. (1956). Forced vibrations of a rigid circular plate on a semi-infinite elastic half-space and on elastic stratum, *Phil. Trans. Roy. Soc. London, Series A*, **248,** 327–368.

Carrier, G. F., M. Krook, and C. E. Pearson (1966). *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York.

Castellani, A. (1970). Foundation compliance effects on earthquake response spectra, *J. Soil Mech. Found. Div., ASCE,* **96,** 1335–1355.

Cloud, W. K. and D. E. Hudson (1968). *Strong-Motion Instrumental Data on the Borrego Mountain Earthquake of 9 April 1968*, Seismological Field Survey, Coast and Geodetic Survey, and Earthquake Engineering Research Laboratory, California Institute of Technology.

Foss, K. A. (1958). Co-ordinates which uncouple the equations of motion of damped linear dynamic systems, *J. Appl. Mech.* **25,** 361–364.

Gladwell, G. M. L. (1968). Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid, *Intern. J. Eng. Sci.* **6,** 591–607.

Housner, G. W. (1957). Interaction of building and ground during an earthquake, *Bull. Seism. Soc. Am.* **47,** 179–186.

Housner, G. W. and A. G. Brady (1963). Natural periods of vibration of buildings, *J. Eng. Mech. Div., ASCE,* **89,** 31–65.

Housner, G. W. and P. C. Jennings (1964). Generation of artificial earthquakes, *J. Eng. Mech. Div., ASCE,* **90,** 113–150.

Hradilek, P. J. and J. E. Luco (1970). Dynamic soil-structure interaction, *IDIEM Technical Report No. 14*, Chile.

Hsieh, T. K. (1962). Foundation vibrations, *Proc. Inst. Civil Engrs.* **22,** 211–225.

Isenberg, J. (1970). Interaction Between Soil and Nuclear Reactor Foundations During Earthquakes, *AJA Tech. Report for Res. Found.*, Univ. Toledo.

Ishizaki, H. and N. Hatakeyama (1960). Experimental and numerical studies on vibrations of buildings, *Proc. World Conf. Earthquake Eng., 2nd, Japan*, 1263–1284.

Jennings, P. C. (1969). Spectrum techniques for tall buildings, *Proc. World Conf. Earthquake Eng., 4th Chile*, A3.61–A3.74.

Liu, S. C. and L. W. Fagel (1971). Earthquake interaction by Fast Fourier Transform, *J. Eng. Mech, Div.*, ASCE, **97,** 1223–1237.

Luco, J. E. (1969). Applications of Singular Integral Equations to the Problem of Forced Vibrations of a Rigid Footing, *Ph.D. Thesis*, University of California, Los Angeles.

Luco, J. E. and R. A. Westmann (1971). Dynamic response of circular footings, *J. Eng. Mech. Div., ASCE,* **97,** 1381–1395.

Merritt, R. G. and G. W. Housner (1954). Effect of foundation compliance on earthquake stresses in multi-story buildings, *Bull. Seism. Soc. Am.* **44,** 551–569.

O'Kelly, M. E. J. (1964). Vibration of Viscously Damped Linear Dynamic Systems, *Ph.D. Thesis,* California Institute of Technology.

Parmelee, R. A. (1967). Building-foundation interaction effects, *J. Eng. Mech. Div., ASCE,* **93,** 131–152.

Parmelee, R. A., D. S. Perelman, and S. L. Lee (1969). Seismic response of multiple-story structure on flexible foundations, *Bull. Seism. Soc. Am.* **59,** 1061–1070.

Rainer, J. H. (1971). Structure-ground interaction in earthquakes, *J. Eng. Mech. Div., ASCE* **97,** 1431–1450.

Rosenberg, L. A. (1965). On the interaction between ground and structure during earthquakes, *Bull. Intern. Inst. Seism. Earthquake Eng.* **2,** 29–36.

Sandi, H. (1960). A theoretical investigation of the interaction between ground and structure during earthquakes, *Proc. World Conf. Earthquake Eng. 2nd, Japan,* 1327–1343.

Sarrazin, M. A. (1970). Soil-Structure Interaction in Earthquake Resistant Design, MIT, Dept. Civil Eng. *Report R70–59.*

Sato, Y. and R. Yamaguchi (1960). Vibration of a building upon an elastic foundation, *Bull. Earthquake Res. Inst., Tokyo Univ.* **38,** 369–383.

Scavuzzo, R. J., J. L. Bailey, and D. D. Raftopoulos (1971). Lateral structure interaction with seismic waves, *J. Appl. Mech.,* **38,** 125–134.

Tajimi, H. (1967). Discussion: Building-foundation interaction effects, *Proc. Paper 5200, J. Eng. Mech. Div., ASCE,* **93,** 294–298.

Thomson, W. (1960). A survey of ground-building vibrations, *Proc. World Conf. Earthquake Eng., 2nd Japan,* 833–846.

Thomson, W. T. and T. Kobori (1963). Dynamical compliance of rectangular foundations on an elastic half-space, *J. Appl. Mech.,* **30,** 579–584.

Veletsos, A. S. and Y. T. Wei (1971). Lateral and rocking vibration of footings, *J. Soil Mech. Found. Div., ASCE* **97,** 1227–1248.

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91109

Institute of Engineering
National University of Mexico
Mexico 20, D.F., Mexico

Manuscript received April 6, 1972