A NOTE ON THE STEADY-STATE RESPONSE OF
AN ELASTIC HALF-SPACE

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ABSTRACT

With reference to the influence of local geology on earthquake ground motions, a more complete analytical formulation is made of the well-known problem of a horizontally stratified, linearly-elastic half-space subjected to vertically traveling, sinusoidal, plane waves. A more general interpretation of a result of Kanai is given, and a recursion formula is derived for computing amplification spectra. Some special properties of the system are pointed out and numerical examples are given.

INTRODUCTION

The influence of local geology on the shaking of the ground is of much interest for its engineering implications. In Mexico City, for example, the spectra of the recorded earthquake motions on the surface of the old lake bed have a prominent peak at a period of 2.5 seconds which is interpreted as showing the influence of local geology. In engineering considerations local geology is usually taken to mean the alluvial layers underlying a site as these would have the most pronounced influence on the seismic waves. An analysis of this problem, taking into account the propagation of the seismic waves from their origin to the site and taking into account the true physical and geometrical properties of the soil deposits and the underlying rock, would be exceedingly difficult. A simplified form of this problem, originally studied by Kanai and Sezawa, considers a set of horizontal, elastic layers of uniform thickness overlying an elastic half-space. The layers are assumed to be excited into motion by a vertically traveling, plane wave which passes from the half-space into the layers, and after reflections and refractions the waves pass back into the half-space. The problem is thus reduced to one-dimensional wave propagation, either transient or steady-state. Although this idealized system differs from the conditions found in nature it is thought that its behavior can provide some insight into the behavior of real systems. In this paper the problem is given analytical formulation which incorporates the work of previous investigations appearing in various scattered publications.

By computing the steady-state response of a layered system to a sinusoidal excitation the frequency-selective property of the system can be clearly exhibited by means of an amplification spectrum which shows how the system amplifies some frequency components and suppresses others. Such studies have been made by Sezawa and Kanai (1930, 1932, 1935a, 1935b), Kanai (1952, 1953), Takahasi (1955), Matthiesen, Duke, Leeds, and Fraser (1964), and Herrera and Rosenblueth (1965). Haskell (1953; 1962) has formulated in terms of transfer matrix the more general problem of a layered system excited by body waves that are incident obliquely rather than vertically. Kanai has treated viscous (Voigt) layered systems with one to three layers. Takahasi developed a graphical technique for finding the response of an N-layered nonviscous system. Matthiesen et al. derived the solution for a viscous (Voigt) N-layered system and the results were applied to study the site characteristics of strong-motion earthquake stations in southern California. Herrera and Rosenblueth derived

the solution for a viscous N-layered system in matrix form. Some conclusions of engineering significance were deduced from the steady-state analysis by Kanai:

(1) A nonviscous system with one layer only has a prominent resonant response while a multi-layered system has less prominent resonant responses.

(2) During earthquakes, there is a dominant period associated with the ground layers. This period is proportional to the total thickness of the layers.

(3) Due to the deformability of the base-rock foundation of the half-space, a portion of wave energy is transmitted back into the foundation. This energy loss is called "geometrical dissipation," (Kanai, 1963, 1964) and, consequently, the resonant response is bounded even if the system is nonviscous.

In this paper, there is re-derived the steady-state response of a linearly viscoelastic layered system in the form of a recursion formula from which several properties of the amplification spectrum can be deduced for a nonviscous system having its stiffnesses increasing with depth. From these conclusions, a more general interpretation can be given to Kanai's first conclusion. Numerical examples are presented to demonstrate the effect of layer parameters on the amplification spectrum, and a tentative conclusion is drawn for viscoelastic systems having gradually increasing stiffnesses and having fixed properties for the top layer and the foundation.

The following analysis is made for incident S waves, but the results can readily be converted to the case of P waves by replacing the corresponding elastic constant and the corresponding viscosity constant.

It should be noted that a more practical approach involves the analysis of transient excitations of more heavily damped layered systems, but this is better done by other methods (Tsai, 1969). Soils are not necessarily uniformly layered or linearly elastic, and all seismic waves are not planar and vertically traveling, so that the real problem is much more complicated than currently used methods of analysis indicate.

MATHEMATICAL ANALYSIS

The linear viscoelastic models most commonly used to describe viscous dissipation are the Maxwell model, the Voigt model, and the standard linear model. Denoting the shear stress and the shear strain by σ and ϵ respectively, the steady-state response is $\epsilon = e^{i\omega t}$ and $\sigma = \mu(\omega)e^{i\omega t}$, and a general stress-strain law can be written for the models as follows (Tsai, 1969):

$$\sigma = \mu(\omega). \tag{1}$$

For the Maxwell model with a "relaxation time" constant β,

$$\mu(\omega) = \mu_0 \frac{i\omega\beta}{1 + i\omega\beta} \tag{2a}$$

where μ_0 is the shearing modulus in the absence of viscosity. For the Voigt model and the standard linear model,

$$\mu(\omega) = \frac{1 + i\omega\tau}{1 + \dfrac{i\omega\tau}{1 + r}} \tag{2b}$$

where τ is the retardation-time constant, and r is a nondimensional parameter, equal

to ∞ for the Voigt model. By definition, the complex shear wave velocity is

$$c(\omega) = \sqrt{\mu(\omega)/\rho}$$

which is equal to $\sqrt{\mu_0/\rho}$ for a nonviscous material, ρ being the density.

Consider a horizontally stratified N-layered linearly elastic system with each layer being homogeneous and isotropic, and obeying the general stress-strain law, equation 1, for steady-state deformation. The layered system is shown in Figure 1 with the top layer indexed 1 and the half-space $N + 1$. A set of N coordinates, z_j, $j = 1, 2, \cdots, N$,
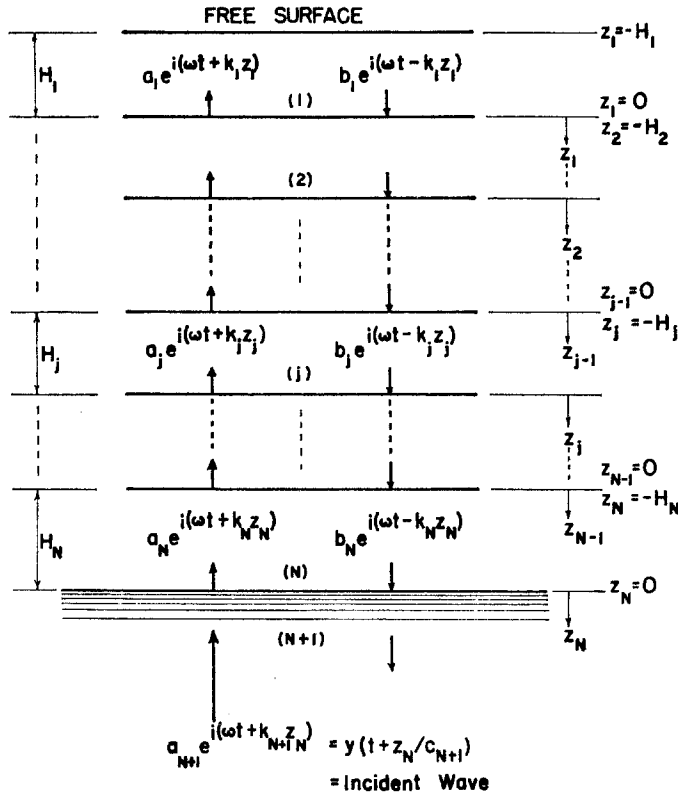

FIG. 1. An N-layer system under steady-state excitation

is also defined as shown. The vertically incident plane S wave

$$y(t + z_N/c_{N+1}) = a_{N+1}e^{i\omega(t+z_N/c_{N+1})} \tag{3}$$

with a_{N+1} being a known amplitude, generates a response, $u_j(z_j, t)$, in the jth layer described by the one-dimensional wave equation

$$c_j^2(\omega) \frac{\partial^2 u_j(z_j, t)}{\partial z_j^2} = \frac{\partial^2 u_j(z_j, t)}{\partial t^2} \qquad j = 1, 2, \cdots, N \tag{4}$$

for $z_j = -H_j$ to $z_j = 0$. The solution of equation (4) can be written as the sum of upward traveling wave and a downward traveling one

$$u_j(z_j, t) = a_j e^{i(\omega t+k_j z_j)} + b_j e^{i(\omega t-k_j z_j)} \tag{5}$$

in which $k_j = \omega/c_j(\omega)$ is the wave number. The amplitudes a_j and b_j are determined by applying the proper boundary conditions at each of the interfaces. At the free surface $z_1 = -H_1$, the stress-free condition yields

$$b_1 = a_1 e^{-2is_1} \tag{6}$$

where it is defined that

$$s_j = k_j H_j \qquad j = 1, 2, \cdots, N. \tag{7}$$

The conditions at the interfaces defined by $z_{j-1} = 0$ and $z_j = -H_j$ are that the displacements are continuous, which gives

$$a_{j-1} + b_{j-1} = a_j e^{-is_j} + b_j e^{is_j}. \tag{8}$$

and that the shearing stresses are continuous, which gives

$$\alpha_{j-1}(a_{j-1} - b_{j-1}) = a_j e^{-is_j} - b_j e^{is_j} \tag{9}$$

In equation (9) α_{j-1} is the impedance ratio between the $(j-1)^{\text{th}}$ and j^{th} layer and is equal to $\rho_{j-1} c_{j-1}/\rho_j c_j$.

From equations (8) and (9), a_j and b_j can be expressed in terms of a_{j-1} and b_{j-1} by

$$\begin{Bmatrix} a_j \\ b_j \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} (1 + \alpha_{j-1})e^{is_j} & (1 - \alpha_{j-1})e^{is_j} \\ (1 - \alpha_{j-1})e^{-is_j} & (1 + \alpha_{j-1})e^{-is_j} \end{bmatrix} \begin{Bmatrix} a_{j-1} \\ b_{j-1} \end{Bmatrix}$$

$$= [S]_j[T]_{j-1} \begin{Bmatrix} a_{j-1} \\ b_{j-1} \end{Bmatrix} \qquad j = 2, \cdots, N \tag{10}$$

where

$$[S]_j = \begin{bmatrix} e^{is_j} & \\ & e^{-is_j} \end{bmatrix} \qquad [T]_j = \frac{1}{2} \begin{bmatrix} 1 + \alpha_j & 1 - \alpha_j \\ 1 - \alpha_j & 1 + \alpha_j \end{bmatrix}. \tag{11}$$

At the lowest interface, where $z_N = 0$, we have

$$\begin{Bmatrix} a_{N+1} \\ b_{N+1} \end{Bmatrix} = [T]_N \begin{Bmatrix} a_N \\ b_N \end{Bmatrix}. \tag{12}$$

Upon repeated use of equation (10), a_j and b_j can be expressed in terms of a_1 and b_1, and, by using equation (6), equation (10) becomes

$$\begin{Bmatrix} a_j \\ b_j \end{Bmatrix} = a_1 e^{-is_1}([S]_j[T]_{j-1}) \cdots ([S]_2[T]_1)[S]_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

For convenience the last equation can be written

$$\begin{Bmatrix} a_j \\ b_j \end{Bmatrix} = a_1 e^{-is_1}[S]_j \begin{Bmatrix} \text{Re}_j + i \, \text{Im}_j \\ \text{Re}_j - i \, \text{Im}_j \end{Bmatrix}. \tag{13}$$

It can be shown by induction that Re_j and Im_j are given by the following recursion formulas

$$Re_j = Re_{j-1} \cos s_{j-1} - Im_{j-1} \sin s_{j-1} \tag{14}$$

$$Im_j = \alpha_{j-1} (Re_{j-1} \sin s_{j-1} + Im_{j-1} \cos s_{j-1}) $$

starting with

$$Re_1 = 1, \qquad Im_1 = 0. \tag{15}$$

Both s_j and α_j are real, and hence Re_j and Im_j are real, only if the layers are non-viscous. For viscoelastic systems, Re_j and Im_j will be complex and their explicit expansions were derived by Tsai (1969).

Substituting equation 13 for $j = N$ into equation (12) gives

$$a_1 = \frac{e^{is_1}}{Re_{N+1} + i \, Im_{N+1}} a_{N+1}. \tag{16}$$

Hence, a_j and b_j are given by equations (13) as

$$\begin{Bmatrix} a_j \\ b_j \end{Bmatrix} = \frac{a_{N+1}}{Re_{N+1} + i \, Im_{N+1}} [S]_j \begin{Bmatrix} Re_j + i \, Im_j \\ Re_j - i \, Im_j \end{Bmatrix} \tag{17}$$

and the response at the surface of the layered system will be

$$u_1(-H_1, t) = \frac{2a_{N+1}}{\sqrt{Re_{N+1}^2 + Im_{N+1}^2}} e^{i(\omega t - \Phi_{N+1})} \tag{18}$$

with

$$\Phi_{N+1}(\omega) = \tan^{-1} \frac{Im_{N+1}}{Re_{N+1}}.$$

We define the following ratio

$$AMP(\omega) = \frac{|u_1(-H_1, t)|}{|2y(t)|} = \frac{|u_1(-H_1, t)|}{2a_{N+1}}$$

$$= \frac{1}{\sqrt{Re_{N+1}^2 + Im_{N+1}^2}} \tag{19}$$

where $y(t)$ is the wave form of the incident wave. The plot of $AMP(\omega)$ as a function of the frequency is the amplification spectrum for the surface response with respect to $2y(t)$. Observe that the double amplitude, $2a_{N+1}$, has been used in the definition of $AMP(\omega)$ because $2a_{N+1}$ would, according to equation (18), be the amplitude of the surface motion of the half-space foundation if there were no superposed layers. By defining

$$G_j(\omega) = \sqrt{Re_j^2 + Im_j^2}$$

and

$$\Phi_j(\omega) = \tan^{-1} \frac{\mathrm{Im}_j}{\mathrm{Re}_j} \qquad (20)$$

the response of the jth layer is

$$u_j(z_j, t) = 2a_{N+1} \, AMP(\omega) \, G_j(\omega) \cos [k_j(H_j + z_j) + \Phi_j]e^{i(\omega t - \Phi_{N+1})}. \quad (21)$$

In the case of viscoelastic layered systems, AMP, Φ_{N+1}, G_j, and Φ_j are complex because Re_j and Im_j are complex, and the explicit expansions for AMP and u_j are given by Tsai (1969).

Because the infinite half-space participates in the motion, the system does not have mode shapes and natural frequencies in the classical sense. However, a characteristic frequency of a nonviscous layered system can be defined as follows. Let ω_n satisfy the relation

$$\mathrm{Re}_{N+1}(\omega_n) = 0 \qquad n = 1, 2, \cdots . \qquad (22)$$

We then have

$$AMP(\omega_n) = \frac{1}{|\, \mathrm{Im}_{N+1}(\omega_n)\,|}$$

$$= \frac{1}{\alpha_N |\, \mathrm{Re}_N \sin s_N + \mathrm{Im}_N \cos s_N\,|_{\omega=\omega_n}} .$$

$AMP(\omega_n)$ is seen to approach infinity as α_N approaches zero. In the limiting case that $\alpha_N = 0$, the foundation is rigid and the system will have resonant response with infinite amplitude at ω_n. Hence, equation (22) defines the characteristic frequencies, ω_n, of a nonviscous system which are simply the natural frequencies of the system with a rigid foundation. Also, equation (22) indicates that the value of ω_n, is independent of the property of the half-space foundation. In most realistic problems, the foundation would be more or less deformable so that α_n will be different from zero and $AMP(\omega_n)$ will always be finite. This results from the fact that a certain amount of energy is lost from the layers by being transmitted back into the deformable foundation. Such an interpretation, first emphasized by Kanai (1963, 1964) is called the "geometric dissipation" of wave energy.

PROPERTIES OF AMP (ω)

A nonviscous layered system whose stiffnesses increase with depth, i.e., $0 < \alpha_j \leqq 1$ for $j = 1, \cdots, N$, is of most practical interest, and several important assertions concerning the nature of the amplification spectrum of such a system can be derived as follows:

Property (A). AMP (ω) is always finite and is greater than or equal to 1.

We first prove that Re_j and Im_j never vanish simultaneously. That this is true is obvious by inspection for the cases of $j = 1$ and 2; for $j \geqq 3$, equation (20) gives

$$\mathrm{Re}_j^2 + \mathrm{Im}_j^2/\alpha_{j-1}^2 = (\mathrm{Re}_{j-1} \cos s_{j-1} - \mathrm{Im}_{j-1} \sin s_{j-1})^2$$

$$+ (\mathrm{Re}_{j-1} \sin s_{j-1} + \mathrm{Im}_{j-1} \cos s_{j-1})^2 = \mathrm{Re}_{j-1}^2 + \mathrm{Im}_{j-1}^2 . \quad (23)$$

Rearranging the right-hand side, equation (23) becomes

$$\mathrm{Re}_j^{\,2} + \mathrm{Im}_j^{\,2}/\alpha_{j-1}^2 = (1 - \alpha_{j-2}^2)\,\mathrm{Re}_{j-1}^2 + \alpha_{j-2}^2\,(\mathrm{Re}_{j-1}^2 + \mathrm{Im}_{j-1}^2/\alpha_{j-2}^2).$$

Upon repeated application of the above recursive relation, we finally arrive at

$$\mathrm{Re}_j^{\,2} + \mathrm{Im}_j^{\,2}/\alpha_{j-1}^2 \prod_{k=1}^{j-2}\alpha_k^{\,2} + \sum_{k=2}^{j-1}\mathrm{Re}_k^{\,2}\,(1 - \alpha_k^{\,2})\left(\prod_{i=k}^{j-2}\alpha_i^{\,2}\right)$$
$$j = 3, \cdots, N + 1. \quad (24)$$

The right-hand side of equation (24) is obviously greater than zero, and hence Re_j and Im_j cannot vanish simultaneously.

On the other hand, adding $\mathrm{Im}_j^{\,2}\,(1\text{--}1/\alpha_{j-1}^2)$ to both sides of equation (23) gives

$$\mathrm{Re}_j^{\,2} + \mathrm{Im}_j^{\,2} = \mathrm{Re}_{j-1}^2 + \mathrm{Im}_{j-1}^2 - \mathrm{Im}_j^{\,2}\,(1 - \alpha_{j-1}^{\,2})/\alpha_{j-1}^2.$$

Applying this recursive relation repeatedly gives

$$\mathrm{Re}_j^{\,2} + \mathrm{Im}_j^{\,2} = 1 - \sum_{k=1}^{j-1}\mathrm{Im}_{k+1}\,(1 - \alpha_k^{\,2})/\alpha_k^{\,2}. \quad (25)$$

Since $\alpha_k^{\,2} \leqq 1$ for $k = 1, 2, \cdots, N$, the sum in equation (25) will always be positive. Equations (24) and (25) lead to the following inequality

$$0 < \mathrm{Re}_j^{\,2} + \mathrm{Im}_j^{\,2} \leqq 1.$$

Setting $j = N + 1$, it immediately follows from the definition of AMP (ω) that

$$\infty > \mathrm{AMP}\,(\omega) \geqq 1. \quad (26)$$

Property (B). The value of AMP (ω_n) has an upper bound equal to $1/(\alpha_1 \cdot \alpha_2 \cdots \alpha_N)$ and a lower bound $1/\alpha_N$.

Making use of equation (23) for $j = N + 1$ gives

$$\mathrm{Re}_{N+1}^2 + \mathrm{Im}_{N+1}^2/\alpha_N^{\,2} = \mathrm{Re}_N^{\,2} + \mathrm{Im}_N^{\,2}.$$

Since $\mathrm{Re}_{N+1}\,(\omega_n)$ vanishes (equation (23)), it follows that

$$\mathrm{Im}_{N+1}^2\,(\omega_n) = \alpha_N^{\,2}\left[\mathrm{Re}_N^{\,2}\,(\omega_n) + \mathrm{Im}_N^{\,2}\,(\omega_n)\right].$$

Making use of equation (25) for $j = N$,

$$\mathrm{Im}_{N+1}^2\,(\omega_n) = \alpha_N^{\,2}\left[1 - \sum_{k=1}^{N-1}\mathrm{Im}_{k+1}^2\,(\omega_n)(1 - \alpha_k^{\,2})/\alpha_k^{\,2}\right] \quad (27)$$
$$\leqq \alpha_N^{\,2}$$

or

$$|\,\mathrm{IM}_{N+1}\,(\omega_n)\,| \leqq \alpha_N. \quad (28)$$

On the other hand, setting $j = N + 1$ in equation (24) gives

$$\text{Im}_{N+1}^2(\omega_n)/\alpha_N^2 = \prod_{k=1}^{N-1} \alpha_k^2 + \sum_{k=2}^{N} \text{Re}_k^2(\omega_n)(1 - \alpha_{k-1}^2)\left(\prod_{i=k}^{N-1} \alpha_i^2\right)$$

$$\geqq \prod_{k=1}^{N-1} \alpha_k^2 \tag{29}$$

or simply

$$|\text{Im}_{N+1}(\omega_n)| \geqq \alpha_1\alpha_2 \cdots \alpha_N .$$

Equation (28) and the last equation together define the bounds for AMP (ω_n) as

$$\frac{1}{\alpha_N} \leqq AMP(\omega_n) \leqq \frac{1}{\alpha_1\alpha_2 \cdots \alpha_N} . \tag{30}$$

Property (C). AMP (ω_n) assumes the minimum value given by property (B) if the characteristic frequency satisfies

$$\sin s_j(\omega_n) = \cos s_N(\omega_n) = 0 \qquad j = 1, \cdots, N - 1 \tag{31}$$

and the maximum value if

$$\cos s_1(\omega_n) = \sin s_j(\omega_n) = 0 \qquad j = 2, \cdots, N. \tag{32}$$

For a single-layered system $(N = 1)$, AMP (ω_n) equals a constant, $1/\alpha_1$, which is both the maximum and minimum itself. For $N \geqq 2$, equation (22) gives

$$\text{Re}_{N+1}(\omega_n) = \text{Re}_N(\omega_n)\cos s_N(\omega_n) - \text{Im}_N(\omega_n)\sin_N(\omega_n) = 0. \tag{33}$$

The characteristic frequency, ω_n, in general does not make AMP (ω_n) a maximum or minimum unless some particular conditions are met by the layer parameters.

(1) From equation (27), there will be a minimum AMP (ω_n) equal to $1/\alpha_N$ only if ω_n satisfies

$$|\text{Im}_{N+1}(\omega_n)| = \alpha_N .$$

The last condition implies that, in equation (27), the following condition should be fulfilled

$$\text{Im}_j(\omega_n) = 0 \qquad j = 2, \cdots N \tag{34}$$

which in turn implies that

$$\sin s_j(\omega_n) = 0 \qquad j = 1, \cdots N - 1. \tag{35}$$

In addition, substituting equation (34) for $j = N$ into equation (33) gives

$$\cos s_N(\omega_n) = 0$$

because, by property (A), $\text{Re}_N(\omega_n)$ cannot vanish. The last condition, together with equation (35), yields equation (31)—the condition under which the minimum value of AMP (ω_n) occurs.

(2) From equation (29), the maximum of AMP (ω_n) will occur if ω_n satisfies

$$| \text{IM}_{N+1}(\omega_n) | = \alpha_1 \alpha_2 \cdots \alpha_N$$

or

$$\text{Re}_j(\omega_n) = 0 \qquad j = 2, \cdots, N \tag{36}$$

which in turn implies that, from the recursion formula for Re_j,

$$\cos s_1(\omega_n) = 0 \quad \text{and} \quad \sin s_j(\omega_n) = 0 \qquad j = 2, \cdots, N-1. \tag{37}$$

Substituting equation (36) for $j = N$ into equation (33) gives the additional condition

$$\sin s_N(\omega_n) = 0.$$

The last condition together with equation (37) constitute the condition, equation (32), under which the maximum of AMP (ω) occurs.

Property (D). A layer is said to be transparent with respect to a particular frequency Ω if the thickness of said layer is equal to an integer multiple of the half-wave length. This property is true for any nonviscous layered system.

First of all, by transparent layer we mean that at a particular frequency Ω the amplification spectrum is completely independent of the parameters of the layer. In other words, AMP (Ω) can be computed as if this layer is absent. Let us consider the jth layer and suppose its thickness to be an integer multiple of the half-wave length at Ω, namely,

$$H_j = \frac{m}{2} \lambda_j(\Omega) \qquad m = 1, 2, \cdots \tag{38}$$

where λ_j is the wave length equal to $2\pi/k_j(\Omega)$. Since equation (38) implies that

$$s_j(\Omega) = k_j(\Omega)H_j = m\pi$$

we have

$$\sin s_j(\Omega) = 0, \qquad \cos s_j(\Omega) = (-1)^m.$$

Substituting the last equation into the recursion formula for Re_{j+1} gives

$$\begin{aligned} \text{Re}_{j+1}(\Omega) &= (-1)^m \text{Re}_j(\Omega) \\ &= (-1)^m [\text{Re}_{j-1}\cos s_{j-1} - \text{Im}_{j-1} \sin s_{j-1}]_{\omega=\Omega}. \end{aligned}$$

Similarly we can obtain

$$\text{Im}_{j+1}(\Omega) = (-1)^m \frac{\rho_{j-1} c_{j-1}}{\rho_{j+1} c_{j+1}} [\text{Re}_{j-1} \sin s_{j-1} + \text{Im}_{j-1} \cos s_{j-1}]_{\omega=\Omega}.$$

Both $\mathrm{Re}_{j+1}(\Omega)$ and $\mathrm{Im}_{j+1}(\Omega)$ are obviously independent of the parameters of the jth layer, which implies that $\mathrm{Re}_{N+1}(\Omega)$ and $\mathrm{Im}_{N+1}(\Omega)$ can be computed as if the jth layer is absent. Hence, property (D) is established.

As already mentioned, Kanai had drawn the conclusion that a single-layer system has prominent resonant amplification while, on the other hand, resonant amplification of a multi-layer system is less prominent. He attributed this behavior of multi-layer systems to the complicated interference of the waves during reflection and transmission across the layer interfaces. Intuitively, since both systems are physically similar and differ from each other only in the layer parameters both would be ex-
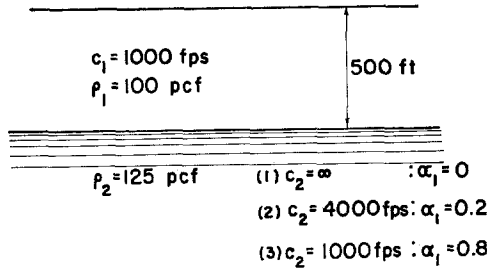
(a) THE SINGLE-LAYER SYSTEM (N=1)
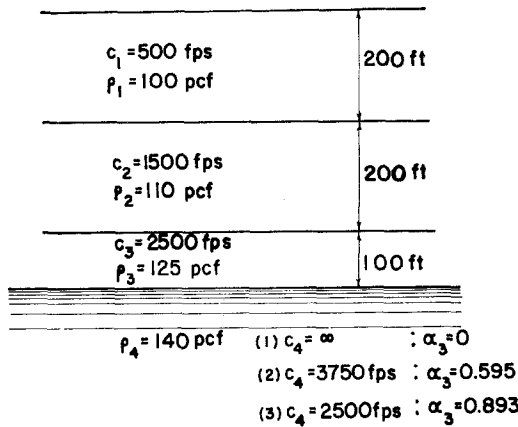
(b) THE TRIPLE-LAYER SYSTEM (N=3)

FIG. 2. Layered systems for demonstrating the effect of α_N.

pected to have resonant responses of comparable prominence, so that Kanai's conclusion seems inconsistent. The following explanation is given for this apparent inconsistency.

Theoretically, property (B) indicates that a nonviscous single-layer system has a regular resonant amplification, AMP (ω_n), equal to $1/\alpha_1$. The value of α_1 depends on the parameters of the system given and, therefore, need not necessarily be small. Hence, the resonant amplifications of a single-layer system need not be prominent. On the other hand, let us consider a nonviscous multi-layer system that has layer stiffnesses increasing with depth. The values of AMP (ω_n) are no longer regular, and

the range of possible variations of AMP (ω_n) is defined by the bounds given in property (B). In general, some resonant amplifications may be prominent while the others are suppressed. Yet, in spite of these possible variations, all the resonant amplifications will be prominent if α_N is sufficiently small. In short, it can be seen that Kanai's conclusion does not hold in general but is valid only for those special systems he analyzed. For illustration, a single-layer system and a triple-layer system, arbitrarily chosen, are shown in Figures 2(a) and 2(b) respectively. For each system, amplification spectra have been computed for three different values of α_N obtained by varying the wave velocity of the foundation. The results are shown in Figure 3, which indicate that α_N has the effect of determining the prominence of the peaks of AMP (ω_n). The values of AMP (ω_n) for both systems are all prominent if α_N is small, but are greatly suppressed if α_N is large. Hence, α_N serves as a measure of the wave energy lost into the foundation due to its deformability.

<center>EFFECT OF LAYER PARAMETERS ON AMP (ω)</center>

Four idealized layered systems were chosen to study the influence of layer parameters, such as the number of layers, layer stiffnesses, viscosities, etc., on the amplification spectrum. The first two systems are double-layered $(N = 2)$. The third system is triple-layered $(N = 3)$, and the fourth quadruple-layered $(N = 4)$. The data for these systems are given in Table 1; they were chosen in such a way that the properties of the top layer and the foundation are the same for all of these systems. Also, the total thickness of the intermediate layers between the top layer and the foundation is 550 feet for all systems. The effect on AMP (ω) were studied by varying the number and the properties of the intermediate layers. In addition, to investigate the effect of material damping, in each layered system the amplification spectra for three different cases were calculated. These three cases were obtained by considering the layer media as

(1) nonviscous elastic solids,
(2) standard linear solids described by the parameters, τ_j and r_j, given in columns
    (5) and (6) respectively of Table 1,
(3) Voigt solids described by the same parameters, $\tau_j (r_j = \infty$ in this case).

The amplification spectra computed for each system are presented in Figures 4 (a) to 4 (d) respectively. Several observations can be made.

(a) The value of the fundamental characteristic frequency, ω_1, is 5 rad/sec for the first system and is 7.2 rad/sec for the second system; whereas both the third and the fourth systems have a value of ω_1 around 6.3 rad/sec.

(b) The spectra for case (2) always lie between the spectra for case (1) and those for case (3).

(c) The spectra for the viscoelastic systems are less sensitive than those for the nonviscous systems to the variation of the parameters of the intermediate layers, and they appear very similar to each other for the cases $N = 3$ and 4. In addition, the spectra for the viscoelastic systems tend to decay with frequency, which is in agreement with theories showing that linear viscoelastic materials always attenuate wave amplitude according to a certain power of wave frequency.

Numerical calculations not shown here indicate that even if the number of intermediate layers is greater than 3 the general characteristics of AMP (ω) will not differ appreciably. Hence, it appears reasonable to conclude that, for a viscous multilayer system with $N \geq 3$ and having layer stiffnesses increasing gradually with depth, if the properties of the top layer and the foundation are fixed, the general charac-

teristics of AMP (ω) do not depend significantly on the properties of the intermediate layers so long as the total thickness of the intermediate layers remains unchanged.
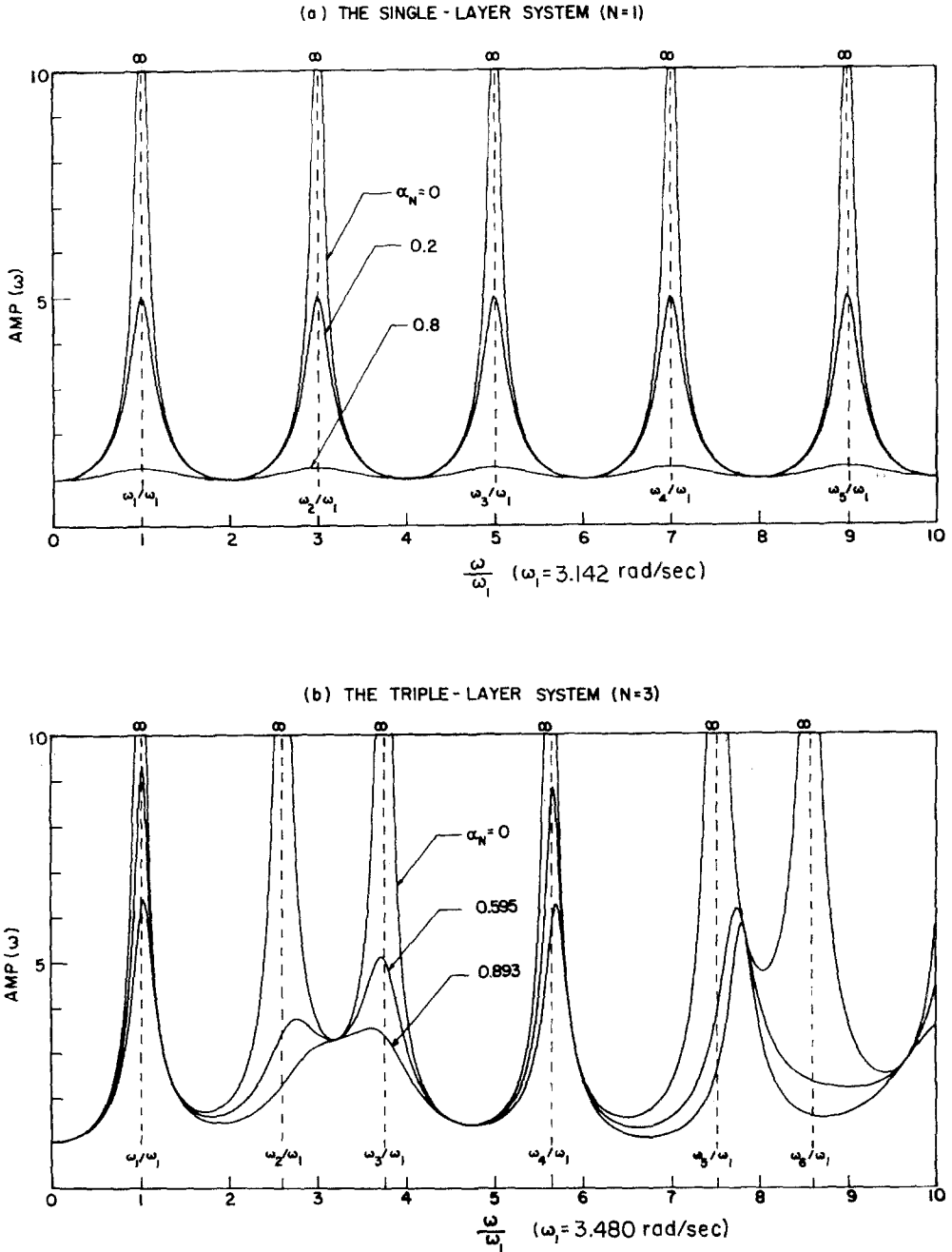

FIG. 3. Amplification spectra for different values of α_N.

For practical problems involving viscous layered systems this implies that inaccuracy in measuring the properties of the intermediate layers will not introduce appreciable error in the computed surface motion. The above conclusion may not be valid for non-

TABLE 1
DATA FOR THE IDEALIZED LAYER SYSTEMS

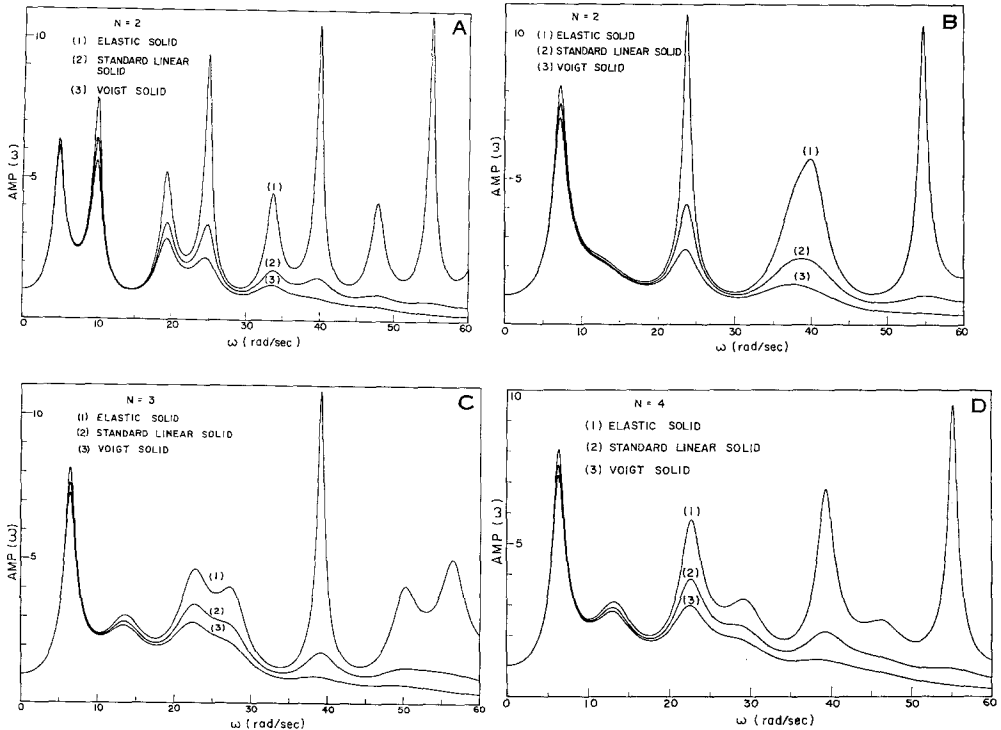| System	(1) Layer No. j	(2) Layer Thickness H_j (ft)	(3) Density ρ_j (pfc)	(4) S Wave Velocity c_j (ft/sec)	(5) Retardation Time τ_j (sec)	(6) r_j	(7) α_j
	1	200	110	1000	0.005	1	0.382
1	2	550	120	2400	0.002	5	0.240
	Base		150	8000	0.	100	
	1	200	110	1000	0.005	1	0.181
2	2	550	135	4500	0.001	20	0.506
	Base		150	8000	0.	100	
	1	200	110	1000	0.005	1	0.382
3	2	200	120	2400	0.002	5	0.474
	3	350	135	4500	0.001	20	0.506
	Base		150	8000	0.	100	
	1	200	110	1000	0.005	1	0.385
	2	150	120	2400	0.002	5	0.658
4	3	175	125	3500	0.0015	10	0.720
	4	225	135	4500	0.001	20	0.506
	Base		150	8000	0.	100	

FIG. 4(a). Amplification spectra.
FIG. 4(b). Amplification spectra.
FIG. 4(c). Amplification spectra.
FIG. 4(d). Amplification spectra.

viscous layered systems because they are more sensitive to the variation of layer parameters.

CONCLUSIONS

The steady-state response of an idealized layered system subjected to vertically incident sinusoidal waves was derived in the form of a recursion formula. The modi-

fying effect of a layered system was expressed in terms of the amplification spectrum, AMP (ω). Four important properties were established regarding the nature of AMP (ω) of a nonviscous system having layer stiffnesses increasing with depth. From these properties, it was concluded that the impedance ratio α_N is a measure of the wave energy lost from the layers into the foundation due to its deformability, and determines the prominence of the resonant response of the system.

Numerical examples were given for studying the effect of layer parameters on the amplification spectrum. It was concluded that for a viscous multi-layer system with $N \geq 3$ having stiffnesses gradually increasing with depth, the amplification spectrum depends primarily on the properties of the top layer and the impedance ratio between the bottom layer and the underlying half-space, and that the intermediate layers have small effect so long as their total thickness is fixed.

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