

The plucked string: an example of non-normal dynamics

David Politzer*

California Institute of Technology, 452-48 Caltech, Pasadena, CA 91125

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Abstract

Motion of a single Fourier mode of the plucked string is an example of transient, free decay of coupled, damped oscillators. It shares the rarely discussed features of the generic case, e.g., possessing a complete set of non-orthogonal eigenvectors and no normal modes, but it can be analyzed and solved analytically by hand in an approximation that is appropriate to musical instruments' plucked strings.

I. BACKGROUND

The transient, free decay of coupled, damped oscillators is not discussed in elementary physics courses and rarely, if ever, in advanced ones. The discussion in advanced physics textbooks is cursory, typically suggesting that one would proceed “just as one might imagine” but that the details are cumbersome. The new features possessed by such systems relative to the well-studied ones are just not part of the basic physics education offered to all students of physical sciences and engineering. Consequently, when analogous aspects arise in particular situations, the people involved sense an aspect of discovery. Sometimes the “newly” discovered perspective has major impact. Although the mechanics of such linear systems has been understood, in principle, for hundreds of years, rediscovery of their special features has occurred even into the 21st Century.

Here are some examples of such rediscoveries. Understanding the $K^o-\bar{K}^o$ meson system¹ in the 1950’s laid the groundwork for the experiments that identified CP and T (time reversal symmetry) violations in the fundamental interactions.^{2,3} Mechanical engineers in the 1960’s were interested in the shaking of buildings,^{4,5} which can exhibit something now known as transient growth. The origin of the characteristic sound of the piano⁶ (in contrast to earlier stringed, keyboard instruments) was elucidated in the 1970’s. Stability analyses in fluid mechanics and analogous problems in applied linear algebra witnessed a major revolution starting in the late 1990’s.^{7,8}

Of course, numerical integration of differential equations has gotten easier and better over the years. And it has often been observed that “non-normal” linear systems sometimes exhibit surprising transient behavior, quite sensitive to parameters and initial conditions. But that is not a substitute for thorough understanding of at least one simple, mechanical system.

There are several related meanings of the term “non-normal.” Here, it refers simply to matrices with complete sets of eigenvectors which are not all orthogonal to one another. They are common elements of a variety of systems including ones that exhibit transient growth where all eigenvectors, individually, decay monotonically.⁷⁻⁹ As described in the following, the total energy of two coupled, damped oscillators decays in time, no matter what the initial conditions or parameter values. However, there are ranges of parameters and initial conditions for which the amplitude and energy of *one* of the oscillators can grow

before eventually decaying. In such cases, the decay of the total system’s energy comes in spurts, rather than a steady one- or two-rate exponential.

II. THE STRING AS TWO COUPLED, DAMPED OSCILLATORS

The ideal string stretched taut between two fixed points has normal modes with frequencies that are integer multiples of the fundamental. In a musical instrument, there are actually two degenerate modes for each frequency, reflecting the possibility of string displacements in the plane transverse to the string direction.

The “fixed” ends are neither fixed nor perfect. That produces a very small coupling between the originally degenerate modes, which splits the frequencies by a small amount (proportional to the coupling) and picks out as normal modes those linear superpositions that diagonalize the perturbation. Again, in the specific context of a stringed instrument, it is important that the splitting be small so that any combination of the two is perceived as a single pitch.

By itself, the vibration of a string produces almost no sound — because the tiny cross section of the string moves almost no air. There must be some further transduction of the string motion to air motion. In acoustic instruments, that is accomplished by linking one end of the string to a sound board. String oscillations force sound board motion, which in turn produces sound. Hence, at least one “fixed” end of the string is not actually fixed. Furthermore, in a good musical instrument, that end motion is the string’s primary loss of energy. And again, that damping must be weak so that the consequent width or spread in frequency due to the damping leaves a single discernible pitch rather than noise.

So, for the present, we focus on a single original frequency. The system is approximated as two initially degenerate oscillators which are weakly coupled and weakly damped. The catch is that the coupling and the damping, represented as matrices in the space of the two initial oscillators, are generally not simultaneously diagonalizable. In particular, we may choose as a basis the up-and-down motion (relative to the sound board) and the side-to-side. In the absence of damping, a small coupling between these two (denoted in the following by ϵ) splits the degenerate frequencies by a fraction ϵ . But arbitrarily small ϵ also rotates the eigenmodes to 45° diagonals with respect to the soundboard.

However, the coupling of vibration to the sound board is typically far more effective for

vertical motion than for horizontal. To make matters as simple as possible, we will consider explicitly small vertical damping (denoted by γ) and no horizontal damping. If the system is started out in one of the modes of the undamped problem, then the non-zero damping will pull it away from that mode and mix in some of the other.

What is it that actually happens?

We choose original restoring forces, the coupling, and the damping so that the system is linear. (E.g., the damping is proportional to the velocities.) With positive damping, the total energy must decrease monotonically with time. It is handy (and virtually essential) to use a complex number representation of the frequencies and vectors; their superposition into real motions at the end is totally parallel to standard treatments of the free decay of the single damped harmonic oscillator.

We will find that there are, indeed, two eigenfrequencies. They describe exponential decay multiplied by sinusoidal oscillation. They have corresponding eigenvectors, which are possible motions that follow their single eigenfrequency. However, these vectors have complex components, which, translated into the real motion of strings, means that their motion in the transverse plane is elliptical rather than strictly linear (as is the case without damping). And, finally, these eigenvectors are not orthogonal. One consequence is that the total energy and the rate of energy dissipation (which is the volume of produced sound in this simple model) are not the sum of two independent, decreasing exponentials.

The spring, mass, and damper analog of this model of a string's single frequency mode is illustrated in Fig. 1. Two identical oscillators are coupled with a weak spring, κ , but

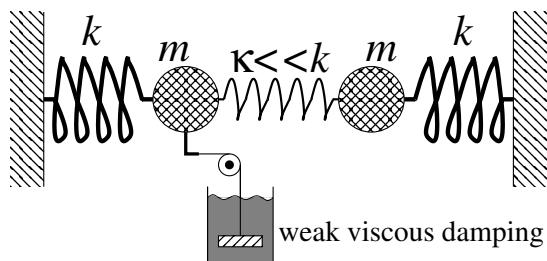


FIG. 1. two oscillator mechanical analog

only one of the oscillators is damped. Newton's second law yields two coupled differential equations in time that are linear in the two displacements. Write the displacements, $x_1(t)$

and $x_2(t)$ in vector form with

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

For the plucked string, $x_1(t)$ is a given mode's vertical motion, and $x_2(t)$ is the horizontal motion.

The equations of motion take the form

$$\ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} - \mathbf{\Gamma} \cdot \dot{\mathbf{x}}, \quad (1)$$

where \mathbf{K} and $\mathbf{\Gamma}$ are 2×2 matrices representing the coupling and damping. (The general mathematical problem would include a mass matrix multiplying $\ddot{\mathbf{x}}$.)

The unit of time can be chosen to put \mathbf{K} into the form

$$\mathbf{K} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}.$$

I.e., for weak coupling and damping, units are chosen such that all of the angular frequencies are close to 1. And the damping, as described above, takes the form

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}.$$

There is no basis in which \mathbf{K} and $\mathbf{\Gamma}$ are both diagonal. A commonly recognized reflection of this is that the commutator

$$[\mathbf{K}, \mathbf{\Gamma}] = \mathbf{K} \cdot \mathbf{\Gamma} - \mathbf{\Gamma} \cdot \mathbf{K} = \begin{pmatrix} 0 & -\epsilon\gamma \\ \epsilon\gamma & 0 \end{pmatrix} \neq 0.$$

(Of course, there are other forms of damping for which the damping and coupling matrices do commute. In such cases, the separation into normal modes is straightforward.)

III. SOLUTION: EIGENVALUES

We seek eigenvalues α and time-independent eigenvectors \mathbf{x}_o such that $e^{\alpha t} \mathbf{x}_o$ is a solution to Eq. (1). Plugging that in yields

$$(\alpha^2 \mathbf{I} + \mathbf{K} + \alpha \mathbf{\Gamma}) \cdot e^{\alpha t} \mathbf{x}_o = 0, \quad (2)$$

where \mathbf{I} is the identity matrix. $\mathbf{x}_o = 0$ is a solution to Eq. (2) but not to the problem at hand. For all other solutions, the matrix factor in Eq. (2) cannot have an inverse, and that requires that its determinant vanish, which is the following:

$$(\alpha^2 + 1)^2 + \gamma\alpha(\alpha^2 + 1) - \epsilon^2 = 0 . \quad (3)$$

This would be easy to solve were either γ or ϵ zero, but as it stands it is a quartic equation. Quartic is the highest order polynomial for which a closed-form solution exists. That solution has been known since the 16th Century, but it is far longer than most people can remember or comprehend. Also it contains a great many nested square and cube roots. Just as the single damped oscillator has cases with radically different qualitative behavior, i.e., over damped, under damped, and critically damped, there are cases here, too — only a great many more.

For weak coupling and weak damping, we can anticipate the structure of the solutions from physical considerations. With $\gamma \ll 1$ (and $\gamma > 0$), the four solutions for α will be two complex conjugate pairs. $\text{Re}[\alpha]$ will be negative, reflecting the monotonic loss of energy. $\text{Im}[\alpha]$ comes in conjugate pairs. These conjugate pair solutions can ultimately be superposed to get real solutions with sines and cosines of t with the same frequency. And this multiplicity of solutions allows fitting of any initial conditions of the two oscillators.

If *both* $\gamma \ll 1$ and $\epsilon \ll 1$, then *all* the frequencies will be near to 1, i.e. $i \text{Im}[\alpha] \approx \pm i$. And this offers a way to approximate Eq. (3) and reduce the algebra problem to a quadratic.¹⁰ In the term $\gamma\alpha(\alpha^2 + 1)$, approximating the first α by $\pm i$ leaves the whole term still as small as ϵ^2 and $(\alpha^2 + 1)^2$, at least in the vicinity of the desired solutions. So, using this approximation, Eq. (3) becomes

$$(\alpha^2 + 1)^2 \pm i\gamma(\alpha^2 + 1) - \epsilon^2 \simeq 0 , \quad (4)$$

whose solutions are

$$\alpha \simeq \pm i \left(1 \pm i\frac{\gamma}{4} \pm \frac{1}{2}\sqrt{\epsilon^2 - \left(\frac{\gamma}{2}\right)^2} \right) . \quad (5)$$

If the three \pm 's were chosen independently, there would appear to be eight solutions. However, the approximation of Eq. (3) by Eq. (4) introduces four extra solutions that do not satisfy the actual approximation. These are easy to identify by their having $\text{Re}[\alpha] > 0$ for $\gamma > 0$. We retain the four α 's with $\text{Re}[\alpha] < 0$.

The separate limits $\epsilon \rightarrow 0$ and $\gamma \rightarrow 0$ recover the previously understood behaviors of single damped oscillators and coupled, undamped oscillators.

There are evidently three qualitatively different regions, even with both $\gamma \ll 1$ and $\epsilon \ll 1$. With $\epsilon > \gamma/2$, the square root term contributes to the oscillation frequency; there are two oscillation frequencies but only one decay rate, which is independent of ϵ . With $\epsilon < \gamma/2$, the square root term effects the two decay rates, and there is no splitting of the oscillation frequency degeneracy. And for $\epsilon \approx \gamma/2$, the frequencies and decay rates of the two eigenmodes are nearly equal.

In the Appendix, evaluations of Eq. (5) for three numerical pairs of ϵ and γ , representative of the three regions, are compared to the exact values that come from solving Eq. (3).

IV. SOLUTION: EIGENVECTORS

Let the components of the four eigenvectors be a and b :

$$\mathbf{x}_o = \begin{bmatrix} a \\ b \end{bmatrix}.$$

(The eigenvalues α and eigenvectors \mathbf{x}_o , with their components a and b , have a four-valued index i to tell which goes with which. That index i is suppressed when that improves clarity.)

The lower component of Eq. (1) tells us that

$$\frac{b}{a} = \frac{-\epsilon}{\alpha^2 + 1} \simeq \frac{-\epsilon}{\pm i\gamma/2 \pm \sqrt{\epsilon^2 - (\gamma/2)^2}}. \quad (6)$$

This specifies the four eigenvectors, one for each α . (Recall that $\alpha^2 + 1 = 0$ only for $\epsilon = 0$ and $\alpha^2 + 1 = \pm\epsilon$ for $\gamma = 0$.) The first expression for b/a with the $=$ sign is exact relative to the initial statement of the problem, i.e., Eq. (1); the approximate solutions to Eq. (4) are used for the \simeq expression. The ratio b/a is of order 1 (because $|\alpha^2 + 1| \ll 1$).

Also, b/a is complex. The phase of each b/a means that in the oscillatory part of the motion corresponding to a single eigenvalue, there is a fixed, non-zero phase between the $x_1(t)$ and the $x_2(t)$. In the language of the plucked string: in the transverse plane, the eigenmotions are elliptical rather than linear (which they would be in the absence of damping).

V. REAL SOLUTIONS AND NON-ORTHOGONALITY

Since there are a variety of precise meanings given to orthogonality and non-normality, it is worth returning in the present context to the original physical problems and constructing

the basis of real eigenfunctions.

For weak damping, the four eigenvalues α are two pairs of complex conjugates. Label them as $\alpha_{\pm 1}$ and $\alpha_{\pm 2}$, where there are two, in general, different, negative real parts, each with a pair of conjugate imaginary parts. The $\alpha_{\pm 1}$'s can be assembled into two real eigenfunctions:

$$\mathbf{y}_{+1}(t) = e^{\alpha_{+1}t} \begin{bmatrix} a_{+1} \\ b_{+1} \end{bmatrix} + e^{\alpha_{+1}^*t} \begin{bmatrix} a_{+1}^* \\ b_{+1}^* \end{bmatrix}$$

$$\mathbf{y}_{-1}(t) = i \left(e^{\alpha_{+1}t} \begin{bmatrix} a_{+1} \\ b_{+1} \end{bmatrix} - e^{\alpha_{+1}^*t} \begin{bmatrix} a_{+1}^* \\ b_{+1}^* \end{bmatrix} \right)$$

$$= e^{\alpha_{+1}t+i\pi/2} \begin{bmatrix} a_{+1} \\ b_{+1} \end{bmatrix} + e^{\alpha_{+1}^*t+i^*\pi/2} \begin{bmatrix} a_{+1}^* \\ b_{+1}^* \end{bmatrix} .$$

These forms use the facts that $\alpha_{-1} = \alpha_{+1}^*$ and $b_{-1}/a_{-1} = b_{+1}^*/a_{+1}^*$ (true in the original, exact formulation). Likewise, there are two real functions $\mathbf{y}_{\pm 2}(t)$ similarly constructed out of the conjugate pair $\alpha_{\pm 2}$. Appropriate superposition of the four $\mathbf{y}(t)$'s can match the two initial positions and velocities.

The normal modes of linearly coupled, undamped oscillators behave themselves, essentially, like a set of uncoupled oscillators. Whatever superposition is determined by the initial conditions remains in force for all time. Many quantities of central importance, such as the kinetic energy, the potential energy, and the total energy of the system are, at any time, just the sum of the contributions from the normal modes. Since these particular quantities are quadratic in the dynamical variables, the reduction to a sum over modes requires that cross terms between the contributions of different modes vanish. And that is the sense in which the normal modes are normal to each other. However, for the generic case of coupled, damped oscillators, such cross terms are non-zero. Hence, there are no normal modes — in spite of there being a complete set of solutions corresponding to the time-dependence eigenvalues.

As long as $\epsilon \neq 0$, for convenience we can choose all four $a_{\pm 1,2} = 1$. Then, for example, at $t = 0$:

$$\mathbf{y}_{+1}(0) \cdot \mathbf{y}_{+2}(0) = 4(1 + \text{Re}[b_{+1}]\text{Re}[b_{+2}])$$

$$\neq 0 \quad \text{for } \gamma/\epsilon \neq 0 .$$

Once again, this result is not particularly surprising, except possibly to those imbued with an overwhelming respect for normal modes. If the coupled, damped system were exactly describable by normal modes, then the total energy would decay steadily as the sum of one or two exponentials. However, if the corresponding undamped system could exhibit beats, then with the addition of very weak dissipation to only one of the (pre-coupled) degrees of freedom, dissipation should likewise come and go at the beat frequency. And that is, indeed, one of the possible generic behaviors.

VI. PLUCKED STRING SOUNDS

Careful measurements made of the sounds of plucked banjo strings were published by Moore and Stephey.¹¹ For one aspect of their experimental survey they damped all strings but the first, plucked it, first in the vertical direction (relative to the banjo head) and then in the horizontal. They did the same for the second string. They recorded the sounds and analyzed them into Fourier components. In Fig. 2 (copied from Fig. 5 of Ref. 11) the (logarithmic) sound intensities for the first three harmonics of each plucked string are displayed as a function of time. Each harmonic acts as a separate nearly degenerate, coupled, damped pair. As those authors noted in their paper, evident are single exponential decays, double exponential decays, and decays with prominent beats modulating a single overall decay rate. In the simple model presented here, sound intensity would be proportional to the power dissipated by the “vertical” oscillator. And the three observed types of behavior of the actual strings match the three qualitative behaviors of solutions of Eq. (4), corresponding to the three regions of parameter space defined by the relative sizes of ϵ and $\gamma/2$.

Even with just the naked eye, some of this behavior is typically visible on a stringed instrument. In particular, there is usually at least one string that after a vigorous pluck exhibits beats. Instead of decreasing steadily, its amplitude gets smaller and larger again a couple or even several times before it dies completely. (Generally, each maximum is smaller than the previous one.)

This phenomenon is actually a very important aspect of banjo sound. The banjo is an instrument where the degeneracy is often four- and even six-fold, not just the two of a single string. That is because, as normally tuned and played, the undamped strings are often in unison or share harmonics (e.g., the second harmonic of one string is degenerate with the

third harmonic of another). And the design of the bridge (the active terminus of the strings) facilitates coupling between between all of the strings.

In the simple model of the plucked string presented, the sound volume is proportional to the rate of energy lost to damping of the vertical motion of the particular string harmonic. The instantaneous value of this lost power is $\mathcal{P}_{\text{inst}} = \gamma \dot{x}_2(t)^2$. In Fig. 3, the log of the several-cycle-averaged $\dot{x}_2(t)^2$ is plotted *verus* time for $\epsilon = 0.01$ and $\gamma = 0.01$ with the $t = 0$ condition that the pluck is purely in the horizontal (undamped) direction. The horizontal motion $x_2(t)$ is the lower component of $\mathbf{x}(t) \equiv \mathbf{y}_{+1}(t) - \mathbf{y}_{+2}(t)$.

The numerical parameter values in the Appendix were chosen for convenience of computer entry, with the consideration that they be small but realistic for stringed instruments. (No effort was made to match the data of Fig. 2.) Of these three pairs, the values used for Fig. 3 are the ones that exhibit beats, i.e., there are two distinct oscillation frequencies with the beat period being distinctly shorter than the damping time. And the horizontal pluck was chosen for display because time must elapse after the pluck before the coupling to the dissipative vertical motion is substantial. Hence, the power dissipated *grows* immediately after the pluck before it subsequently decays. With a vertical pluck, the dissipation would be evident from the start but would also exhibit the beats between the two eigenfrequencies. The dissipated power for the other domains of ϵ - γ space would look like single or double exponential decay without beats.

Appendix A: Exact/approximate eigenvalue comparison

Modern computer math packages include the exact solution for the roots of the general quartic polynomial. Hence, their numerical evaluations of Eq. (3) are unassailable. The table below gives comparison of the approximate eigenvalues from Eq. (4) to the exact eigenvalues from Eq. (3) for $\epsilon = 0.01$ and $\gamma = 0.01, 0.02, \text{ and } 0.03$.

TABLE I. Eigenvalue exact/approximate comparison

ϵ	γ	method	Re[α]	$i\text{Im}[\alpha]$
0.01	0.01	exact	-0.002504	$\pm i$ 0.995679
			-0.002496	$\pm i$ 1.00431
0.01	0.01	approx	-0.0025	$\pm i$ 0.995670
			-0.0025	$\pm i$ 1.00433
0.01	0.02	exact	-0.005352	$\pm i$ 0.9996
			-0.004648	$\pm i$ 1.0003
0.01	0.02	approx	-0.005	$\pm i$
			-0.005	$\pm i$
0.01	0.03	exact	-0.01309021	$\pm i$ 0.999856
			-0.00190978	$\pm i$ 1.00001
0.01	0.03	approx	-0.01309017	$\pm i$
			-0.00190983	$\pm i$

* politzer@theory.caltech.edu

¹ M. Gell-Mann and A. Pais, “Behavior of neutral particles under charge conjugation,” Phys. Rev. **97** (5) 1387–1389 (1955).

² J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, “Evidence for the 2π decay of the K_2^0 meson,” Phys. Rev. Lett. **13** (4) 138–140 (1964).

³ P. K. Kabir, *The CP Puzzle*, Academic, New York (1968).

⁴ T. K. Caughey, “Classical Normal Modes in Damped Linear Dynamic Systems,” J. Appl. Mech. **27E** 269-271 (1960).

⁵ T. K. Caughey and M. E. J. O’Kelly, “Classical normal modes in damped linear dynamic systems,” J. Appl. Mech. **32** (3), 583-588 (1965).

⁶ Gabriel Weinreich, “Coupled piano strings,” J. Acoust. Soc. Am. **62** (6) 1474–1484 (1977).

⁷ P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows*, Springer-Verlag, New York, 2001.

- ⁸ L. N. Trefethen and Mark Embree, *Spectra and Pseudospectra – The Behaviour of Nonnormal Matrices and Operators*, Princeton University Press, Princeton and Oxford, 2005.
- ⁹ Rahul Bale and Rama Govindarajan, “Transient growth and why we should care about it,” *Resonance* **15** (5) 441–457 (2010); this article offers a fine, elementary introduction to the subjects addressed in Ref.s 7 and 8.
- ¹⁰ The analysis of coupled, damped strings in Ref. 6 likewise uses near degeneracy as a key approximation. However, the methods and tools of analysis used there assume considerably greater sophistication on the part of the reader. Also, with somewhat different goals, several of the very general features are not remarked upon as such.
- ¹¹ Laurie A. Stephey and Thomas R. Moore, “Experimental investigation of an American five-string banjo,” *J. Acoust. Soc. Am.* **124** (5) 3276–3283 (2008).

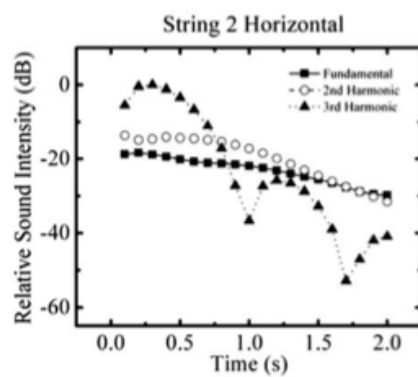
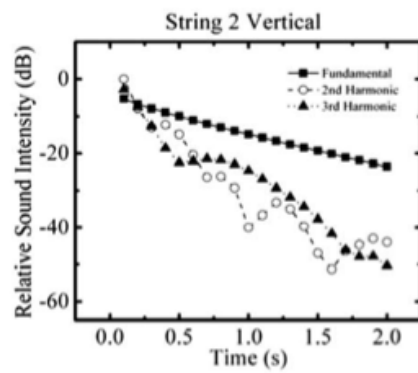
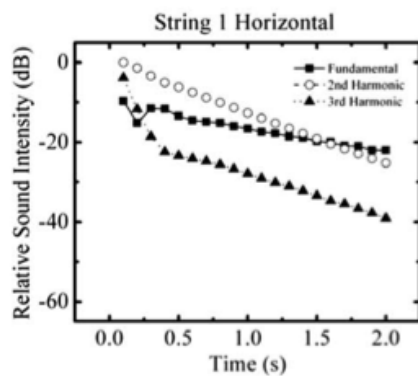
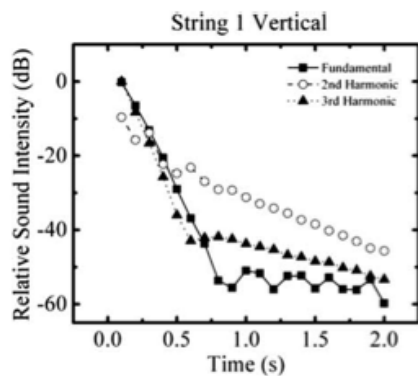


FIG. 2. Measured loudness in dB (log scale) *vs.* time for string harmonics, copied from Ref. 11

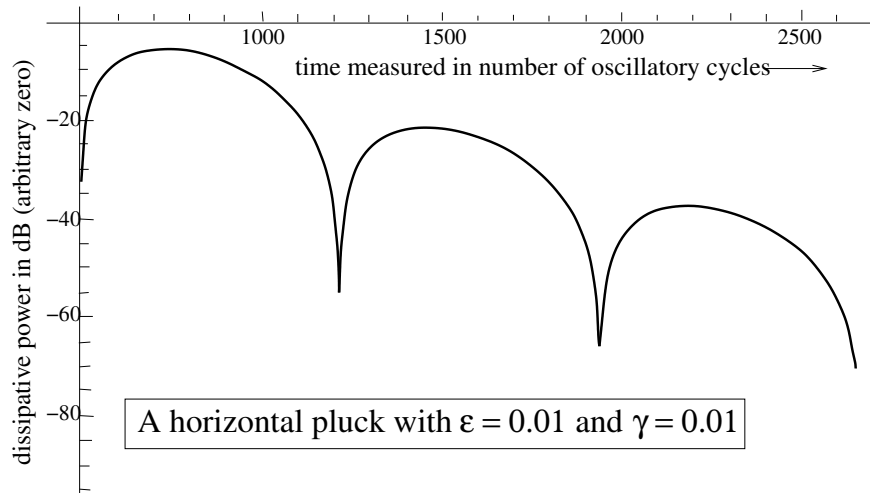


FIG. 3. The calculated “sound” of a particular horizontal pluck