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Angular momentum generation by parity violation

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We generalize our holographic derivation of spontaneous angular momentum generation in $2+1$ dimensions in several directions. We consider cases when a parity-violating perturbation responsible for the angular momentum generation can be nonmarginal (while in our previous paper we restricted to a marginal perturbation), including all possible two-derivative interactions, with parity violations triggered both by gauge and gravitational Chern-Simons terms in the bulk. We make only a minimal assumption about the bulk geometry that it is asymptotically AdS, respects the Poincaré symmetry in $2+1$ dimensions, and has a horizon. In this generic setup, we find a remarkably concise and universal formula for the expectation value of the angular momentum density, to all orders in the parity violating perturbation.

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I. INTRODUCTION

The spontaneous generation of angular momentum and of an edge current are typical phenomena in parity-violating physics (see, for example, [1–5]). For a given interacting system, whether spontaneous generation of angular momentum does occur, and if yes, the precise value, are important dynamical questions for which a universal answer (applicable to generic parity-violating systems) does not appear to exist. A famous example is helium 3-A, in which case there has been a long controversy about the value of its angular momentum (see e.g. [1,6]). The controversy highlights the importance of finding exactly solvable models, especially strongly interacting systems, through which one could extract generic lessons. Holographic systems are ideal laboratories for this purpose.

In a previous paper [7], we initiated exploration of these phenomena in holographic systems.[1] There, for technical simplicity, we restricted to parity violation effected by turning on a marginal pseudoscalar operator, and considered only the Schwarzschild and Reissner-Nordström geometries. In this paper, we generalize the results to parity violation through a relevant scalar operator, and to general bulk black hole geometries.

More explicitly, we consider a $(2+1)$-dimensional boundary field theory with a $U(1)$ global symmetry, which is described by classical gravity (together with various matter fields) in a four-dimensional, asymptotically anti–de Sitter spacetime (AdS_4). We consider two representative bulk mechanisms for parity violation, with a gravitational Chern-Simons interaction [19]

$$\alpha_{\mathrm{CS}} \int \vartheta R \wedge R, \tag{1.1}$$

or an axionic coupling [20,21]

$$\beta_{\mathrm{CS}} \int \vartheta F \wedge F, \tag{1.2}$$

where R is the Riemann curvature two-form, F is field strength for the bulk gauge field A_a dual to the $U(1)$ global current, and ϑ is a pseudoscalar dual to a boundary relevant pseudoscalar operator $\mathcal{O}$. α_{CS} and β_{CS} are some constants.

The parity symmetry is broken explicitly if a source is turned on for $\mathcal{O}$ corresponding to turning on a nonnormalizable mode for the pseudoscalar field ϑ. Alternatively, the parity can be spontaneously broken when $\mathcal{O}$ develops an expectation value in which case the bulk field ϑ is normalizable. In both situations if we put the system in a finite box (i.e., parity-violation terms are nonzero only inside the box), the spontaneous generation of angular momentum is always accompanied by an edge current. We emphasize that the source or expectation value for $\mathcal{O}$ is taken to be homogeneous along boundary directions. An angular momentum density is generated, despite the boundary quantum state and the corresponding bulk geometry being homogeneous and isotropic.

It may appear puzzling how a homogeneous and isotropic bulk geometry can give rise to a nonzero angular momentum, as directly applying the standard AdS/CFT dictionary to such a geometry will clearly yield a zero value. The key idea, following [7], is to consider a small and slightly *inhomogeneous* perturbation $\delta\vartheta$ around the

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[1]See [8–18] for other discussions of parity-violating effects in holographic systems and in $(2+1)$-dimensional field theories.

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background value of ϑ, which results in a nonzero momentum current density δT_{0i}.[2] To leading order in the derivative expansion along the boundary directions, T_{0i} depends linearly on $\epsilon_{ij}\partial_j\delta\vartheta$. Now let us consider a configuration of $\delta\vartheta$ which is homogeneous along boundary spatial directions inside a big box but vanishes outside. Then δT_{0i} is only nonvanishing at the edge of the box, but remarkably such an edge current generates an angular momentum proportional to the volume of the box

$$\delta J = \epsilon_{ij}\int d^2x x_i \delta T_{0j} \propto V_{\rm box}\delta\vartheta \tag{1.3}$$

resulting in a nonzero angular momentum density $\delta\mathcal{L}$ which survives even when we take the size of the box to infinity. Thus in the homogeneous limit, the angular momentum density $\delta\mathcal{L}$ arises from the global effect of an edge current, which explains why it is not visible from the standard local analysis of the stress tensor.

When ϑ is dual to a marginal operator, $\delta\vartheta$ is independent of radial direction of AdS and $\delta\mathcal{L}$ is given by $\delta\vartheta$ times a constant which can be easily integrated to find the value of $\mathcal{L}$ for a finite ϑ. But for ϑ dual to a boundary relevant operator, $\delta\vartheta$ has a nontrivial radial evolution (which simply reflects that a relevant operator flows), and the relation between $\delta\mathcal{L}$ and $\delta\vartheta$ involves a somewhat complicated radial integral over various bulk fields. Remarkably, this relation can be written as a total variation in the space of gravity solutions, which can then be easily integrated to yield a closed expression for $\mathcal{L}$ at a finite ϑ.

More explicitly, we consider a most general bulk metric consistent with translational and rotational symmetries along boundary directions, which can be written in a form

$$ds^2 = \frac{\ell^2}{z^2}(-f(z)dt^2 + h(z)dz^2 + (dx^i)^2) \tag{1.4}$$

with $z = 0$ as the boundary. Matter fields include $\vartheta(z)$, $A_t(z)$, and possibly others. We denote z_0 as the horizon of the metric. Note that in the coordinate choice of (1.4) z_0 is inversely proportional to the square root of the entropy density s, i.e. $z_0 \propto s^{-\frac{1}{2}}$, and serves as an IR cutoff scale[3] of the boundary system. For the axionic coupling (1.2) we find that the angular momentum density can be written as

$$\mathcal{L} = -\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\mu^2\vartheta(z_0) + \frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\int_0^{z_0} dz (A_t(z)-\mu)^2\vartheta'(z) \tag{1.5}$$

where μ is the chemical potential, ℓ is the AdS radius, and $\kappa^2 = 8\pi G_4$. For gravitational CS coupling (1.1), we find that

$$\mathcal{L} = -\frac{4\pi^2\alpha_{\rm CS}\ell^2}{\kappa^2}T^2\vartheta(z_0) + \frac{\alpha_{\rm CS}\ell^2}{4\kappa^2}\int_0^{z_0} dz\left(\frac{f'^2}{fh}\right)\vartheta' \tag{1.6}$$

where T is the temperature.

Equations (1.5)–(1.6) are universal in the bulk sense that they have the same form in terms of bulk gauge fields or metric components, independent of the specific form of bulk actions, geometries and possible other matter fields. But they are not universal in the boundary sense as it appears that they cannot be further reduced to expressions in terms of boundary quantities only.

When ϑ is dual to a marginal operator at the boundary, $\vartheta(z)$ is constant in the bulk and its value can be identified as the coupling of $\mathcal{O}$. Then for both (1.5) and (1.6), $\mathcal{L}$ is given by the first term, reproducing our earlier results in [7]. These expressions are now universal also in the boundary sense, valid for any boundary theory with a gravity dual. In Sec. IV, we will present a preliminary explanation of this universal behavior from the perspective of the boundary conformal field theory (CFT). We hope to explore this point in future.

For ϑ dual to a relevant operator, $\vartheta(z)$ can be interpreted as the running coupling for the corresponding boundary operator $\mathcal{O}$, with z as the renormalization group (RG) length scale. In this case, the first term of (1.5) and (1.6) is proportional to the running coupling evaluated at the IR cutoff scale z_0. The second term of (1.5) and (1.6) has the form of the beta function (given by ϑ') for $\mathcal{O}$ integrated over the RG trajectory all the way to the IR cutoff. This indicates that in the case of a relevant operator, despite being an IR quantity, the angular momentum receives contribution from all scales. The simplicity of the integration kernel in these equations may suggest a possible simple boundary interpretation which should be explored further.

Another interesting phenomenon associated to parity violation in $2+1$ dimensions is the Hall viscosity [22]. It turns out that, in quantum Hall states, there is a close relation between the Hall viscosity and the angular momentum density [5,23–25]. It would be interesting to understand how universal such a relation is. In a forthcoming paper, we will discuss this issue from the holographic perspective. We will apply the prescription of [12] to identify models where the Hall viscosity is nonzero and compare its value with the angular momentum density.

For the remainder of this paper, we will use the following. Latin letters stand for $(3+1)$-dimensional spacetime indices, greek letters stand for $(2+1)$-dimensional indices on the boundary, latin letters in the middle of the alphabet $(i, j, k, \ldots)$ stand for 2-dimensional spatial indices on the boundary and $\partial^2 \equiv \partial_x^2 + \partial_y^2$. The metric is denoted via g_{ab} with signature $(-,+,+,+)$ in the bulk, and via $h_{\alpha\beta}$ on the boundary; the Einstein summation convention and geometric units with $\hbar = c = 1$ are assumed, unless otherwise specified; we denote $\kappa^2 = 8\pi G_4$.

After posting this paper on the arXiv e-print server, it was pointed out by K. Landsteiner and by a referee of this paper

[2]We use latin letters in the middle of the alphabet $(i, j, k, \ldots)$ to denote two-dimensional spatial indices on the boundary.

[3]Physically, it can be interpreted as characterizing the correlation length of the boundary system.

that the spontaneous generation of the edge current and of the angular momentum in $2+1$ dimensions discussed in this paper may be related to the chiral magnetic effect [26–28] and axial magnetic effect [29–31] in $3+1$ dimensions. Prompted by their suggestions, we found that the effects in $3+1$ dimensions and $2+1$ dimensions are indeed related by dimensional reduction when the parity-violating perturbation is *marginal*, which was the focus of our previous paper [7]. For completeness, we added Sec. IV to discuss the relation. The purpose of this paper is to generalize our results to the case when the parity-violating perturbation is *relevant*, and the discussion in Sec. IV is not immediately applicable. There may exist a generalization of the chiral magnetic effect and axial magnetic effect in $3+1$ dimensions which correspond to dimensional oxidation of the effects studied in this paper.

II. AXIONIC COUPLING

In this section we consider a scalar field ϑ coupled to a Maxwell field via an axionic coupling, $\vartheta^* F^{ab} F_{ab}$. We first explicitly work out the angular momentum for a simple setup, and then generalize the results to general gravity theories.

A. Angular momentum

1. Small perturbations

Consider the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[R - \frac{1}{2}(\partial\vartheta)^2 - V(\vartheta) - \ell^2 F^{ab}F_{ab} - \ell^2\beta_{\rm CS}\vartheta^* F^{ab}F_{ab}\Big], \tag{2.1}$$

with $\beta_{\rm CS}$ a coupling constant, and ϑ dual to a relevant (or marginal) pseudoscalar boundary operator. We assume that the background geometry is asymptotically AdS with ℓ the AdS radius. The equations of motion are

$$R_{ab} - \frac{1}{2}\partial_a\vartheta\partial_b\vartheta - \frac{1}{2}g_{ab}V(\vartheta) - 2\ell^2\Big(F_{ca}F^c{}_b - \frac{1}{4}g_{ab}F^2\Big) = 0, \tag{2.2}$$

$$\frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\vartheta) - V'(\vartheta) - \beta_{\rm CS}\ell^{2*}FF = 0 \tag{2.3}$$

$$\partial_a[\sqrt{-g}(F^{ab} + \beta_{\rm CS}\vartheta^* F^{ab})] = 0. \tag{2.4}$$

A most general solution describing the boundary in a static, homogeneous, isotropic state can be written as

$$g^{(0)}_{ab}dx^a dx^b = \frac{\ell^2}{z^2}(-f(z)dt^2 + h(z)dz^2 + (dx^i)^2),$$
$$\vartheta = \vartheta(z), \qquad A_a = A_t(z)\delta^t_a. \tag{2.5}$$

The AdS boundary lies at $z=0$ with

$$f(z)\to 1, \qquad h(z)\to 1, \qquad z\to 0 \tag{2.6}$$

and

$$A_t(z=0) = \mu \tag{2.7}$$

where μ is the chemical potential. We assume that there is a horizon as $z = z_0$, where $f(z)$ has a simple zero and $h(z)$ has a simple pole. The temperature is given by

$$T = \frac{1}{4\pi}\sqrt{f'(z_0)h^{-1'}(z_0)}. \tag{2.8}$$

Here are some background equations of motion which will be important below. The t component of the background Maxwell equation can be integrated to give

$$A_t{}'(z) = Q\sqrt{f(z)h(z)} \tag{2.9}$$

with Q the charge density. The tt and ii components of the background Einstein equations can be used to obtain

$$4\sqrt{fh}Q^2 = \left(\frac{f'}{z^2\sqrt{fh}}\right)'. \tag{2.10}$$

As discussed in the Introduction, to compute the angular momentum, we consider a small and slightly inhomogeneous perturbation $\delta\vartheta(z,x^i)$ around the background value $\vartheta(z)$. Such a perturbation will clearly also induce perturbations of the metric and gauge field,

$$g_{ab} = g^{(0)}_{ab} + \frac{\ell^2}{z^2}\delta g_{ab}, \tag{2.11}$$

$$A_a = A_t(z)\delta^t_a + \delta A_a(z,x^i). \tag{2.12}$$

The metric and gauge field perturbations will be assumed to be normalizable, while $\delta\vartheta$ can be either normalizable or non-normalizable. We will also make the following gauge choice:

$$\delta A_z = 0, \qquad \delta g_{zt} = 0. \tag{2.13}$$

To find the angular momentum, we first compute T_{ti}, which in turn requires us to find δg_{ti}. Since $\delta\vartheta$ is small we can work at the linear order in all perturbations, and since we will eventually take $\delta\vartheta$ to be homogeneous, it will be enough to keep only terms with at most one boundary spatial derivative (for details on the derivative expansion in holographic fluid dynamics see for instance [32,33]).

We now proceed with the computation in detail. The ti component of the Einstein equations reads

$$\left(\frac{f'}{f}+\frac{h'}{h}+\frac{4}{z}\right)\partial_z\delta g_{ti}-2\partial_z^2\delta g_{ti}=8z^2A_t'\partial_z\delta A_i \tag{2.14}$$

while the i component of the Maxwell equations reads

$$\partial_z\left(\frac{\sqrt{f}\partial_z\delta A_i}{\sqrt{h}}+Q\delta g_{ti}\right)+\beta_{\rm CS}\epsilon_{ij}(\sqrt{fh}Q\partial_j\delta\vartheta-\vartheta'\partial_j\delta A_t)=0. \tag{2.15}$$

Due to the presence of $\delta\vartheta$ and δA_t in (2.15), Eqs. (2.14)–(2.15) do not close between themselves, which implies that solving δg_{ti} explicitly will be a very complicated task, if possible at all.[4] Fortunately as we will see it turns out to be unnecessary to do so.

Integrating (2.15) from the horizon to z we find that

$$\partial_z\delta A_i(z,x^k)=\frac{\sqrt{h(z)}}{\sqrt{f(z)}}\left[-Q\delta g_{ti}(z,x^k)+\beta_{\rm CS}\epsilon_{ij}\int_{z0}^{z}dw(\vartheta'\partial_j\delta A_t-A_t{}'\partial_j\delta\vartheta)\right] \tag{2.16}$$

where we have assumed that $\partial_z\delta A_i(z,x^k)$ is nonsingular at the horizon.

Plugging Eqs. (2.16) into (2.14), and using (2.10) we find that

$$\partial_z\left[\frac{f^{\frac{3}{2}}(z)}{z^2\sqrt{h(z)}}\partial_z\left(\frac{\delta g_{ti}(z,x^k)}{f(z)}\right)\right]=4\beta_{\rm CS}\epsilon_{ij}A_t'(z)\int_{z_0}^{z}dw[A_t'\partial_j\delta\vartheta-\vartheta'\partial_j\delta A_t]. \tag{2.17}$$

The above equation implies that despite the mixing between δA_i and δg_{ti}, the combination $\frac{1}{f}\delta g_{ti}$ remains "massless." Writing $g_{ab}=g_{ab}^{(0)}+g_{ab}^{(1)}$ with $g_{ab}^{(1)}=\frac{\ell^2}{z^2}\delta g_{ab}$, we note that $\frac{1}{f}\delta g_{ti}$ in fact corresponds to $(g^{(1)})_i^t$.

Integrating Eq. (2.18) from the boundary $z=0$ to the horizon z_0, we find that

$$\frac{f^{\frac{3}{2}}(z)}{z^2\sqrt{h(z)}}\partial_z\left(\frac{\delta g_{ti}(z,x^k)}{f(z)}\right)\bigg|_{z=0}=4\beta_{\rm CS}\epsilon_{ij}\int_{z_0}^{0}dzA_t'(z)\int_{z_0}^{z}dw[A_t'\partial_j\delta\vartheta-\vartheta'\partial_j\delta A_t] \tag{2.18}$$

where we have used that at the horizon

$$\delta g_{ti}(z_0,x^i)=0 \tag{2.19}$$

and $\partial_z\delta g_{ti}$ is regular there. Equation (2.19) is analogous to the well-known statement that A_t vanishes at black hole horizons, and is similarly most transparent in Euclidean signature, where a nonzero δg_{ti} at the shrinking time cycle indicates a delta-function contribution to the Einstein tensor. It can be also shown directly from consistency of various components of Einstein equations (see Appendix A).

Now consider the left-hand side of (2.18). With δg_{ti} normalizable, i.e.

$$\delta g_{ti}(z,x^l)=G_i^{(3)}(x^l)z^3+\mathcal{O}(z^4), \tag{2.20}$$

we find

$$3G_i^{(3)}=4\beta_{\rm CS}\epsilon_{ij}\int_{z_0}^{0}dzA_t'(z)\int_{z_0}^{z}dw[A_t'(w)\partial_j\delta\vartheta(w)-\vartheta'(w)\partial_j\delta A_t(w)]. \tag{2.21}$$

Using the standard formulas as in [34–36] (see also Appendix B and the Appendix of [7]), the boundary stress-energy tensor is

$$\delta T_{ti}=\frac{3\ell^2}{2\kappa^2}G_i^{(3)}=-\epsilon_{ij}\partial_j\delta\Phi \tag{2.22}$$

where

$$\delta\Phi=\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\int_0^{z_0}dzA_t'(z)\int_{z_0}^{z}dw[A_t'\delta\vartheta-\vartheta'\delta A_t]=\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\int_{z_0}^{0}dw[A_t'\delta\vartheta-\vartheta'\delta A_t](A_t-\mu). \tag{2.23}$$

In the second equality above we have exchanged the order of integration to perform one integral and used that $A_t(0)=\mu$.

Now consider a configuration of $\delta\vartheta$ which is homogeneous along boundary spatial directions inside a big box but vanishes outside. The above δT_{0i} is nonvanishing only at the edge of the box, but generates an angular momentum proportional to the volume of the box, resulting in an angular momentum density

$$\delta\mathcal{L}=2\delta\Phi. \tag{2.24}$$

We now take the box size to infinity, with $\delta\vartheta$ and δA_t homogeneous everywhere with no dependence on x^i.

2. Angular momentum density

Equations (2.22)–(2.24) apply to infinitesimal variations $\delta\vartheta$ and δA_t around (2.5). To compute $\mathcal{L}$ for (2.5), we need to integrate (2.23) along some trajectory in the space of field configurations from a configuration with $\vartheta=0$ (and thus $\mathcal{L}=0$) to (2.5), i.e. schematically

[4]Note that equations for $\delta\vartheta$ and δA_t are rather complicated. This is especially the case for more general action (2.41).

$$\Phi = \int_{\vartheta=0}^{\vartheta} \delta\Phi \tag{2.25}$$

from which we then find

$$T_{ti} = -\epsilon_{ij}\partial_j\Phi, \qquad \mathcal{L} = 2\Phi. \tag{2.26}$$

At first sight this appears to be an impossible task as solving δA_t in terms of $\delta\vartheta$ is complicated and so is integration over field space as A_t in general also has nontrivial ϑ dependence.

Remarkably, Eq. (2.23) can be written as a total derivative δ in the field configuration space. Choosing a trajectory in configuration space with a fixed μ (i.e. $\delta\mu = 0$) we can rewrite (2.23) as

$$\begin{aligned}\delta\Phi &= \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\int_{z_0}^{0} dw[B'\delta\vartheta - \vartheta'\delta B] \\ &= \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\int_{z_0}^{0} dw[(B\delta\vartheta)' - \delta(B\vartheta')]\end{aligned} \tag{2.27}$$

where $B = A_t^2 - 2\mu A_t$ and in the second line we have used that for arbitrary functions F and G

$$(F\delta G)' - \delta(FG') = F'\delta G - \delta FG'. \tag{2.28}$$

Recall that A_t is zero at the horizon and equal to μ at the boundary. Evaluating the total derivative and taking δ operation outside the integral for the second term, Eq. (2.27) becomes

$$\delta\Phi = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\delta\left[-\mu^2\vartheta(0) + \int_0^{z_0} dw(A_t^2 - 2\mu A_t)\vartheta'\right]. \tag{2.29}$$

Note that in exchanging the order of δ with the integration, there is a term proportional to δz_0, which, however, vanishes as $A_t(z_0) = 0$. Now (2.29) is a total variation and we conclude that

$$\Phi = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\left[-\mu^2\vartheta(0) + \int_0^{z_0} dw(A_t^2 - 2\mu A_t)\vartheta'\right]. \tag{2.30}$$

The above equation can also be slightly rewritten as

$$\Phi = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\left[-\mu^2\vartheta(z_0) + \int_0^{z_0} dw(A_t - \mu)^2\vartheta'\right]. \tag{2.31}$$

Note that Eqs. (2.30)–(2.31) also apply to inhomogeneous configurations as far as the spatial variations are sufficiently small.

When ϑ is dual to a marginal operator, ϑ is constant in the bulk with $\vartheta(0) = \vartheta(z_0) = \vartheta$, and the second term in (2.30) or (2.31) drops out. We then recover the result of [7],

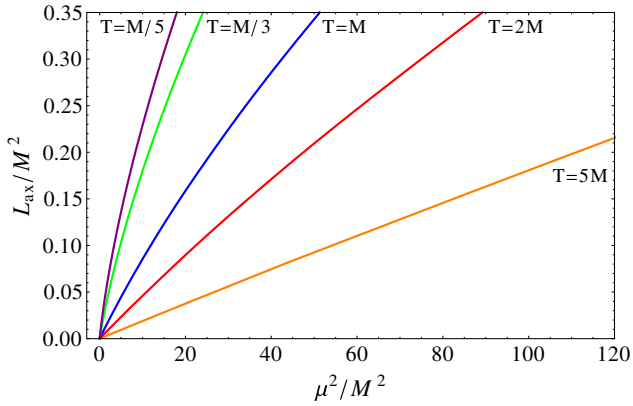

FIG. 1 (Color online) (color online). Angular momentum density as a function of μ^2/M^2 for axionic coupling and non-normalizable scalar field in a quadratic potential with $m^2 = -2$.

$$\mathcal{L} = -\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\mu^2\vartheta. \tag{2.32}$$

For a general relevant operator, the second term in (2.30) or (2.31) is nonzero and the angular momentum density will receive contribution from integration over the bulk full spacetime. In terms of boundary language, the angular momentum receives contributions from degrees of freedom at all scales. Also note that for a relevant operator $\vartheta(0) = 0$, so in (2.30) the sole contribution comes from the second term.

3. An explicit example

We now consider an explicit example. For simplicity we take $V(\vartheta) = \frac{1}{2}m^2\vartheta^2$ with $m^2 = -2$. Thus ϑ is dual to a relevant boundary operator $\mathcal{O}$ in $d = 3$ with $\Delta = 2$. We will consider a solution (2.5) in which ϑ is non-normalizable, i.e. ϑ has the asymptotic behavior near the boundary

$$\vartheta(z) = Mz + O(z^2), \qquad z \to 0 \tag{2.33}$$

where M is a parameter of dimension mass. The solution (2.5) then describes a boundary theory flow upon turning on a relevant perturbation $\int d^3xM\mathcal{O}$, with M interpreted as the bare coupling. Since we are considering the system at a finite density/finite temperature, the flow is cut off at some infrared scale characteristic of finite density/finite temperature physics. In the coordinate system we are using in (2.5), such a scale should correspond to location of the horizon $z_0 \propto s^{-\frac{1}{2}}$ with s the entropy density.

We present plots of the axionic angular momentum as a function of μ^2/M^2 in Figs. 1 and 2 and as a function of μ^2/MT in Figs. 3 and 4. We exhibit the two terms entering Eq. (1.5), as well as the total angular momentum, in Figs. 2 and 4. We note that in the large T regime the angular momentum density grows as $L_{\rm ax} \propto \mu^2 M/T$. This is expected from the general structure of Eq. (1.5) since

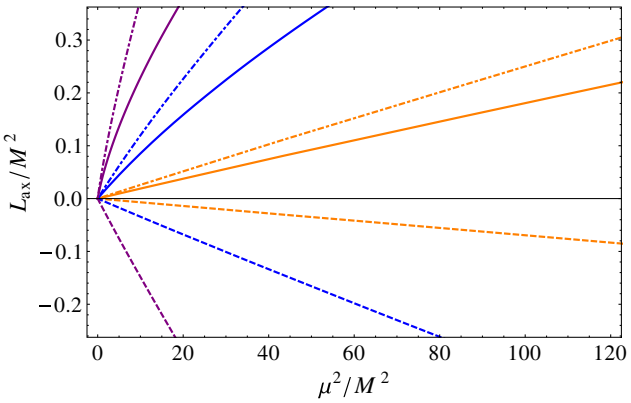

FIG. 2 (Color online) (color online). Angular momentum density as a function of μ^2/M^2 for axionic coupling and non-normalizable scalar field in a quadratic potential with $m^2 = -2$ for $T = 5M$ (orange), $T = M$ (blue) and $T = M/5$ (purple). The total angular momentum is represented by solid lines, the first term in Eq. (1.5) by dot-dashed lines and the second term in Eq. (1.5) by dashed lines.

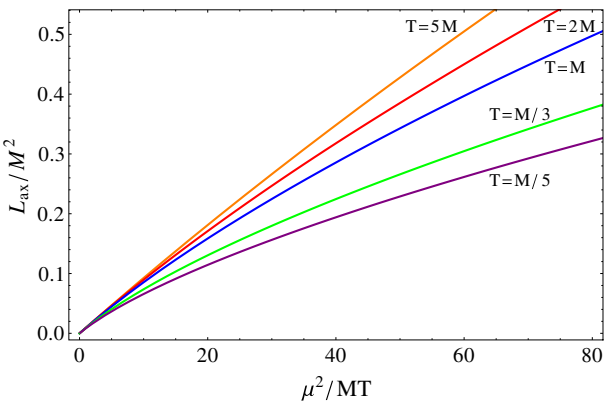

FIG. 3 (Color online) (color online). Angular momentum density as a function of μ^2/MT for axionic coupling and non-normalizable scalar field in a quadratic potential with $m^2 = -2$.

roughly speaking the angular momentum is proportional to A_t^2 and ϑ, while the gauge field is proportional to μ^2 plus corrections and the scalar field is proportional to M/T plus corrections. When $T \to 0$, the angular momentum tends to a finite constant. We also remark that out of the three contributions represented in Figs. 2 and 4, the second term in Eq. (1.5) varies almost linearly with μ^2/MT over the interval we have considered.

B. Electric edge current

Another interesting phenomenon associated to the axionic coupling is a spontaneous generation of the electric current dual to the bulk gauge field. As we will see in Sec. IV, this is closely related to the angular momentum generation when the scalar field ϑ is dual to a marginal operator.

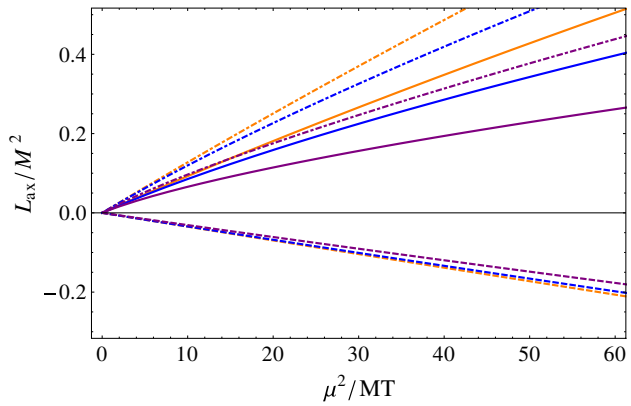

FIG. 4 (Color online) (color online). Angular momentum density as a function of μ^2/MT for axionic coupling and non-normalizable scalar field in a quadratic potential with $m^2 = -2$ for $T = 5M$ (orange), $T = M$ (blue) and $T = M/5$ (purple). The total angular momentum is represented by solid lines, the first term in Eq. (1.5) by dot-dashed lines and the second term in Eq. (1.5) by dashed lines.

The expectation value of the current is defined in terms of the normalizable mode of the bulk gauge field as

$$\delta j_i = -\frac{4\ell^2}{2\kappa^2}\lim_{z\to 0}\frac{\delta A_i}{z} = -\frac{4\ell^2}{2\kappa^2}(\delta A_i)'|_{z=0}. \tag{2.34}$$

Evaluating (2.16) at the boundary we obtain (with δg_{ti} normalizable)

$$\partial_z \delta A_i(z=0, x^k) = -\beta_{\rm CS}\epsilon_{ij}\partial_j \int_0^{z_0} dw[\vartheta'\delta A_t - A_t{}'\delta\vartheta] \tag{2.35}$$

leading to

$$\delta j_i = -\epsilon_{ij}\partial_j \delta\chi \tag{2.36}$$

with

$$\delta\chi = \frac{2\ell^2\beta_{\rm CS}}{\kappa^2}\int_{z_0}^{0} dw[\vartheta'\delta A_t - A_t'\delta\vartheta]. \tag{2.37}$$

Again using (2.28), the above equation can be written as a total variation in the space of configurations

$$\begin{aligned}\delta\chi &= -\frac{2\ell^2\beta_{\rm CS}}{\kappa^2}\delta\left[\mu\vartheta(0) + \int_0^{z_0} dw\vartheta' A_t\right] \\ &= \frac{2\ell^2\beta_{\rm CS}}{\kappa^2}\delta\left[\int_0^{z_0} dw\vartheta A_t'\right].\end{aligned} \tag{2.38}$$

We thus find an electric current

$$j_i = -\epsilon_{ij}\partial_j\chi \tag{2.39}$$

with

$$\chi = \frac{2\ell^2\beta_{\rm CS}}{\kappa^2}\left[\int_0^{z_0} dw\vartheta A_t'\right]. \tag{2.40}$$

C. Bulk universality

The results obtained in the previous subsections extend without modification to most general two-derivative theories of the form

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[R - \frac{1}{2}G^{IJ}(\vartheta^K)\partial_a\vartheta^I\partial^a\vartheta^J - V(\vartheta^K) - \ell^2 Z^{PQ}(\vartheta^K)F^P_{ab}F^{Qab} - \ell^2\beta_{\rm CS}C^{PQ}(\vartheta^K)^*F^{Pab}F^Q_{ab}\Big]. \tag{2.41}$$

In the above I, J, K label different scalar fields, while P, Q label different vector fields, and G^{IJ}, Z^{PQ} and C^{PQ} are functions of scalar fields ϑ^K. They are symmetric and assumed to be invertible. We consider a metric of the form (2.5) with

$$\vartheta^I = \vartheta^I(z), \qquad A^P_a = A^P_t(z)\delta^t_a, \qquad A^P_t(0) = \mu^P \tag{2.42}$$

where μ^P is the chemical potential for boundary conserved current J^P dual to A^P_a.

The discussion exactly parallels that of Sec. II A so below we will simply list the counterparts of the key equations there.

Background equations of motion (2.9)–(2.10) now become

$$A^{P\prime}_t(z) = (Z^{-1})^{PR}Q^R\sqrt{f(z)h(z)} \tag{2.43}$$

with Q^R the charge density for J^R and

$$4\sqrt{fh}(Z^{-1})^{PR}Q^PQ^R = \left(\frac{f'}{z^2\sqrt{fh}}\right)'. \tag{2.44}$$

As before we consider general small perturbations generated by a small and slow-varying $\delta\vartheta^I(z, x^i)$ and make the gauge choice

$$\delta A^P_z = 0, \qquad \delta g_{zt} = 0. \tag{2.45}$$

Equations (2.14) and (2.15) then generalize respectively to

$$\left(\frac{f'}{f} + \frac{h'}{h} + \frac{4}{z}\right)\partial_z\delta g_{ti} - 2\partial_z^2\delta g_{ti} = 8z^2Z^{PQ}A^{P\prime}_t\partial_z A^Q_i \tag{2.46}$$

and

$$\partial_z\left(\frac{\sqrt{f(z)}Z^{PQ}\partial_z\delta A^Q_i}{\sqrt{h(z)}} + Q^P\delta g_{ti}\right) + \beta_{\rm CS}\epsilon_{ij}\partial_j(\delta C^{PQ}A^{Q\prime}_t - C^{PQ\prime}\delta A^Q_t) = 0. \tag{2.47}$$

Then identical manipulations as before lead to (2.26) with

$$\Phi = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\left[-\mu^P\mu^Q C^{PQ}(0) + \int_0^{z_0} dw(A^P_tA^Q_t(w) - 2\mu^PA^Q_t(w))C^{PQ\prime}(w)\right] \tag{2.48}$$

or equivalently

$$\Phi = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\left[-\mu^P\mu^Q C^{PQ}(z_0) + \int_0^{z_0} dw(A^P_t - \mu^P)(A^Q_t - \mu^Q)C^{PQ\prime}\right]. \tag{2.49}$$

III. GRAVITATIONAL CHERN-SIMONS TERM

In this section, we consider the induced stress tensor and angular momentum density for bulk theories where parity violation is generated by the gravitational Chern-Simons coupling, ϑ^*RR. We will first consider a simple example with a relevant scalar operator and then generalize the discussion to generic theories. The discussion is similar to that of the last section, so we will be briefer.

A. Relevant scalar field

Consider the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\vartheta)^2 - V(\vartheta) - \frac{\alpha_{\rm CS}\ell^2}{4}\vartheta^*RR\right] \tag{3.1}$$

where $\alpha_{\rm CS}$ is a constant and ϑ is dual to a relevant (or marginal) pseudoscalar boundary operator. In (3.1)

$$^*RR = {}^*R^{abcd}R_{bacd}, \qquad {}^*R^{abcd} = \frac{1}{2}\epsilon^{cdef}R^{ab}{}_{ef} \tag{3.2}$$

and ϵ^{abcd} is the totally antisymmetric tensor with $\epsilon^{012z} = 1/\sqrt{-g}$. The equations of motion are

$$R_{ab} - \frac{1}{2}\partial_a\vartheta\partial_b\vartheta - \frac{1}{2}g_{ab}V(\vartheta) = \alpha_{\rm CS}\ell^2C_{ab},$$
$$\frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\vartheta) - \frac{\alpha_{\rm CS}\ell^2}{4}{}^*RR = 0 \tag{3.3}$$

where $C^{ab} \equiv \nabla_c(\nabla_d\vartheta^*R^{c(ab)d})$.

We again consider a solution of the form (2.5) (without the gauge field). The strategy is the same as before. We

consider a small and slowly varying perturbation $\delta\vartheta(z, x^i)$ and work out the momentum response δT_{ti} to order $O(\epsilon)$ where the power ϵ counts the number of spatial derivatives of $\delta\vartheta$. We then write the resulting expression as a total variation in the space of field configurations which enables us to find the angular momentum associated with (2.5).

We will choose a gauge where $\delta g_{tz} = \delta g_{xx} = \delta g_{yy} = 0$. With the Einstein equations schematically reading

$$\mathrm{LHS}_{ab} = \alpha_{\mathrm{CS}}\ell^2 C_{ab} \tag{3.4}$$

we note that in this gauge,

$$\mathrm{LHS}_{ti} = \frac{-z^2}{2\sqrt{f(z)h(z)}}\partial_z\left[\frac{f^{\frac{3}{2}}(z)}{z^2\sqrt{h(z)}}\partial_z\left(\frac{\delta g_{ti}(z,x^k)}{f(z)}\right)\right] \tag{3.5}$$

and δg_{zi}, δg_{xy} are all at least of order $O(\epsilon)$. We then find to $O(\epsilon)$, C_{ti} can be written as

$$C_{ti} = \frac{z^2}{\ell^2\sqrt{fh}}\epsilon_{ij}\partial_j\delta\Psi \tag{3.6}$$

with

$$\delta\Psi = K' + \left(\frac{f'^2}{8fh}\right)'\delta\vartheta - \frac{f'^2\vartheta'\delta g_{tt}}{8f^2h} + \frac{f'\vartheta'\delta g_{tt}{}'}{4fh} + \frac{f'^2\vartheta'\delta g_{zz}}{8fh^2} \tag{3.7}$$

and

$$K = \frac{ff'h' + h(f'^2 - 2ff')}{8fh^2}\delta\vartheta - \frac{f'\vartheta'\delta g_{zz}}{8h^2} + \frac{f'\vartheta'\delta g_{tt}}{8fh} - \frac{\vartheta'\delta g_{tt}{}'}{4h}. \tag{3.8}$$

Then following similar manipulations as in (2.18)–(2.22) we find that

$$\delta T_{ti} = -\epsilon_{ij}\partial_j\delta\Phi \tag{3.9}$$

with

$$\delta\Phi = \frac{\alpha_{\mathrm{CS}}\ell^2}{\kappa^2}\int_{z_0}^{0}\delta\Psi dz. \tag{3.10}$$

Note that $\delta g_{tt} = -\delta f$ and $\delta g_{zz} = \delta h$ and $\delta\Psi$ can be further written as

$$\delta\Psi = K' + \left(\frac{f'^2\delta\vartheta}{8fh}\right)' - \delta\left(\frac{f'^2}{8fh}\vartheta'\right). \tag{3.11}$$

It can then be immediately checked that the boundary terms coming from K are all zero with the assumption of the asymptotic behavior

$$f(z) = 1 + \#z^{2+2\alpha} + \cdots, \qquad h(z) = 1 + \#z^{2\beta} + \cdots \tag{3.12}$$

where $\alpha > 0$, $\beta > 0$. We then note further that

$$\int_0^{z_0} dz\left[\left(\frac{f'^2\delta\vartheta}{8fh}\right)' - \delta\left(\frac{f'^2}{8fh}\vartheta'\right)\right] = \delta\left(\int_0^{z_0} dz\left(\frac{f'^2}{8fh}\right)'\vartheta\right) - \delta\left(\frac{f'^2}{8fh}\bigg|_{z_0}\right)\vartheta(z_0) \tag{3.13}$$

where the second term is proportional to δT, and thus vanishes if we choose a path in configuration space such that $\delta T = 0$. Collecting the above we thus find $\delta\Phi$ is a total variation with

$$\Phi = -\frac{\alpha_{\mathrm{CS}}\ell^2}{\kappa^2}\int_0^{z_0} dz\left(\frac{f'^2}{8fh}\right)'\vartheta \tag{3.14}$$

$$= -\frac{2\pi^2\alpha_{\mathrm{CS}}\ell^2}{\kappa^2}T^2\vartheta(z_0) + \frac{\alpha_{\mathrm{CS}}\ell^2}{8\kappa^2}\int_0^{z_0} dz\left(\frac{f'^2}{fh}\right)\vartheta'. \tag{3.15}$$

The angular momentum is thus given by

$$\mathcal{L} = 2\Phi. \tag{3.16}$$

For a marginal ϑ, ϑ is independent of z and only the first term in (3.22) is present. We then find a universal result which is independent of specific forms of f and h

$$\mathcal{L} = -\frac{4\pi^2\alpha_{\mathrm{CS}}\ell^2}{\kappa^2}T^2\vartheta. \tag{3.17}$$

B. Generalizations

The above discussion can be immediately generalized to theories of the form

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}G^{IJ}(\vartheta^K)\partial_a\vartheta^I\partial^a\vartheta^J - V(\vartheta^K) - \ell^2 Z^{PQ}(\vartheta^K)F^P_{ab}F^{Qab} - \frac{\alpha_{\mathrm{CS}}\ell^2}{4}C(\vartheta^K)^*RR\right]. \tag{3.18}$$

Fixing the gauge $A^P_z = 0$, one finds that

$$\partial_z\delta A^P_i(z, x^k) = -Q^P\sqrt{\frac{h(z)}{f(z)}}\delta g_{ti}(z, x^k). \tag{3.19}$$

From (3.19) one then finds that the ti component of the Einstein equations can again be written as

$$\mathrm{LHS}_{ti} = \alpha_{\mathrm{CS}}\ell^2 C_{ti} \tag{3.20}$$

with LHS_{ti} given by (3.5) and C_{ti} by (3.6)–(3.8) except that everywhere in C_{ti} the pseudoscalar ϑ is replaced by $C(\vartheta^I)$. In this case we thus find that

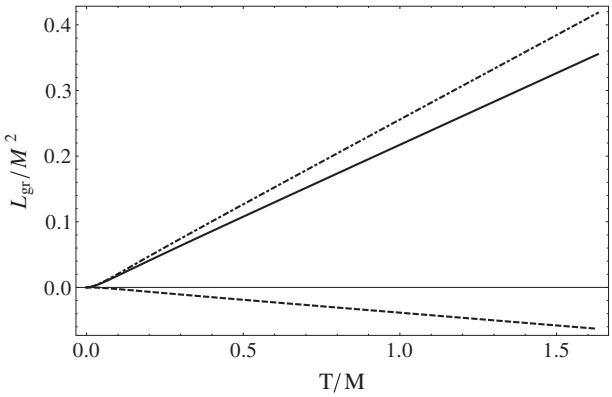

FIG. 5. Angular momentum density as a function of T/M for gravitational Chern-Simons coupling and non-normalizable scalar field in a quadratic potential with $m^2 = -2$, at $\mu = 0$. The total angular momentum is represented by solid lines, the first term in Eq. (1.6) by dot-dashed lines and the second term in Eq. (1.6) by dashed lines.

$$\Phi = -\frac{\alpha_{\rm CS}}{\kappa^2}\int_0^{z_0} dz\left(\frac{f'^2}{8fh}\right)' C(\vartheta^I) \tag{3.21}$$

$$= -\frac{2\pi^2\alpha_{\rm CS}\ell^2}{\kappa^2}T^2 C(\vartheta^I(z_0)) + \frac{\alpha_{\rm CS}\ell^2}{8\kappa^2}\int_0^{z_0} dz\left(\frac{f'^2}{fh}\right)C(\vartheta^I)' \tag{3.22}$$

We also note in passing that in this case there is no electric edge current as

$$\delta j_i^P = -\frac{4\ell^2}{2\kappa^2}\lim_{z\to 0}\frac{\delta A_i^P}{z} = 0 \tag{3.23}$$

where we have used (3.19) and that $\delta g_{ti} \sim O(z^3)$.

C. An explicit example

We now examine an explicit example. For simplicity we once again consider the setup of Sec. II A 3 with $V(\vartheta) = \frac{1}{2}m^2\vartheta^2$, $m^2 = -2$ and ϑ non-normalizable with M the scalar source. We exhibit plots of the gravitational angular momentum as a function of T/M in Fig. 5, with the two terms entering Eq. (1.6) presented separately. We remark that the plots are almost linear, which can be understood from the general structure of Eq. (1.5) as follows: the geometric factor under the integral is roughly proportional to T^2 to leading order, while the scalar field is proportional to M/T at leading order, making the overall leading order dependence $L_{\rm gr} \propto MT$.

IV. RELATION TO THE CHIRAL MAGNETIC EFFECT AND THE AXIAL MAGNETIC EFFECT

When the scalar field is marginal it is possible to relate our results to the chiral magnetic effect and to the axial magnetic effect in $3+1$ dimensions [26–31] via dimensional reduction, as we now explain.

In $3+1$ dimensions, the gauge anomaly,

$$\partial_\alpha j^\alpha = \frac{b_{\rm CS}}{4}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{4.1}$$

is known to cause spontaneous generation of the corresponding current,

$$j^i = b_{\rm CS}\mu\epsilon^{ijk}F_{jk}, \tag{4.2}$$

and of the momentum density,

$$T^{0i} = \frac{b_{\rm CS}}{2}\mu^2\epsilon^{ijk}F_{jk}, \tag{4.3}$$

where i, j, $k = 1, 2, 3$ are spatial directions in $3+1$ dimensions. These effects are called the chiral magnetic effect for j^i and the axial magnetic effect for T^{0i}. (The formulas derived in [27] in the Landau frame contain terms in higher powers of μ. The formulas in the above are in the laboratory frame [28].)

In comparison, the Chern-Simons term in our bulk action in $3+1$ dimensions,

$$S_{\rm CS} = -\frac{\beta_{\rm CS}\ell^2}{2\kappa^2}\int d^4x\sqrt{-g}\vartheta^* F^{ab}F_{ab}, \tag{4.4}$$

gives rise to an anomalous divergence of the current j^α on the boundary in $2+1$ dimensions as

$$\partial_\alpha j^\alpha = \frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\epsilon^{\alpha\beta\gamma}\partial_\alpha\vartheta F_{\beta\gamma}, \tag{4.5}$$

where $F_{\beta\gamma}$ is the background gauge field for the boundary CFT. Since it is the dimensional reduction of the chiral anomaly (4.1) in $3+1$ dimensions, where the scalar field ϑ in the bulk is identified with the extra component $\vartheta = A_3$ and $F_{i3} = \partial_i\vartheta$, we expect effects corresponding to the chiral magnetic effect (4.2) and to the axial magnetic effect (4.8) to be

$$j^i = 2b_{\rm CS}\mu\epsilon^{ij}\partial_j\vartheta,$$
$$T^{0i} = b_{\rm CS}\mu^2\epsilon^{ij}\partial_j\vartheta, \tag{4.6}$$

where we should identify $b_{\rm CS} = \beta_{\rm CS}\ell^2/\kappa^2$.

We can also include effects due to the axial-gravitational anomalies. In $3+1$ dimensions, the axial-gravitational anomaly,

$$\partial_\alpha j^\alpha = \frac{a_{\rm CS}}{8\pi^2}\epsilon^{\gamma\delta\eta\theta}R^{\alpha\beta}{}_{\eta\theta}R_{\beta\alpha\gamma\delta}, \tag{4.7}$$

is known to generate the momentum current

$$T^{0i} = \frac{a_{\rm CS}}{2}T^2\epsilon^{ijk}F_{jk}, \tag{4.8}$$

but not the current j^α itself. The corresponding effect in $2+1$ dimensions should be

$$T^{0i} = a_{\rm CS} T^2 \epsilon^{ij} \partial_j \vartheta, \tag{4.9}$$

with the identification, $a_{\rm CS} = 2\pi^2 \alpha_{\rm CS} \ell^2/\kappa^2$.

The dimensional reduction of the chiral magnetic effect and axial magnetic effect, (4.6) and (4.9), are in agreement with Eqs. (2.39) and (2.40) and consistent with results in our previous paper [7], where the scalar field ϑ is dual to a marginal operator on the boundary CFT.

The main results in this paper, however, are for ϑ dual to a relevant operator, which cannot be obtained by dimensional reduction of a massless gauge field in $4+1$ dimensions. There may be a generalization of the chiral magnetic effect and of the axial magnetic effect in $3+1$ dimensions which would correspond to dimensional oxidation of the effects studied in this paper, and we leave this possibility for future investigation.

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Note added.—When this paper was almost complete, we received the paper [37], in which holographic models with nonzero angular momentum and Hall viscosity are discussed. Their models are different from those discussed in this paper.

APPENDIX A: BOUNDARY CONDITION AT THE HORIZON

The zt component of the Einstein equations reads

$$f'(z)\partial_i \delta g_{ti}(z, x^i) - f(z)\partial_i \delta g'_{ti}(z, x^i) = 0 \tag{A1}$$

which can be integrated to give

$$\partial_i \delta g_{ti}(z, x^i) = f(z) W(x^i). \tag{A2}$$

Since $f(0) = 1$ and we choose δg_{ti} to be a normalizable perturbation, we must have $W(x^i) = 0$ so we conclude

$$\partial_i \delta g_{ti}(z, x^i) = 0. \tag{A3}$$

Using the ii component of the background Einstein equations the ti component of the Einstein equations reads

$$\begin{aligned} &+ 2zfh^2 \epsilon_{ij} \partial_j (\partial_x \delta g_{ty} - \partial_y \delta g_{tx}) - 2zfh\delta g''_{ti} + (zfh' \\ &\quad + zhf' + 4fh)\delta g'_{ti} - 8z^3 fhA'_t \delta A'_i(z, x^i) = 0. \end{aligned} \tag{A4}$$

Using (A3) $\epsilon_{ij}\partial_j(\partial_x \delta g_{ty} - \partial_y \delta g_{tx}) = -\partial^2 \delta g_{ti}$ with $\partial^2 = \partial^i \partial_i$ this is

$$\begin{aligned} &- 2zfh^2 \partial^2 \delta g_{ti} + (zfh' + zhf' + 4fh)\delta g'_{ti} \\ &\quad - 2zfh\delta g''_{ti} - 8z^3 fhA'_t \delta A'_i = 0. \end{aligned} \tag{A5}$$

We now count the divergences in (A5), using that near the horizon

$$h(z) = \frac{\mathcal{K}}{f(z)} + K_0 + K_1(z - z_0) + \cdots, \tag{A6}$$

$$h'(z) = \mathcal{K}\frac{f'(z)}{f^2(z)} + K_1 + \cdots, \tag{A7}$$

with $\mathcal{K}$ an arbitrary constant. Since the gauge and scalar fields do not diverge at the horizon we obtain the lhs of the Einstein equations to be

$$-\frac{1}{2}\partial^2 \delta g_{ti}(z_0, x^i) = 0 \tag{A8}$$

and imposing the boundary condition $\delta g_{ti}(z_0, x^i) \to 0$ at spatial infinity we conclude

$$\delta g_{ti}(z_0, x^i) = 0. \tag{A9}$$

APPENDIX B: REGULARIZATION AND RENORMALIZATION

Consider the action

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[&R - \frac{1}{2}G^{IJ}(\vartheta^K)\partial_a \vartheta^I \partial^a \vartheta^J \\ &- V(\vartheta^K) - \ell^2 Z^{PQ}(\vartheta^K) F^P_{ab} F^{Qab} + S_{cs}\Big], \end{aligned} \tag{B1}$$

where either

$$S_{\rm cs} = -\ell^2 \beta_{\rm CS} C^{PQ} (\vartheta^K)^* F^{Pab} F^Q_{ab} \tag{B2}$$

or

$$S_{\rm cs} = -\frac{\alpha_{\rm CS}\ell^2}{4} \vartheta^{I=0*} RR. \tag{B3}$$

Note the gravitational Chern-Simons term can always be written in this form via field redefinition.

A priori, there are four possible contributions that need to be accounted for: the usual Gibbons-Hawking-York boundary term, a term arising from the variation of the (axionic or gravitational) Chern-Simons term $S_{\rm cs}$, potential additional terms that must be added for the Dirichlet boundary-value problem to be well-defined and local counterterms (see the Appendix of [7] for details). Thus, we can write

$$T^{\rm bdy}_{\alpha\beta} = \frac{1}{2\kappa^2}(2K_{\alpha\beta} - 2h_{\alpha\beta}K + T^{\rm cs}_{\alpha\beta} + T^{\rm reg}_{\alpha\beta} - T^{\rm ct}_{\alpha\beta}). \tag{B4}$$

The CFT stress-energy tensor is obtained by computing the boundary stress-energy tensor $T^{\rm bdy}_{\alpha\beta}$ on a plane at finite z parallel to the boundary, multiplying by an appropriate power of z (z^{-1} in our case for the stress-energy tensor with both indices down) and taking the $z \to 0$ limit, according to the standard AdS/CFT dictionary (see e.g. [34–36]).

Let us first concentrate on possible Chern-Simons and regularization contributions to the boundary stress-energy tensor. As explained in the Appendix of [7], the gravitational Chern-Simons term does not contribute to the boundary stress-energy tensor and also does not require additional regularization terms. Similarly, the axionic Chern-Simons term is topological, so under the variation we consider it will not contribute to the boundary stress-energy tensor, nor will it require regularization terms.

We are thus left to analyze possible counterterms. For planar boundaries there is a standard counterterm obtained by adding a cosmological constant term on the boundary, which does not depend on the presence of scalar fields. In addition, there can be scalar-field dependent counterterms, which we can schematically write by adding

$$\sqrt{-h} H(\vartheta^I) \tag{B5}$$

to the action, with H some function and h_{ab} the induced metric,

$$h_{ab} = g_{ab} - n_a n_b, \qquad n_a = \frac{1}{g^{zz}} \delta^z_a. \tag{B6}$$

The scalar field counterterms contribute

$$T^{{\rm ct},\vartheta}_{ti} \sim H(\vartheta^I) h_{ti} \tag{B7}$$

to the ti component of $T^{\rm bdy}_{\alpha\beta}$. However, since we are considering the metric perturbations to be normalizable

$$h_{ti} \sim \mathcal{O}(z) \tag{B8}$$

near the boundary. Furthermore, $H(\vartheta^I)$ cannot contain marginal scalar fields, so it must consist entirely of scalar fields decaying as some positive power of z towards the boundary. Since the counterterms must vanish in the absence of any scalar field $H(\vartheta^I)$ must be proportional to at least one positive power of ϑ^I, which introduces at least one more positive power of z in $T^{{\rm ct},\vartheta}_{ti}$. Thus

$$T^{{\rm ct},\vartheta}_{ti} \sim \mathcal{O}(z^{1+\gamma}), \qquad \gamma > 0 \tag{B9}$$

and the scalar field counterterms decay at least one power of z^γ too fast near the boundary to contribute to the CFT stress-energy tensor.

We are thus left with the usual boundary stress-energy tensor in the ti component,

$$T^{\rm bdy}_{ti} = \frac{1}{\kappa^2}\left(K_{ti} - h_{ti}K - \frac{2}{\ell} h_{ti}\right). \tag{B10}$$

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