Supplementary Material for "Topological Response Theory of Abelian Symmetry-Protected Topological Phases in Two Dimensions"

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(Dated: February 25, 2014)

I. ABELIAN CHERN-SIMONS-HIGGS THEORY

In the main text we have argued that the gauged SPT phases can be described by an Abelian Chern-Simons theory with the $\mathbb{U}(1)$ gauge fields Higgsed. In this section we provide derivations of the flux statistics in the Abelian Chern-Simons-Higgs theory.

Recall that after we integrate out the gapped degrees of freedom in the SPT phases protected by a symmetry group $G=\prod_{\alpha=1}^k \mathbb{Z}_{m_\alpha}$, we arrive at the following Lagrangian density:

$$\mathcal{L}_{\text{CSH}}=\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\tilde{K}_{\alpha\beta}\partial_\nu A_\lambda^\beta-\sum_\alpha\Big[\frac{1}{2}\big|(\partial_\mu-im_\alpha A_\alpha)\varphi_\alpha)\big|^2-V(\varphi_\alpha)\Big]. \tag{1}$$

Here A_α are external $\mathbb{U}(1)^k$ gauge fields, which we take to be semi-classical, adiabatically varying background fields.

We assume that the amplitudes of the Higgs fields are fixed by the potential energy term $V(\varphi_\alpha)$ and only the phase degrees of freedom remain. First we take this energy scale to be infinite. Heuristically speaking, the gauging is in the "weak" sense and the fluxes are semi-classical objects. Then A field does not have its own dynamics. In fact, it should be thought as being slaved to the vortex current $j_{v,\alpha}$. In other words, we can integrate over A but with a contraint $\frac{1}{\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda=j_v^\mu$ enforced. With this in mind we can concentrate on the Chern-Simons part and introduce Lagrange multipliers a_μ to resolve the constraint:

$$\mathcal{L}_{\text{eff}}=\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\tilde{K}_{\alpha\beta}\partial_\nu A_\lambda^\beta+a_\mu^\alpha\left(\frac{m_\alpha}{\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_{\alpha\lambda}-j_{v,\alpha}^\mu\right). \tag{2}$$

Now we can integrate out A_μ yielding a Chern-Simons action for the gauge fields a_μ:

$$\mathcal{L}_{\text{eff}}=-\frac{m_\alpha\tilde{K}^{-1}_{\alpha\beta}m_\beta}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu^\alpha\partial_\nu a_\lambda^\beta-a_\mu^\alpha j_{v,\alpha}^\mu. \tag{3}$$

We can then compute the braiding statistics of the fluxes directly from the action.

Now we step back and derive the full dynamical gauge theory. We write $\varphi_\alpha=v_\alpha e^{i\theta_\alpha}$ and substitute into the Lagrangian density:

$$\mathcal{L}_{\text{eff}}=\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\tilde{K}_{\alpha\beta}\partial_\nu A_\lambda^\beta-\sum_\alpha\frac{v_\alpha^2}{2}\Big(m_\alpha A_\mu^\alpha-\partial_\mu\theta_\alpha\Big)^2. \tag{4}$$

First we perform the Hubbard-Stratonovich transformation of the quadratic term $\propto(mA-\partial\theta)^2$ and write

$$\mathcal{L}_{\text{eff}}=\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\tilde{K}_{\alpha\beta}\partial_\nu A_\lambda^\beta-\sum_\alpha\Big[\frac{1}{v_\alpha^2}\xi_\alpha^2-\xi_\alpha^\mu\Big(m_\alpha A_\mu^\alpha-\partial_\mu\theta_\alpha\Big)\Big]. \tag{5}$$

Here ξ_μ is the Hubbard-Stratonovich field. Then decompose the phase field as $\theta_\alpha=\eta_\alpha+\zeta_\alpha$ where η_α is the smooth part of the phase fluctuation and ζ_α is the singular(vortex) part determined by j_v. Integrate out the smooth part of the phase fields η_α, we obtain the constraint $\partial_\mu\xi_\alpha^\mu=0$, which can be resolved as $\xi_\alpha^\mu=\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu b_{\alpha\lambda}$. So we obtain the following dual representation

$$\mathcal{L}=\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\tilde{K}_{\alpha\beta}\partial_\nu A_\lambda^\beta+\frac{m_\alpha}{2\pi}\varepsilon^{\mu\nu\lambda}A_\mu^\alpha\partial_\nu b_\lambda^\alpha+\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}j_\mu\partial_\nu A_{\alpha\lambda}+\varepsilon^{\mu\nu\lambda}j_{v,\mu}\partial_\nu b_{\alpha\lambda}-\sum_\alpha\frac{1}{4\pi^2v_\alpha^2}(\partial_\mu b_{\alpha\nu}-\partial_\nu b_{\alpha\mu})^2. \tag{6}$$

Here we have also included the charge current j coupled to A. The Maxwell term is less relevant than the Chern-Simons type terms and can be safely neglected. The remaining action is a doubled Chern-Simons theory. We can pack the theory into a K-matrix:

$$K = \begin{pmatrix} 0 & \mathbf{m} \\ \mathbf{m} & \tilde{\mathbf{K}} \end{pmatrix}. \tag{7}$$

Here $\mathbf{m} = [m_1, m_2, \dots]$ is a diagonal matrix. The statistics of thecharge and flux excitations can be easily computed by taking the inverse of K. This also establishes formally the connection between the SPT phases and the Abelian intrinsic topological order given by the K matrix given in (7).

II. INTRINSIC TOPOLOGICAL ORDER AND CLASSIFICATION OF TOPOLOGICAL RESPONSES

We elaborate on the nonequivalence between the classification of the gauged SPT phases as a "topological response" theory and the classification of the intrinsic topological order. As emphasized in the main text, the classification of the response theory is not completely equivalent to the classification of the intrinsic topological order defined by the Chern-Simons-Higgs theory. In classifying the intrinsic topological orders, all the gauge fluxes are regarded as dynamical deconfined objects. Two (Abelian) topological phases are equivalent as long as they have the same quasiparticle braiding matrices (so-called T and S matrices), regardless of how the quasiparticles are labeled. This kind of equivalence relation is nothing but the $\mathbb{GL}(N,\mathbb{Z})$ equivalence for the $\mathbf{K}$ matrix.

To illustrate the difference, first we start from $G = \mathbb{Z}_n$. As disucussed in the main text, the n different SPTs after being gauged result in the following n gauge theories:

$$\mathbf{K} = \begin{pmatrix} 0 & n \\ n & 2p \end{pmatrix}, p = 0, 1, \dots, n-1. \tag{8}$$

One might wonder these gauge theories are all distinct. However, this is not generally true. In fact, for every odd $n \geq 5$, we have the following $\mathbb{GL}(2,\mathbb{Z})$ equivalence between $p = 2$ and $p = \frac{n+1}{2}$:

$$W^T \begin{pmatrix} 0 & n \\ n & 4 \end{pmatrix} W = \begin{pmatrix} 0 & n \\ n & n+1 \end{pmatrix}, W = \begin{pmatrix} -2 & -1 \\ n & \frac{n+1}{2} \end{pmatrix}. \tag{9}$$

This is just one example and there could be more "collapse" for general n.

Our second example is the gauge group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. The dynamical gauged theory is given by the following $\mathbf{K}$ matrices:

$$\mathbf{K} = \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{2}_{2\times 2} \\ \mathbf{2}_{2\times 2} & \tilde{K} \end{pmatrix}, \tilde{K} = \begin{pmatrix} k & l \\ l & p \end{pmatrix}, k, p \in \{0, 2\}, l \in \{0, 1\}. \tag{10}$$

The possible choices of k, p and l yields all the $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{U}(1)) = \mathbb{Z}_2^3$ cohomology classes.

In fact, the 8 classes listed in (10) reduce to only 4 under generic $\mathbb{SL}(4,\mathbb{Z})$ equivalence. In terms of the $\tilde{K}$ matrices, there are only the following four different intrinsic topological orders:

$$\begin{gathered} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}. \end{gathered} \tag{11}$$

Here $\sim$ denotes the equivalence of the derived intrinsic topological orders.

In the response theory, we are not allowed to permute the gauge fluxes from different subgroups of the symmetry group since they correspond to different physical symmetries.

III. EQUIVALENCE BETWEEN FERMIONIC AND BOSONIC SPT PHASES WITH $\mathbb{Z}_m$ SYMMETRY

We demonstrate directly that when m is odd all $\mathbb{Z}_m$ fermionic SPT phase are equivalent to bosonic ones. Let us consider the bulk Chern-Simons theory of $\mathbb{Z}_m$ fermionic SPT phase

$$\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{q}_g = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{12}$$

Here g denotes the generator of the $\mathbb{Z}_m$ symmetry. We then add a trivial phase given by $\mathbf{K} = \sigma_z$ with trivial symmetry transformation on the edge bosons $\mathbf{q}'_g = \begin{pmatrix} p \\ p \end{pmatrix}$ where $p \in \mathbb{Z}$. We pack the whole system into a 4×4 $\mathbf{K}$ given by $\mathbf{K} = \sigma_z \otimes \mathbf{1}_{2\times 2}$.

We then perform the following $\mathbb{SL}(4, \mathbb{Z})$ transformation

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}. \tag{13}$$

Under $\mathbf{W}$ the $\mathbf{K}$ matrix becomes $\mathbf{K} = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_x \end{pmatrix}$. So the first two components are describing fermionic systems and the last two bosonic ones. We denote the symmetry vector for the collectively as $\tilde{\mathbf{q}}_g = (q_1, q_2, p, p)^T$. The edge modes in the new basis are denoted by

$$\tilde{\boldsymbol{\phi}} = \begin{pmatrix} \phi_1^f \\ \phi_2^f \\ \phi_1^b \\ \phi_2^b \end{pmatrix} = \begin{pmatrix} \phi_1 - \phi'_1 - \phi'_2 \\ \phi_2 \\ \phi'_1 + \phi'_2 \\ \phi_1 - \phi'_2 \end{pmatrix} \tag{14}$$

Under the $\mathbb{SL}(4, \mathbb{Z})$ transformation $\mathbf{W}$, the vector $\tilde{\mathbf{q}}_g \to \mathbf{W}^{-1}\tilde{\mathbf{q}}_g = (q_1 - 2p, q_2, 2p, q_1 - p)^T$.

If $q_1 - q_2$ is even, we let $p = \frac{q_1 - q_2}{2}$ and then $\tilde{\mathbf{q}}_g = (q_2, q_2, q_1 - q_2, \frac{q_1+q_2}{2})^T$. Thus the fermionic SPT is equivalent to a bosonic one with $\mathbf{K} = \sigma_x, \mathbf{q}_g^b = (q_1 - q_2, \frac{q_1+q_2}{2})^T$.

If $q_1 - q_2$ is odd, it seems that $p = \frac{q_1 - q_2}{2}$, being a half integer, is not physical. Here the fermionic nature plays a crucial. This is most easily understood from the edge modes. The edge modes $\boldsymbol{\phi}' = (\phi'_1, \phi'_2)^T$ transforms under the $\mathbb{Z}_m$ symmetry as

$$U_g \boldsymbol{\phi}' U_g^\dagger = \boldsymbol{\phi}' + \frac{2\pi p}{m} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{15}$$

When p is a half integer, we have

$$U_g^m \boldsymbol{\phi}' (U_g^\dagger)^m = \boldsymbol{\phi}' + 2\pi p \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \boldsymbol{\phi}' + \pi \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{16}$$

which is projectively the identity in a fermionic system.

We now turn to the corresponding bosonic SPT. We notice that

$$U_g^m \boldsymbol{\phi}^b (U_g^\dagger)^m = \boldsymbol{\phi}^b + \pi \begin{pmatrix} 0 \\ q_1 + q_2 \end{pmatrix} \tag{17}$$

which is not consistent with $U_g^m = 1$ in the bosonic case. Again the identity transformation in a fermionic system can be realized projectively:

$$\begin{aligned} \phi_1 &\to \phi_1 + \pi \\ \phi_2 &\to \phi_2 + \pi, \end{aligned} \tag{18}$$

which means that

$$\phi_2^b = \phi_1 - \phi'_2 \to \phi_2^b + \pi \tag{19}$$

is also an identity transformation. So we can freely add $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$ to $\boldsymbol{\phi}^b$ and as a result U_g^m can differ by $\begin{pmatrix} 0 \\ m\pi \end{pmatrix}$, which makes the derived relation legitimate for odd m, but not for even m. We therefore prove that the fermionic SPT phases with $\mathbb{Z}_m$ are all equivalent to bosonic ones when m is odd.