FRENKEL-GROSS' IRREGULAR CONNECTION AND HEINLOTH-NGÔ-YUN'S ARE THE SAME

XINWEN ZHU

We show that the irregular connection on $\mathbb{G}_m$ constructed by Frenkel-Gross ([FG]) and the one constructed by Heinloth-Ngô-Yun ([HNY]) are the same, which confirms the Conjecture 2.14 of [HNY].

The proof is simple, modulo the big machinery of quantization of Hitchin's integrable systems as developed by Beilinson-Drinfeld ([BD]). The idea is as follows. Let $\mathcal{E}$ be the irregular connection on $\mathbb{G}_m$ as constructed by Frenkel-Gross. It admits a natural oper form. We apply the machinery of Beilinson-Drinfeld to produce an automorphic D-module on the corresponding moduli space of G-bundles, with Hecke eigenvalue $\mathcal{E}$. We show that this automorphic D-module is equivariant with respect to the unipotent group $I(1)/I(2)$ (see [HNY] for the notation) against the non-degenerate additive character Ψ. By the uniqueness of such D-modules on the moduli space, one knows that the automorphic D-module constructed using the Beilinson-Drinfeld machinery is the same as the automorphic D-module explicitly constructed by Heinloth-Ngô-Yun. Since the irregular connection on $\mathbb{G}_m$ constructed in [HNY] is by definition the Hecke-eigenvalue of this automorphic D-module, it is the same as $\mathcal{E}$.

1. Recollection of [BD]

We begin with the review of the main results of Beilinson-Drinfeld ([BD]). We take the opportunity to describe a slightly generalized (and therefore weaker) version of [BD] in order to deal with the level structures.

Let G be a simple, simply-connected complex Lie group, with Lie algebra $\mathfrak{g}$ and the Langlands dual Lie algebra ${}^L\mathfrak{g}$. Let X be a smooth projective algebraic curve over $\mathbb{C}$. For every closed point $x \in X$, let $\mathcal{O}_x$ be the completed local ring of X at x and let F_x be its fractional field. Let $D_x = \mathrm{Spec}\mathcal{O}_x$ and $D_x^\times = \mathrm{Spec}F_x$. In what follows, for an affine (ind-)scheme T, we denote by $\mathrm{Fun}\, T$ the (pro)-algebra of regular functions on T.

Let $\mathcal{G}$ be an integral model of G over X, i.e. $\mathcal{G}$ is a (fiberwise) connected smooth affine group scheme over X such that $\mathcal{G}_{\mathbb{C}(X)} = G_{\mathbb{C}(X)}$, where $\mathbb{C}(X)$ is the function field of X. Let $\mathrm{Bun}_\mathcal{G}$ be the moduli stack of $\mathcal{G}$-torsors on X. The canonical sheaf $\omega_{\mathrm{Bun}_\mathcal{G}}$ is a line bundle on $\mathrm{Bun}_\mathcal{G}$. As G is assumed to be simply-connected, we have

Lemma 1. *There is a unique line bundle $\omega^{1/2}_{\mathrm{Bun}_\mathcal{G}}$ over $\mathrm{Bun}_\mathcal{G}$, such that $(\omega^{1/2}_{\mathrm{Bun}_\mathcal{G}})^{\otimes 2} \simeq \omega_{\mathrm{Bun}_\mathcal{G}}$.*

Now we assume that $\mathrm{Bun}_\mathcal{G}$ is "good" in the sense of Beilinson-Drinfeld, i.e.

$$\dim T^*\mathrm{Bun}_\mathcal{G} = 2\dim \mathrm{Bun}_\mathcal{G}.$$

In this case one can construct the D-module of the sheaf of critically twisted (a.k.a. $\omega^{1/2}_{\mathrm{Bun}_\mathcal{G}}$ twist) differential operators on the smooth site $(\mathrm{Bun}_\mathcal{G})_{sm}$ of $\mathrm{Bun}_\mathcal{G}$, denoted by $\mathcal{D}'$. Let $D' = (\underline{\mathrm{End}}\mathcal{D}')^{op}$ be the sheaf of endomorphisms of $\mathcal{D}'$ as a twisted D-module.

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Then D' is a sheaf of associative algebra on $(\mathrm{Bun}_{\mathcal{G}})_{sm}$ and $D' \simeq (D')^{op}$. For more details, we refer to [BD, §1].

Recall the definition of opers on a curve (cf. [BD, §3]). Let $\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times)$ be the ind-scheme of $^L\mathfrak{g}$-opers on the punctured disc $D_x^\times$. Then there is a natural ring homomorphism

$$h_x : \mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times) \to \Gamma(\mathrm{Bun}_{\mathcal{G}}, D'). \tag{1.1}$$

Let us briefly recall its definition. Let $\mathrm{Gr}_{\mathcal{G},x}$ be the affine Grassmannian, which is an ind-scheme classifying pairs $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on X and β is a trivialization of $\mathcal{F}$ away from x. Then we have $\mathrm{Gr}_{\mathcal{G},x} \simeq G(F_x)/K_x$, where $K_x = \mathcal{G}(\mathcal{O}_x)$. Let $\mathcal{L}_{\mathrm{crit}}$ be the pullback of the line bundle $\omega^{1/2}_{\mathrm{Bun}_{\mathcal{G}}}$ on $\mathrm{Bun}_{\mathcal{G}}$ to $\mathrm{Gr}_{\mathcal{G},x}$, and let δ_e be the delta D-module on $\mathrm{Gr}_{\mathcal{G},x}$ twisted by $\mathcal{L}_{\mathrm{crit}}$. Let

$$\mathrm{Vac}_x := \Gamma(\mathrm{Gr}_{\mathcal{G},x}, \delta_e)$$

be the vacuum $\hat{\mathfrak{g}}_{\mathrm{crit},x}$-module at the critical level.

Remark 1.1. The module Vac_x is not always isomorphic to $\mathrm{Ind}^{\hat{\mathfrak{g}}_{\mathrm{crit},x}}_{\mathrm{Lie}\,K_x+\mathbb{C}\mathbf{1}}(\mathrm{triv})$, due to the twist by $\mathcal{L}_{\mathrm{crit}}$. For example, if K_x is an Iwahori subgroup,

$$\mathrm{Vac}_x = \mathrm{Ind}^{\hat{\mathfrak{g}}_{\mathrm{crit},x}}_{\mathrm{Lie}\,K_x+\mathbb{C}\mathbf{1}}(\mathbb{C}_{-\rho}),$$

is the Verma module of highest weight $-\rho$ ($-\rho$ is anti-dominant w.r.t. the chosen K_x).

Let $\mathrm{Bun}_{\mathcal{G},x}$ be the scheme classifying pairs $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on X and β is a trivialization of $\mathcal{F}$ on $D_x = \mathrm{Spec}\mathcal{O}_x$. It admits a $(\hat{\mathfrak{g}}_{\mathrm{crit},x}, K_x)$ action, and $\mathrm{Bun}_{\mathcal{G},x}/K_x \simeq \mathrm{Bun}_{\mathcal{G}}$. Now applying the standard localization construction to the Harish-Chandra module Vac_x (cf. [BD, §1]) gives rise to

$$\mathrm{Loc}(\mathrm{Vac}_x) \simeq \mathcal{D}'$$

as critically twisted D-modules on $\mathrm{Bun}_{\mathcal{G}}$. Recall that the center $\mathfrak{Z}_x$ of the category of smooth $\hat{\mathfrak{g}}_{\mathrm{crit},x}$-modules is isomorphic to $\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times)$ by the Feigin-Frenkel isomorphism ([BD, §3.2], [F]). The mapping h_x then is the composition

$$\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times) \simeq \mathfrak{Z}_x \to \mathrm{End}(\mathrm{Vac}_x) \to \mathrm{End}(\mathrm{Loc}(\mathrm{Vac}_x)) \simeq \Gamma(\mathrm{Bun}_{\mathcal{G}}, D').$$

If $\mathcal{G}$ is unramified at x, then h_x factors as

$$h_x : \mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times) \twoheadrightarrow \mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_x) \simeq \mathrm{End}(\mathrm{Vac}_x) \to \Gamma(\mathrm{Bun}_{\mathcal{G}}, D'),$$

where $\mathrm{Op}_{^L\mathfrak{g}}(D_x)$ is the scheme (of infinite type) of $^L\mathfrak{g}$-opers on D_x.

The mappings h_x can be organized into a horizontal morphism h of $\mathcal{D}_X$-algebras over X (we refer to [BD, §2.6] for the generalities of $\mathcal{D}_X$-algebras). Let us recall the construction. By varying x on X, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},x}$ form an ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ formally smooth over X. Let $\pi : \mathrm{Gr}_{\mathcal{G}} \to X$ be the projection and $e : X \to \mathrm{Gr}_{\mathcal{G}}$ be the unital section given by the trivial $\mathcal{G}$-torsor. Let δ_e be the delta D-module along the section e twisted by $\mathcal{L}_{\mathrm{crit}}$. Then we have a chiral algebra

$$\mathcal{V}ac_X := \pi_!(\delta_e).$$

over X whose fiber over x is Vac_x.

Lemma 2. *The sheaf $\mathcal{V}ac_X$ is flat as an $\mathcal{O}_X$-module.*

For any chiral algebra $\mathcal{A}$ over a curve, one can associate the algebra of its endomorphisms, denoted by $\mathcal{E}nd(\mathcal{A})$. As sheaves on X,

$$\mathcal{E}nd(\mathcal{A}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}),$$

where Hom is taken in the category of chiral $\mathcal{A}$-modules. Obviously, $\mathcal{E}nd(\mathcal{A})$ is an algebra by composition. Less obviously, there is a natural chiral algebra structure on $\mathcal{E}nd(\mathcal{A}) \otimes \omega_X$ which is compatible with the algebra structure. Therefore, $\mathcal{E}nd(\mathcal{A})$ is a commutative $\mathcal{D}_X$-algebra. If $\mathcal{A}$ is $\mathcal{O}_X$-flat, there is a natural injective mapping $\mathcal{E}nd(\mathcal{A})_x \to \mathrm{End}(\mathcal{A}_x)$ which is not necessarily an isomorphism in general, where $\mathrm{End}(\mathcal{A}_x)$ is the endomorphism algebra $\mathcal{A}_x$ as a chiral $\mathcal{A}$-module. However, this is an isomorphism if there is some open neighborhood U containing x such that $\mathcal{A}|_U$ is constructed from a vertex algebra. We refer to [R] for details of the above discussion.

Let $U \subset X$ be an open subscheme such that $\mathcal{G}|_U \simeq G \times U$, then by the above generality, the Feigin-Frenkel isomorphism gives rise to

$$\mathrm{Spec}\ \mathcal{E}nd(\mathcal{V}ac_U) \simeq \mathrm{Op}_{^L\mathfrak{g}}|_U,$$

where $\mathrm{Op}_{^L\mathfrak{g}}$ is the $\mathcal{D}_X$-scheme over X, whose fiber over $x \in X$ is the scheme of ${}^L\mathfrak{g}$-opers on D_x. Recall that for a commutative $\mathcal{D}_U$-algebra $\mathcal{B}$, we can take the algebra of its horizontal sections $H_\nabla(U, \mathcal{B})$ (or so-called conformal blocks) [BD, §2.6], which is usually a topological commutative algebra. For example,

$$\mathrm{Spec} H_\nabla(U, \mathrm{Op}_{^L\mathfrak{g}}) = \mathrm{Op}_{^L\mathfrak{g}}(U)$$

is the ind-scheme of ${}^L\mathfrak{g}$-opers on U ([BD, §3.3]). As $H_\nabla(U, \mathcal{E}nd(\mathcal{V}ac_U)) \to H_\nabla(X, \mathcal{E}nd(\mathcal{V}ac_X))$ is surjective, we have a closed embedding

$$\mathrm{Spec} H_\nabla(X, \mathcal{E}nd(\mathcal{V}ac_X)) \to \mathrm{Op}_{^L\mathfrak{g}}(U).$$

Let $\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}}$ denote the image of this closed embedding. This is a subscheme (rather than an ind-scheme) of $\mathrm{Op}_{^L\mathfrak{g}}(U)$.

On the other hand, as argued in [BD, §2.8], the mapping h_x is of crystalline nature so that it induces a mapping of $\mathcal{D}_X$-algebras

$$h : \mathcal{E}nd(\mathcal{V}ac_X) \to \Gamma(\mathrm{Bun}_{\mathcal{G}}, D') \otimes \mathcal{O}_X, \tag{1.2}$$

which induces a mapping of horizontal sections

$$h_\nabla : H_\nabla(X, \mathcal{E}nd(\mathcal{V}ac_X)) \to \Gamma(\mathrm{Bun}_{\mathcal{G}}, D'). \tag{1.3}$$

Therefore, (1.3) can be rewrite as a mapping

$$h_\nabla : \mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}} \to \Gamma(\mathrm{Bun}_{\mathcal{G}}, D'). \tag{1.4}$$

We recall the characterization $\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}}$.

Lemma 3. *Let $X \setminus U = \{x_1, \ldots, x_n\}$. Assume that the support of Vac_{x_i} (as an $\mathfrak{Z}_{x_i}$-module) is $Z_{x_i} \subset \mathrm{Op}_{^L\mathfrak{g}}(D_{x_i}^\times)$ (i.e. $\mathrm{Fun}(Z_{x_i}) = \mathrm{Im}(\mathrm{Op}_{^L\mathfrak{g}}(D_x^\times) \to \mathrm{End}(\mathrm{Vac}_x))$). Then*

$$\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}} \simeq \mathrm{Op}_{^L\mathfrak{g}}(U) \times_{\prod_i \mathrm{Op}_{^L\mathfrak{g}}(D_{x_i}^\times)} \prod Z_{x_i}.$$

The mapping (1.4) is a quantization of a classical Hitchin system. Namely, there is a natural filtration ([BD, §3.1]) on the algebra $\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(U)$ whose associated graded is the algebra of functions on the classical Hitchin space

$$\mathrm{Hitch}(U) = \bigoplus_i \Gamma(U, \Omega^{d_i+1})$$

where d_is are the exponent of $\mathfrak{g}$ and Ω is the canonical sheaf of X. On the other hand, there is a natural filtration on $\Gamma(\mathrm{Bun}_{\mathcal{G}}, D')$ coming from the order of the differential operators. Then (1.4) is strictly compatible with the filtration and the associated graded map gives rise to the classical Hitchin map

$$h^{cl} : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Hitch}(U).$$

Remark 1.2. The above map h^{cl} factors through certain closed subscheme $\mathrm{Hitch}(X)_{\mathcal{G}} \subset \mathrm{Hitch}(U)$ whose algebra of functions is the associated graded of $\mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}}(X)_{\mathcal{G}}$.

The following theorem summarizes the main results of [BD].

Theorem 4. *Let $\chi \in \mathrm{Op}_{{}^L\mathfrak{g}}(X)_{\mathcal{G}} \subset \mathrm{Op}_{{}^L\mathfrak{g}}(U)$ be a closed point, which gives rise to a ${}^L\mathfrak{g}$-oper $\mathcal{E}$ on U. Let $\varphi_\chi : \mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}}(X)_{\mathcal{G}} \to \mathbb{C}$ be the corresponding homomorphism of $\mathbb{C}$-algebras. Then*

$$\mathrm{Aut}_{\mathcal{E}} := (\mathcal{D}' \otimes_{\mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}}(X)_{\mathcal{G}},\varphi_\chi} \mathbb{C}) \otimes \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1/2}$$

is a Hecke-eigensheaf on $\mathrm{Bun}_{\mathcal{G}}$ with respect to $\mathcal{E}$ (regarded as a ${}^LG$-local system).

Remark 1.3. The statement of the about theorem is weaker than the main theorem in [BD] in two aspects: (i) if $\mathcal{G}$ is the constant group scheme (the unramified case), then $\mathrm{Op}_{{}^L\mathfrak{g}}(X)_{\mathcal{G}} = \mathrm{Op}_{{}^L\mathfrak{g}}(X)$ is the space of ${}^L\mathfrak{g}$-opers on X. In this case, Beilinson and Drinfeld proved that

$$\mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}}(X) \simeq \Gamma(\mathrm{Bun}_G, D')$$

and therefore $\mathrm{Aut}_{\mathcal{E}}$ is always non-zero in this case; (ii) in the unramified case, the automorphic D-module $\mathrm{Aut}_{\mathcal{E}}$ is holonomic.

The proofs of both assertions are based on the fact that the classical Hitchin map is a complete integrable system. If the level structure of $\mathcal{G}$ is not deeper than the Iwahori level structure (or even the pro-unipotent radical of the Iwahori group), then by the same arguments, the above two assertions still holds. However, it is not obvious from the construction that $\mathrm{Aut}_{\mathcal{E}}$ is non-zero for the general deeper level structure, although we do conjecture that this is always the case. In addition, for arbitrary $\mathcal{G}$, the automorphic D-modules constructed as above will in general not be holonomic. This is the reason that we need to use a group scheme different from [HNY] in what follows.

2.

Now we specialize the group scheme $\mathcal{G}$. Let G be a simple, simply-connected complex Lie group, of rank ℓ. Let us fix $B \subset G$ a Borel subgroup and B^- an opposite Borel subgroup. The unipotent radical of B (resp. B^-) is denoted by U (resp. U^-). Following [HNY], we denote by $\mathcal{G}(0,1)$ the group scheme on $\mathbb{P}^1$ obtained from the dilatation of $G \times \mathbb{P}^1$ along $B^- \times \{0\} \subset G \times \{0\}$ and $U \times \{\infty\} \subset G \times \{\infty\}$. Following *loc. cit.*, we denote $I(1) = \mathcal{G}(0,1)(\mathcal{O}_\infty)$.

Let $\mathcal{G}(0,2) \to \mathcal{G}(0,1)$ be the group scheme over $\mathbb{P}^1$ so that they are isomorphic away from ∞ and $\mathcal{G}(0,2)(\mathcal{O}_\infty) = I(2) := [I(1), I(1)]$. Then $I(1)/I(2) \simeq \prod_{i=0}^{\ell} U_{\alpha_i}$, where α_i are simple affine roots, and U_{α_i} are the corresponding root groups. Let us choose for each α_i an isomorphism $\Psi_i : U_{\alpha_i} \simeq \mathbb{G}_a$. Then we obtain a well-defined morphism

$$\Psi : I(1) \to I(1)/I(2) \simeq \prod_{i=0}^{\ell} U_{\alpha_i} \simeq \prod \mathbb{G}_a \stackrel{\mathrm{sum}}{\to} \mathbb{G}_a.$$

Let $I_\Psi := \ker \Psi \subset I(1)$.

As explained in *loc. cit.*, there is an open substack of $\mathrm{Bun}_{\mathcal{G}(0,2)}$, which is isomorphic to $\mathbb{G}_a^{\ell+1}$. For the application of Beilinson-Drinfeld's construction, it is convenient to consider $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$, where $\mathcal{G}(0,\Psi)\to\mathcal{G}(0,1)$ is an isomorphism away from ∞ and $\mathcal{G}(0,\Psi)(\mathcal{O}_\infty)=I_\Psi\subset I(1)=\mathcal{G}(0,1)$. Then $\mathrm{Bun}_{\mathcal{G}(0,2)}$ is a torsor over $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ under the group $I_\Psi/I(2)\cong\mathbb{G}_a^\ell$ and there is an open substack of $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ isomorphic to $\mathbb{G}_a$.

Lemma 5. *The stack* $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ *is good in the sense of* [BD, §1.1.1].

Proof. Since $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ is a principal bundle over $\mathrm{Bun}_{\mathcal{G}(0,1)}$ under the group $I(1)/I_\Psi\simeq\mathbb{G}_a$, it is enough to show that $\mathrm{Bun}_{\mathcal{G}(0,1)}$ is good. It is well-known in this case $\mathrm{Bun}_{\mathcal{G}(0,1)}$ has a stratification by elements in the affine Weyl group of G and the stratum corresponding to w has codimension $\ell(w)$ and the stabilizer group has dimension $\ell(w)$. Therefore $\mathrm{Bun}_{\mathcal{G}(0,1)}$ is good. □

Let S_w denote the preimage in $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ of the stratum in $\mathrm{Bun}_{\mathcal{G}(0,1)}$ corresponding to w. Then $S_1\simeq\mathbb{A}^1$, and for a simple reflection s, $S_1\cup S_s\simeq\mathbb{P}^1$. In particular, any regular function on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ is constant.

Let us describe $\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}(0,\Psi)}$ in this case.

At $0\in\mathbb{P}^1$, $K_0=\mathcal{G}(0,\Psi)(\mathcal{O}_0)$ is the the Iwahori subgroup I^{op} of $G(F_0)$, which is $\mathrm{ev}^{-1}(B^-)$ under the evaluation map $\mathrm{ev}:G(\mathcal{O}_0)\to G$, and

$$\mathrm{Vac}_0=\mathrm{Ind}_{\mathrm{Lie}\,I^{op}+\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\mathrm{crit},x}}(\mathbb{C}_{-\rho}).$$

is just the Verma module $\mathbb{M}_{-\rho}$ of highest weight $-\rho$ ($-\rho$ is anti-dominant w.r.t. B^-), and it is known ([F, Chap. 9]) that $\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_0^\times)\to\mathrm{End}(\mathbb{M}_{-\rho})$ induces an isomorphism

$$\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_0)_{\varpi(0)}\simeq\mathrm{End}(\mathbb{M}_{-\rho}),$$

where $\mathrm{Op}_{^L\mathfrak{g}}(D_0)_{\varpi(0)}$ is the scheme of $^L\mathfrak{g}$ opers on D_0 with regular singularities and zero residue. Let us describe this space in concrete terms. Let $f=\sum_i X_{-\alpha_i}$ be the sum of root vectors $X_{-\alpha_i}$ corresponding negative simple roots $-\alpha_i$ of $^L\mathfrak{g}$. After choosing a uniformizer z of the disc D_0, $\mathrm{Op}_{^L\mathfrak{g}}(D_0)_{\varpi(0)}$ is the space of operators of the form

$$\partial_z+\frac{f}{z}+{}^L\mathfrak{b}[[z]]$$

up to $^LU(\mathcal{O}_0)$-gauge equivalence.

At $\infty\in\mathbb{P}^1$, $K_\infty=\mathcal{G}(0,\Psi)(\mathcal{O}_\infty)=I_\Psi$. Denote

$$\mathbb{W}_{univ}=\mathrm{Vac}_\infty=\mathrm{Ind}_{\mathrm{Lie}\,I_\Psi+\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\mathrm{crit},x}}(\mathrm{triv}).$$

It is known ([FF, Lemma 5]) that $\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_\infty^\times)\to\mathrm{End}(\mathbb{W}_{univ})$ factors as

$$\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_\infty^\times)\twoheadrightarrow\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(D_\infty)_{1/h}\hookrightarrow\mathrm{End}(\mathbb{W}_{univ}),$$

where $\mathrm{Op}_{^L\mathfrak{g}}(D_\infty)_{1/h}$ is the scheme of opers with slopes $\le 1/h$ (as ^LG-local systems) and h is the Coxeter number of ^LG. To give a concrete description of this space, let us complete f to an $\mathfrak{sl}_2$-triple $\{e,\gamma,f\}$ with $e\in{}^LB$. Let $^L\mathfrak{g}^e$ be the centralizer of e in $^L\mathfrak{g}$, and decompose $^L\mathfrak{g}^e=\oplus_{i=1}^\ell{}^L\mathfrak{g}^e_i$ according to the principal grading by γ. Let $d_i=\deg{}^L\mathfrak{g}^e_i$. Then after choosing a uniformizer z on the disc D_∞, $\mathrm{Op}_{^L\mathfrak{g}}(D_\infty)_{1/h}$ is the space of operators of the form

$$\partial_z+f+\sum_{i=1}^{\ell-1}+z^{-d_i-1}({}^L\mathfrak{g}^e_i)[[z]]+z^{-d_\ell-2}({}^L\mathfrak{g}^e_\ell)[[z]]$$

up to $^LU(\mathcal{O}_\infty)$-gauge equivalence.

Therefore, $\mathrm{Op}_{^L\mathfrak{g}}(X)_{\mathcal{G}(0,\Psi)}$ is isomorphic to

$$\mathrm{Op}_{^L\mathfrak{g}}(X)_{(0,\mathrm{RS}),(\infty,1/h)} := \mathrm{Op}_{^L\mathfrak{g}}(D_\infty)_{1/h}\times_{\mathrm{Op}_{^L\mathfrak{g}}(D_\infty^\times)}\mathrm{Op}_{^L\mathfrak{g}}(\mathbb{G}_m)\times_{\mathrm{Op}_{^L\mathfrak{g}}(D_0^\times)}\mathrm{Op}_{^L\mathfrak{g}}(D_0)_{\varpi(0)}.$$

As observed in [FG], $\mathrm{Op}_{^L\mathfrak{g}}(X)_{(0,\mathrm{RS}),(\infty,1/h)} \simeq \mathbb{A}^1$. Indeed, let z be the global coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. Then the space of such opers are of the form

$$\nabla = \partial_z + \frac{f}{z} + \lambda e_\theta,$$

where f is the sum of root vectors corresponding to negative simple roots and e_θ is a root vector corresponding to the highest root θ.

According to §1, there is a ring homomorphism

$$h_\nabla : \mathbb{C}[\lambda] \to \Gamma(\mathrm{Bun}_{\mathcal{G}(0,\Psi)}, D').$$

Let us describe this mapping more explicitly. Recall that there is an action of $I(1)/I_\Psi \simeq \mathbb{G}_a$ on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$, and therefore the action of $\mathbb{G}_a$ induces an algebra homomorphism

$$a : U(\mathrm{Lie}I(1)/I_\Psi) \to \Gamma(\mathrm{Bun}_{\mathcal{G}(0,\Psi)}, D').$$

Lemma 6. *We have $h_\nabla(\lambda) = a(\xi)$ for some non-zero element $\xi \in \mathrm{Lie}I(1)/I_\Psi \simeq \mathbb{C}$.*

Proof. Consider the associated graded $h^{cl} : \mathrm{gr}\,\mathbb{C}[\lambda] \to \Gamma(T^*\mathrm{Bun}_{\mathcal{G}(0,\Psi)}, \mathcal{O})$, which is the classical Hitchin map. Recall that the filtration on $\mathbb{C}[\lambda]$ comes from the existence of $\hbar$-opers, and therefore the symbol of λ is identified with a coordinate function on

$$\begin{aligned}&\mathrm{Hitch}(X)_{\mathcal{G}(0,\Psi)}\\ \simeq\ & \bigoplus_{i=1}^{\ell-1}\Gamma(\mathbb{P}^1, \Omega^{d_i+1}((d_i)\cdot 0 + (d_i+1)\cdot\infty)\bigoplus\Gamma(\mathbb{P}^1, \Omega^{d_\ell+1}((d_\ell)\cdot 0 + (d_\ell+2)\cdot\infty))\\ \simeq\ & \mathbb{A}^1.\end{aligned}$$

On the other hand, it is easy to identify the Hitchin map with the moment map associated to the action of $I(1)/I_\Psi$ on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$. Therefore, $h_\nabla(\lambda) = a(\xi) - c$ for some constant c. Up to normalization, we can assume that $d\Psi(\xi) = 1$. We show that $c = 0$. Indeed, consider the automorphic D-module $\mathrm{Aut} = \mathcal{D}'/\mathcal{D}'\lambda$ on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$. It is $I(1)/I_\Psi$-equivariant against $c\Psi$, with eigenvalue the local system on $\mathbb{G}_m$ represented by the connection $\partial_z + \frac{f}{z}$ by Theorem 4, which is regular singular. However, if $c \neq 0$, by [HNY, Theorem 4(1)], the eigenvalue for this Aut should be irregular at ∞. Contradition. □

Finally, for any $\chi \in \mathrm{Op}_{^L\mathfrak{g}}(X)_{(0,\mathrm{RS}),(\infty,1/h)}$ given by $\lambda = c$, $\mathrm{Aut}_{\mathcal{E}} = \mathcal{D}'/\mathcal{D}'(\lambda - c)$ is a D-module on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$, equivariant against $(I(1)/I_\Psi, c\Psi)$. By the uniqueness of such D-modules on $\mathrm{Bun}_{\mathcal{G}(0,\Psi)}$ (same argument as in [HNY, Lemma 2.3]), this must be the same as the automorphic D-module as constructed in [HNY]. We are done.

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