PERIOD INTEGRALS AND THE RIEMANN-HILBERT CORRESPONDENCE

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ABSTRACT. A tautological system, introduced in [16][17], arises as a regular holonomic system of partial differential equations that govern the period integrals of a family of complete intersections in a complex manifold X , equipped with a suitable Lie group action. A geometric formula for the holonomic rank of such a system was conjectured in [4], and was verified for the case of projective homogeneous space under an assumption. In this paper, we prove this conjecture in full generality. By means of the Riemann-Hilbert correspondence and Fourier transforms, we also generalize the rank formula to an arbitrary projective manifold with a group action.

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1. INTRODUCTION

Let G be a connected algebraic group over a field k of characteristic zero. Let X be a projective G -variety and let $\mathcal{L}$ be a very ample G -linearized invertible sheaf over X which gives rise to a G -equivariant embedding

$$X \rightarrow \mathbb{P}(V),$$

where $V = \Gamma(X, \mathcal{L})^\vee$. Let $r = \dim V$. We assume that the action of G on X is locally effective, i.e. $\ker(G \rightarrow \operatorname{Aut}(X))$ is finite. Let $\mathbb{G}_m$ be the multiplicative group acting on V by homotheties. Let $\hat{G} = G \times \mathbb{G}_m$, whose Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus ke$, where e acts on V by identity. We denote by $Z : \hat{G} \rightarrow \operatorname{GL}(V)$ be the corresponding group representation, and $Z : \hat{\mathfrak{g}} \rightarrow \operatorname{End}(V)$ be the corresponding Lie algebra representation. In particular, $Z(e)$ is the Euler vector field on V . Note that under our assumptions, $Z : \hat{\mathfrak{g}} \rightarrow \operatorname{End}(V)$ is injective.

Let $\hat{i} : \hat{X} \subset V$ be the cone of X , defined by the ideal $I(\hat{X})$. Let $\beta : \hat{\mathfrak{g}} \rightarrow k$ be a Lie algebra homomorphism. Then a *tautological system* as defined in [16][17] is the cyclic D -module on $V^\vee$

$$\tau(G, X, \mathcal{L}, \beta) = D_{V^\vee} / D_{V^\vee} J(\hat{X}) + D_{V^\vee}(Z(\xi) + \beta(\xi), \xi \in \hat{\mathfrak{g}}),$$

where

$$J(\hat{X}) = \{\hat{D} \mid D \in I(\hat{X})\}$$

is the ideal of the commutative subalgebra $\mathbb{C}[\partial] \subset D_{V^\vee}$ obtained by the Fourier transform of $I(\hat{X})$ (see §A for the review of the Fourier transform and in particular (A.6) for the notation).

We recall the main motivation for studying tautological systems. Let X' be a compact complex manifold (not necessarily algebraic), such that the complete linear system of anti-canonical divisors in X' is base point free. Let $\pi : \mathcal{Y} \rightarrow B := \Gamma(X', \omega_{X'}^{-1})_{sm}$ be the family of smooth CY hyperplane sections $Y_a \subset X'$, and let $\mathbb{H}^{top}$ be the Hodge bundle over B whose fiber at $a \in B$ is the line $\Gamma(Y_a, \omega_{Y_a}) \subset H^{n-1}(Y_a)$, where $n = \dim X'$. In [17], the period integrals of this family are constructed by giving a canonical trivialization of $\mathbb{H}^{top}$. Let $\Pi = \Pi(X')$ be the period sheaf of this family, i.e. the locally constant sheaf generated by the period integrals (Definition 1.1 [17].)

Let $V = \Gamma(X', \omega_{X'}^{-1})^\vee$, X be the image of the natural map $X' \rightarrow \mathbb{P}(V)$, and $\mathcal{L} = \mathcal{O}_X(1)$. Let G be a connected algebraic group acting on X .

Theorem 1.1. *The period integrals of the family $\pi : \mathcal{Y} \rightarrow B$ are solutions to*

$$\tau = \tau(G, X, \mathcal{L}, \beta_0),$$

where β_0 is the Lie algebra homomorphism which vanishes on $\mathfrak{g}$ and $\beta_0(e) = 1$.

This was proved in [16] for X a partial flag variety, and in full generality in [17], where the result was also generalized to hyperplane sections of general type.

We note that when X' is a projective homogeneous manifold of a semisimple group G , in which case we have $X = X'$, τ is amenable to explicit descriptions. For example, one description says that the tautological system can be generated by the vector fields corresponding to the linear G action on $V^\vee$, and a twisted Euler vector field, together with a set of quadratic differential operators corresponding to the defining relations of X in $\mathbb{P}(V)$ under the Plücker embedding. The case where X is a Grassmannian has been worked out in detail [16]. Furthermore, when the middle primitive cohomology $H^n(X)_{prim} = 0$, it is also known that the system τ is complete, i.e. the solution sheaf coincides with the period sheaf [4].

We now return to a general tautological system τ . Applying an argument of [12], we find that if G acts on X by finitely many orbits, and if the character D-module on $\hat{G}$

$$\mathcal{L}_\beta := D_{\hat{G}}/D_{\hat{G}}(\xi + \beta(\xi), \xi \in \hat{\mathfrak{g}})$$

on $\hat{G}$ is regular singular, then τ is regular holonomic. See [16] Theorem 3.4(1). In this case, if $X = \sqcup_{l=1}^r X_l$ is the decomposition into G -orbits, then the singular locus of τ is contained in $\cup_{l=1}^r X_l^\vee$. Here $X_l^\vee \subset V^\vee$ is the conical variety whose projectivization $\mathbb{P}(X_l^\vee)$ is the projective dual to the Zariski closure of X_l in X . From now on we assume that G acts on X by finitely many orbits, and $\mathcal{L}_\beta$ is regular singular. Note that the latter assumption is always satisfied when G is reductive.

Let us now turn to the main problem studied in this paper. In the well-known applications of variation of Hodge structures in mirror symmetry, it is important to decide which solutions of our differential system come from period integrals. By Theorem 1.1, the period sheaf is a subsheaf of the solution sheaf of a tautological system. Thus an important problem is to decide when the two sheaves actually coincide, i.e. when τ is complete. If τ is not complete, how much larger is the solution sheaf relative to the period sheaf? From Hodge theory, we know that (see Proposition 6.3 [4]) the rank of the period sheaf is given by the dimension of the middle vanishing cohomology of the smooth hypersurface Y_a . Therefore, to answer those questions, it is clearly desirable to know precisely the holonomic rank of τ . For a brief overview of known results on these questions in a number of special cases, see Introduction in [4].

Conjecture 1.2. (Holonomic rank conjecture) Let X be an n -dimensional projective homogeneous space of a semisimple group G . The solution rank of $\tau = \tau(G, X, \omega_X^{-1}, \beta_0)$ at the point $a \in V^\vee$ is given by $\dim H_n(X - Y_a)$.

In [4], the following is proved,

Theorem 1.3. *Assume that the natural map*

$$\mathfrak{g} \otimes \Gamma(X, \omega_X^{-r}) \rightarrow \Gamma(X, T_X \otimes \omega_X^{-r})$$

is surjective for each $r \geq 0$. Then conjecture 1.2 holds.

In this paper, we prove this in full generality.

Theorem 1.4. *Conjecture 1.2 holds.*

This will be proved in §2. There are at least two immediate applications of this result. First we can now compute the solution rank for τ for *generic* $a \in V^\vee$.

Corollary 1.5. *The solution rank of τ at a smooth hyperplane section a is*

$$\dim H^n(X)_{\text{prim}} + \dim H^{n-1}(Y_a) - \dim H^{n+1}(X).$$

where the first term is the middle primitive cohomology of $X = G/P$ with $n = \dim X$.

The last two terms of the rank above can be computed readily in terms of the semisimple group G and the parabolic subgroup P by the Lefschetz hyperplane and the Riemann-Roch theorems. (See Example 2.4 [16]). A second application of Theorem 1.4 is to find certain exceptional points a in $V^\vee$ where the solution sheaf of τ degenerates “maximally”.

Definition 1.1. A nonzero section $a \in V^\vee = \Gamma(X, \omega_X^{-1})$ is called a rank 1 point if the solution rank of τ at a is 1. In other words, $\text{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee, a}) \cong \mathbb{C}$.

Corollary 1.6. *Any projective homogeneous variety admits a rank 1 point.*

We will construct these rank 1 points in two explicit but different ways. The first, which works for $G = SL_l$, is a recursive procedure that produces such a rank 1 point by assembling rank 1 points from lower step flag varieties, starting from Grassmannians, and by repeatedly applying Theorem 1.4. The second way, which works for any semisimple group G , is by using a well-known B -invariant stratification of the flag variety G/B to produce an open stratum in $X = G/P$ with a one dimensional middle degree cohomology. The complement of this stratum is an anticanonical divisor, hence a rank 1 point of X by Theorem 1.4.

The geometric formula in Conjecture 1.2 appears to go well beyond the context of homogeneous spaces. Theorem 1.4 can be seen as a special case of the following much more general theorem. Consider a smooth projective G -variety X with $\mathcal{L} = \omega_X^{-1}$ very ample. Set $\tau = \tau(G, X, \mathcal{L}, \beta)$ and $V^\vee = \Gamma(X, \mathcal{L})$. We introduce some more notations. Let $\mathbb{L}^\vee$ be the total space of $\mathcal{L}$ and $\mathring{\mathbb{L}}^\vee$ be the complement of the zero section. Let

$$ev : V^\vee \times X \rightarrow \mathbb{L}^\vee, \quad (a, x) \mapsto a(x)$$

be the evaluation map, and $\mathbb{L}^\perp := \ker(ev)$. Finally let

$$\pi^\vee : U := V^\vee \times X - \mathbb{L}^\perp \rightarrow V^\vee.$$

Note that this is the complement of the universal family of hyperplane sections $\mathbb{L}^\perp \rightarrow V^\vee$, $(a, x) \mapsto a$. Put

$$D_{X, \beta} = (D_X \otimes k_\beta) \otimes_{U\hat{\mathfrak{g}}} k,$$

where k_β is the 1-dimensional $\hat{\mathfrak{g}}$ -module given by the character β (see §A for the notations). We now state the main results of this paper.

Theorem 1.7. *For $\beta(e) = 1$, there is a canonical isomorphism*

$$\tau \simeq H^0 \pi_+^\vee(\mathcal{O}_{V^\vee} \boxtimes D_{X, \beta})|_U.$$

Corollary 1.8. *Suppose G acts on X by finitely many orbits, and $k = \mathbb{C}$. Then the solution rank of τ at $a \in V^\vee$ is given by $\dim H_c^n(U_a, \text{Sol}(D_{X,\beta})|_{U_a})$, where $U_a = X - Y_a$.*

More generally, we have

Theorem 1.9. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, there is a canonical isomorphism*

$$\tau \simeq H^0 \pi_+^\vee \text{ev}^!(D_{\mathbb{L}^\vee, \beta})[1 - r].$$

In addition to proving Conjecture 1.2 as a special case, Theorem 1.7 can also be used to derive the well-known formula for the solution rank of a GKZ system [7] at generic point a . But since Corollary 1.8 holds for arbitrary $a \in V^\vee$, it holds in particular for a corresponding to the union of all T -invariant divisors in X (which is anticanonical). In this case, Theorem 1.7 implies that a is a rank 1 point – a result of [9] based on Gröbner basis theory but motivated by applications to mirror symmetry. Thus Theorem 1.7 and Corollary 1.8 interpolate a result of [7] and [9] by unifying the rank formula at generic point and at those exceptional rank 1 points, and at the same time, generalize them to an arbitrary G -variety.

We now outline the paper. In §2, we prove Theorem 1.7 and a number of its consequences, including Theorem 1.4. We also describe explicitly the “cycle-to-period” map $H_n(X - Y_a) \rightarrow \text{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee, a})$ as a result of Theorem 1.7, and use it to answer a question recently communicated to us by S. Bloch. While §2 deals only with the case $\beta(e) = 1$, we remove this assumption in §§3-5. In §3, we study the $!$ -fibers of τ , and describe some vanishing results at the special point $a = 0$. We describe the geometric set up in §4 for proving Theorem 1.9. The key step of the proof, involving an exact sequence for τ , is done in §5. Finally, we apply our results to construct rank 1 points for partial flag varieties in the case $G = SL_l$ in §§7-8, and for general semisimple groups in §6. The appendix §A collects some standard facts on D-modules.

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2. CY HYPERPLANE SECTIONS

We begin with Theorem 1.7 : X is a G -variety with $\mathcal{L} = \omega_X^{-1}$ very ample, and $\beta(e) = 1$. This is in fact a special case of the more general Theorem 1.9 and therefore can be also

obtained by the methods introduced in later sections. However, we decide to deal with this case first for several reasons. On the one hand, the proof given here is different from the later method and is more direct. On the other hand, Theorem 1.7 already contains a subcase when $\beta|_{\mathfrak{g}} = 0$, i.e. $\beta = \beta_0$, which is important to mirror symmetry.

Let $n = \dim X$. Let $U = V^\vee \times X - V(f)$, where $V(f) = \mathbb{L}^\perp$ is the universal hyperplane section, so that $U_a = X - V(f_a)$ where $V(f_a) = Y_a$, the zero locus of the section $f_a \equiv a \in V^\vee$. Let $\pi^\vee : U \rightarrow V^\vee$ denote the projection. The restriction of β to $\mathfrak{g}$ is still denoted by β when no confusion arises. Put $D_{X,\beta} = (D_X \otimes k_\beta) \otimes_{U_{\mathfrak{g}}} k$. Note that if G acts on X by finitely many orbits, then $D_{X,\beta}$ is (G, β) -equivariant holonomic D-module on X , and

$$\mathcal{N} := (\mathcal{O}_{V^\vee} \boxtimes D_{X,\beta})|_U$$

is a holonomic D-module on U .

Theorem 2.1. *There is a canonical isomorphism $\tau \simeq H^0 \pi_+^\vee \mathcal{N}$.*

Corollary 2.2. *If $\beta|_{\mathfrak{g}} = 0$. There is a canonical surjective map*

$$\tau \rightarrow H^0 \pi_+^\vee \mathcal{O}_U.$$

Proof. Note that there is always a surjective map $D_{X,0} = D_X/D_X \mathfrak{g} \rightarrow D_X/D_X T_X = \mathcal{O}_X$. The corollary follows. $\square$

Corollary 2.3. *If $\beta|_{\mathfrak{g}} = 0$ and X is a homogeneous G -variety, then $\tau \simeq H^0 \pi_+^\vee \mathcal{O}_U$.*

Proof. If X is homogeneous, $\mathfrak{g} \otimes \mathcal{O}_X \rightarrow T_X$ is surjective. Therefore $D_{X,0} = D_X/D_X T_X = \mathcal{O}_X$. $\square$

Note that this corollary implies that the tautological system in this case, which is a priori defined as a D-module by generators and relations, is of geometric origin, i.e. itself is a Gauss-Mannin connection.

We turn to the solution sheaf of τ via the Riemann-Hilbert correspondence. Assume G acts on X by finitely many orbits. Let us write $\mathcal{F} = \text{Sol}(D_{X,\beta})$. This is a perverse sheaf on X . In the case X is a homogenous G -variety and $\beta|_{\mathfrak{g}} = 0$, then $\mathcal{F} = \mathbb{C}[n]$.

Corollary 2.4. *Let $k = \mathbb{C}$ and $a \in V^\vee$. Then the solution rank of τ at a is given by $\dim H_c^n(U_a, \mathcal{F}|_{U_a})$.*

Proof. To simplify the notation, we assume that X is homogeneous and $\beta|_{\mathfrak{g}} = 0$. According to the Riemann-Hilbert correspondence, $\text{Sol}(\tau) = {}^p R^0 \pi_!^\vee \mathbb{C}_U[n+r]$, where ${}^p R^0 \pi_!^\vee$ denotes the 0th perverse cohomology of $\pi_!^\vee$. Then the non-derived solution sheaf $\text{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})$ is $H^{-r}({}^p R^0 \pi_!^\vee \mathbb{C}[n+r])$, the $(-r)$ th (standard) sheaf cohomology of ${}^p R^0 \pi_!^\vee \mathbb{C}[n+r]$. However, as $R\pi_!^\vee \mathbb{C}[n+r]$ lives in positive perverse degrees, $H^{-r}({}^p R^0 \pi_!^\vee \mathbb{C}[n+r]) = H^{-r} R\pi_!^\vee \mathbb{C}[n+r] = R^n \pi_!^\vee \mathbb{C}_U$. The claim follows. $\square$

Corollary 2.5. *Conjecture 1.2 holds.*

Now we prove Theorem 2.1. We will assume $\beta|_{\mathfrak{g}} = 0$ to simplify notations.

Let us write $\mathcal{R} := D_{V^\vee}/D_{V^\vee} J(\hat{X})$, which is a left $D_{V^\vee}$ -module. Observe that for any $\xi \in \hat{\mathfrak{g}}$,

$$D_V I(\hat{X}) Z^\vee(\xi) \subset D_V I(\hat{X}),$$

and therefore

$$D_{V^\vee} J(\hat{X}) Z(\xi) \subset D_{V^\vee} J(\hat{X}).$$

Therefore, $D_{V^\vee} J(\hat{X})$ can be regarded as a *right* $\hat{\mathfrak{g}}$ -module, on which $\xi \in \hat{\mathfrak{g}}$ acts via the right multiplication by $Z(\xi)$. Accordingly, $\mathcal{R}$ is also a right $\hat{\mathfrak{g}}$ -module. In addition, by definition we have

$$\tau = (\mathcal{R} \otimes k_\beta) \otimes_{\hat{\mathfrak{g}}} k,$$

where k_β is the 1-dimensional representation of $\hat{\mathfrak{g}}$ defined by β .

Let $\{a_i\}$ be a basis of V and $\{a_i^*\}$ the dual basis. Observe that as $\mathcal{O}_{V^\vee}$ -modules, one can write

$$\mathcal{R} \simeq \mathcal{O}_{V^\vee} \otimes S,$$

where $S = k[\partial_{a_i}]/J(\hat{X}) \simeq \mathcal{O}_V/I(\hat{X})$ is identified with the homogeneous coordinate ring of $\hat{X}$, and $\mathcal{O}_{V^\vee}$ acts on the first factor¹. If we convert the right action of $\hat{\mathfrak{g}}$ on $\mathcal{R}$ described above to a left action α , then α will be the sum of the following two actions: the first is the action of $\hat{\mathfrak{g}}$ on the second factor through $Z : \hat{\mathfrak{g}} \rightarrow \text{End} V \rightarrow \text{End} S$, which is denoted by α_1 ; to describe the second action α_2 , observe that the natural multiplication map

$$(V \otimes V^\vee) \otimes (\mathcal{O}_{V^\vee} \otimes S) \rightarrow (\mathcal{O}_{V^\vee} \otimes S),$$

induces $V \otimes V^\vee \rightarrow \text{End} \mathcal{R}$ and α_2 is via $Z : \hat{\mathfrak{g}} \rightarrow V \otimes V^\vee \rightarrow \text{End}(\mathcal{R})$. Explicitly, if we write $a \otimes b \in \mathcal{O}_{V^\vee} \otimes S$, then

$$\alpha_1(\xi)(a \otimes b) = a \otimes Z(\xi)(b).$$

Let's write $Z(\xi) = \sum_{ij} \xi_{ij} a_i \otimes a_j^*$. Then

$$\alpha_2(\xi)(a \otimes b) = \sum_{ij} \xi_{ij} a a_i \otimes b a_j^*.$$

¹The $D_{V^\vee}$ -module structure on $\mathcal{R}$ is given as follows: ∂_{a_i} acts on $\mathcal{R}$ as $\partial_{a_i} \otimes 1 + 1 \otimes a_i^*$.

Let $f = \sum a_i \otimes a_i^* \in \mathcal{R}$, which can be regarded as the universal section of the line bundle $\mathcal{O}_{V^\vee} \boxtimes \mathcal{L}$ over $V^\vee \times X$. Recall that $U = V^\vee \times X - \mathbb{L}^\perp$. Then

$$\mathcal{O}_U = (\mathcal{O}_{V^\vee} \otimes S(\hat{X}))_{(f)}$$

is the homogeneous localization of $\mathcal{R}$ with respect to f , where the degree of $a \otimes b \in \mathcal{O}_{V^\vee} \otimes S(\hat{X})$ is the degree of b in the graded ring $S(\hat{X})$. As $\mathcal{L}^{-1} = \omega_X$, f^{-1} can be regarded as a rational section of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$, regular on U . Then $\mathcal{O}_U f^{-1}$ can be identified with the regular sections of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$ over U . In other words,

$$(2.1) \quad \mathcal{O}_U f^{-1} \simeq \omega_{U/V^\vee} = (\mathcal{O}_{V^\vee} \boxtimes \omega_X)|_U.$$

Therefore, it is equipped with a $(D_{V^\vee} \boxtimes D_X^{op})|_U$ module structure (see [5, VI, §3] for the definition of right D_X module structure on ω_X). As $\mathfrak{g}$ maps to the vector fields on X , $\mathcal{O}_U f^{-1}$ is a $D_{V^\vee} \times \mathfrak{g}$ -module. We will describe this structure more explicitly. First, we describe the $D_{V^\vee}$ -module structure. Let θ be a vector field on $V^\vee$, and $\xi \in \mathfrak{g}$. It is enough to describe $\theta(f^{-1})$ and $(f^{-1})\xi$. Let us write $Z(\xi) = \sum \xi_{ij} a_i \otimes a_j^*$ as before.

Lemma 2.6. *We have*

$$\theta(f^{-1}) = \frac{\sum_i \theta(a_i) \otimes a_i^*}{f^2} \in \mathcal{O}_U f^{-1},$$

and

$$(f^{-1})\xi = \frac{\sum \xi_{ij} a_i \otimes a_j^*}{f^2} \in \mathcal{O}_U f^{-1}.$$

Proof. Let $v \in V^\vee$, regarded as a section of $\mathcal{L}$. Then v^{-1} is a rational section of ω_X , and $\omega = 1 \otimes v^{-1}$ is a rational section of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$, obtained by pullback of a rational section of ω_X . Note that $g = (1 \otimes v)/f \in \mathcal{O}_U$, and we can write $f^{-1} = g(1 \otimes v^{-1})$. By definition, for a vector field θ on $V^\vee$, $\theta(\omega) = 0$, and for $\xi \in \mathfrak{g}$, $\xi(\omega) = -1 \otimes \text{Lie}_\xi v^{-1}$, where $\text{Lie}_\xi : \omega_X \rightarrow \omega_X$ is the Lie derivative (see [5, VI, §3] for the definition of right D-module structures on ω_X). Therefore

$$\theta(f^{-1}) = \theta(g)\omega, \quad (f^{-1})\xi = -\xi(g)\omega - g(1 \otimes \text{Lie}_\xi v^{-1}).$$

Note that

$$\theta(g) = \theta\left(\frac{1}{\sum a_i \otimes \frac{a_i^*}{v}}\right) = \frac{\sum \theta(a_i) \otimes \frac{a_i^*}{v}}{(\sum a_i \otimes \frac{a_i^*}{v})^2} = g^2 \sum \theta(a_i) \otimes \frac{a_i^*}{v} = g \frac{\sum \theta(a_i) \otimes a_i^*}{f}.$$

Therefore, the first equation holds. On the other hand

$$\xi(g) = \xi\left(\frac{1 \otimes v}{f}\right) = \frac{1 \otimes Z(\xi)(v)}{f} - \frac{(1 \otimes v) \sum \xi_{ij} a_i \otimes a_j^*}{f^2}.$$

To prove the second, we need to understand $\text{Lie}_\xi v^{-1}$. We consider the general situation. Let X be a Fano projective variety. Assume that $\mathcal{L} = \omega_X^{-1}$ is very ample, and $X \rightarrow \mathbb{P}(V)$ be the closed embedding where $V = \Gamma(X, \mathcal{L})^\vee$. Then $\mathfrak{g} = \Gamma(X, T_X)$ is a Lie algebra and $\mathcal{L}$ is naturally $\mathfrak{g}$ -linearized. Therefore, V is a natural $\mathfrak{g}$ -module with action $Z : \mathfrak{g} \rightarrow \text{End}(V)$. As $Z(\xi) = \sum \xi_{ij} a_i \otimes a_j^*$, we have $Z(\xi)(v) = \sum \xi_{ij} a_i(v) a_j^*$ for $v \in V^\vee$. On the other hand, recall that $\mathfrak{g}$ acts on ω_X by Lie derivatives. Note that for $v \in V^\vee$, v^{-1} can be regarded as a rational section of ω_X .

Lemma 2.7. *Let $\xi \in \mathfrak{g}, 0 \neq v \in V^\vee$ and write $Z(\xi) = \sum \xi_{ij} a_i \otimes a_j^*$. Then*

$$\text{Lie}_\xi v^{-1} = -\frac{Z(\xi)(v)}{v} v^{-1}.$$

Now Lemma 2.6 follows. $\square$

Note that explicitly, the $D_{V^\vee} \times \mathfrak{g}$ -module structure on $\mathcal{O}_U f^{-1}$ can be described as follows. Let $\theta = \partial_{a_i}$ be a vector field on $V^\vee$ and $\xi = \sum \xi_{ij} a_i \otimes a_j^* \in \mathfrak{g}$, $m = \frac{1}{f^{k+1}}(a \otimes b) \in \mathcal{O}_U f^{-1}$, where $a \in \mathcal{O}_{V^\vee}$ and $b \in S$ is homogeneous of degree k , then

$$\begin{aligned} \partial_{a_i}(m) &= \frac{\partial_{a_i}(a) \otimes b}{f^{k+1}} + (-1)^{k+1}(k+1) \frac{a \otimes b a_i^*}{f^{k+2}}, \\ (m)\xi &= -\frac{1}{f^{k+1}}(a \otimes Z(\xi)(b)) + \frac{k+1}{f^{k+2}} \left(\sum \xi_{ij} a a_i \otimes b a_j^* \right). \end{aligned}$$

We extend this to a $\hat{\mathfrak{g}}$ -module by requiring that e acts by zero on $\mathcal{O}_U f^{-1}$.

Now, we have the following technical lemma. Recall that $\beta(e) = 1$.

Lemma 2.8. *The map $\phi : \mathcal{R} \otimes \beta \rightarrow \mathcal{O}_U f^{-1}$ given by*

$$\phi(a \otimes b) = \frac{(-1)^k k!}{f^{k+1}} a \otimes b$$

is a $D_{V^\vee} \times \hat{\mathfrak{g}}$ -module homomorphism. In addition, it induces an isomorphism

$$\tau = (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k \simeq (\mathcal{O}_U f^{-1}) \otimes_{\hat{\mathfrak{g}}} k = (\mathcal{O}_U f^{-1}) \otimes_{\mathfrak{g}} k.$$

Proof. A direct calculation shows that ϕ is a $D_{V^\vee} \times \hat{\mathfrak{g}}$ -module homomorphism. Namely, we know that ∂_{a_i} acts on $\mathcal{R}$ by $\partial_{a_i} \otimes 1 + 1 \otimes a_i^*$. Therefore,

$$\phi(\partial_{a_i}(a \otimes b)) = \phi(\partial_{a_i}(a) \otimes b + a \otimes b a_i^*) = \frac{(-1)^k k!}{f^{k+1}} (\partial_{a_i}(a) \otimes b) + \frac{(-1)^{k+1} (k+1)!}{f^{k+2}} (a \otimes b a_i^*),$$

which is the same as $\partial_{a_i} \phi(a \otimes b)$. The $\mathfrak{g}$ -equivariance can be checked similarly.

Clearly ϕ is surjective, with the kernel spanned by $(k+1)a \otimes b + f(a \otimes b)$ for b homogeneous of degree k . But $(a \otimes b)\alpha(e) = (k+1)a \otimes b + f(a \otimes b)$. The lemma is proved. $\square$

To apply this lemma, recall the definition of $\pi_+^\vee$ for $\pi^\vee : U \rightarrow V^\vee$ a smooth morphism of algebraic varieties. As $\pi^\vee$ is an affine morphism,

$$\pi_+^\vee \mathcal{N} = \Omega_{U/V^\vee}^\bullet \otimes \mathcal{N}[\dim X].$$

In particular,

$$H^0 \pi_+^\vee \mathcal{N} = \text{coker}((\mathcal{O}_{V^\vee} \boxtimes \Omega_X^{\dim X-1} \otimes D_X \otimes_{U_{\mathfrak{g}}} k)|_U \rightarrow (\mathcal{O}_{V^\vee} \boxtimes \omega_X \otimes D_X \otimes_{U_{\mathfrak{g}}} k)|_U).$$

As $\text{coker}(\Omega_X^{\dim X-1} \otimes D_X \rightarrow \omega_X \otimes D_X) = \omega_X$ as right D_X -modules, $H^0 \pi_+^\vee \mathcal{N}$ is exactly $(\mathcal{O}_{V^\vee} \boxtimes \omega_X)|_U \otimes_{U_{\mathfrak{g}}} k \simeq \tau$. This completes the proof of Theorem 2.1.

We continue to let X be a general smooth projective G -variety, and let $\beta(e) = 1$ and $\beta|_{\mathfrak{g}} = 0$, and consider some consequences of Theorem 2.1. By taking the solution sheaves on both sides in Corollary 2.2, we get an injective map

$$H_n(X - V(f_a)) \rightarrow \text{Hom}(\tau, \mathcal{O}_{V^\vee, a}).$$

This gives an explicit lower bound for the solution rank of τ at any point a . It is interesting to compare this with the “cycle-to-period” map constructed in [17] when $k = \mathbb{C}$. The latter can be described in differential geometric terms as follows. For $\mathcal{L} = \omega_X^{-1}$, the $D_{V^\vee} \times \mathfrak{g}$ -module isomorphism (2.1) is given, at each point a , by

$$\frac{1}{f_a} \mapsto \frac{\Omega}{f_a},$$

where Ω is the canonical G -invariant $\mathbb{G}_m$ -horizontal form of degree $\dim X$ on $\mathbb{L}^\vee - \{0\}$ ([17, §4]). Here $\frac{\Omega}{f_a}$ is a $\mathbb{G}_m$ -invariant meromorphic form on the principal $\mathbb{G}_m$ -bundle $\pi^\vee : (\mathbb{L}^\vee - \{0\}) \rightarrow X$, and thus can be viewed as a meromorphic form on the base X with pole along $V(f_a)$. It follows that

$$\tau = \mathcal{O}_U f^{-1} \otimes_{\mathfrak{g}} k \simeq \omega_{U/V^\vee} \otimes_{\mathfrak{g}} k, \quad 1 \mapsto \frac{\Omega}{f_a}$$

as $D_{V^\vee}$ -modules. But τ is generated by 1, so $\omega_{U/V^\vee} \otimes_{\mathfrak{g}} k$ is generated by $\frac{\Omega}{f}$ under the Gauss-Manin connection.

Corollary 2.9. *The cycle-to-period map $H_n(X - V(f_a)) \rightarrow \text{Hom}(\tau, \mathcal{O}_{V^\vee, a})$,*

$$\gamma \mapsto \langle \gamma, \frac{\Omega}{f_a} \rangle = \int_\gamma \frac{\Omega}{f_a},$$

is injective.

Proof. For $\gamma \in \text{Hom}(H^0 \pi_+^\vee \mathcal{O}_U, \mathcal{O}_{V^\vee, a}) = H_n(X - V(f_a))$, its image is a $D_{V^\vee}$ -linear function $\tau \simeq \omega_{U/V^\vee} \otimes_{\mathfrak{g}} k \rightarrow \mathcal{O}_{V^\vee, a}$. Since τ_a is generated by $\frac{\Omega}{f_a}$, if $\langle \gamma, \frac{\Omega}{f_a} \rangle = 0$ then $\langle \gamma, \tau_a \rangle = 0$, hence $\gamma = 0$. $\square$

We also have the following interesting topological consequence, which answers a question S. Bloch communicated to us.

Corollary 2.10. *Let $a \in V^\vee$. Then for a' close to a , the map $H_n(X - V(f_a)) \rightarrow H_n(X - V(f_{a'}))$ induced by parallel transport is injective.*

Proof. Since any analytic solution to τ at a extends to some neighborhood of a , the map $\text{Hom}(\tau, \mathcal{O}_{V^\vee, a}) \rightarrow \text{Hom}(\tau, \mathcal{O}_{V^\vee, a'})$ given by analytic continuation is injective. Now the claim follows from Corollary 2.9. $\square$

Remark 2.1. The two preceding corollaries rely only on the fact that $\omega_{U/V^\vee} \otimes_{\mathfrak{g}} k$ is generated by Ω/f as $D_{V^\vee}$ -module, which can be shown without using τ at all. Thus the two results are general facts about the D-module $H\pi_+^\vee \mathcal{N}$, as apposed to τ specifically.

3. !-FIBERS OF τ

In the following three sections, we consider τ when $\beta(e)$ is not necessarily 1. Here we will give a formula of the !-fibers of τ at $a \in V^\vee$. For $a \in V^\vee$, let $i_a : \{a\} \rightarrow V^\vee$ be the inclusion and for simplicity, let us write

$$\tau_a^! = i_a^! \tau.$$

This is a complex of vector spaces and our goal is to give an expression of this complex.

Recall that

$$\tau = (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k.$$

As a result, we have

$$\tau_a^! = k_a \otimes_{\mathcal{O}_{V^\vee}}^L ((\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k)[- \dim V],$$

where $k_a = \mathcal{O}_{V^\vee} / \mathfrak{m}_a$ is the residual field at a , and $\mathfrak{m}_a$ is the maximal ideal of $\mathcal{O}_{V^\vee}$ corresponding to a .

The advantage of this expression of $\tau_a^!$ is that we can first calculate $k_a \otimes_{\mathcal{O}_{V^\vee}}^L \mathcal{R}$ as a (complex of) right $\hat{\mathfrak{g}}$ -modules, and then taking the Lie algebra coinvariants. Namely, we have the Koszul resolution of k_a , which gives the complex that calculates $\tau_a^!$

$$(3.1) \quad \tau_a^! = \left(\bigwedge V \otimes \mathcal{O}_{V^\vee} \otimes S \right) \otimes_{\hat{\mathfrak{g}}} (-\beta).$$

where $V \otimes \mathcal{O}_{V^\vee} \rightarrow \mathcal{O}_{V^\vee}$ is given by $v \otimes 1 \mapsto v - v(a)$. In general, this complex is difficult to compute. However, when $a = 0$, this is more tractable.

In fact, at $a = 0$, for the degree r -term, we have

$$(3.2) \quad H^r \tau_0^! \simeq H_0(\hat{\mathfrak{g}}, S \otimes \beta),$$

where $\mathfrak{g}$ acts on S via $Z : \hat{\mathfrak{g}} \rightarrow \text{End} V \rightarrow \text{End} S$, $Z(\xi) = \sum_{ij} \xi_{ij} a_i \otimes a_j^*$.

Corollary 3.1. *Assume that $\beta(e) \notin \mathbb{Z}_{\leq 0}$. Then $H^r \tau_0^! = 0$.*

Proof. The homothety $\mathbb{G}_m$ acts on S by nonnegative weights. Therefore, if $\beta(e) \notin \mathbb{Z}_{\leq 0}$, the coinvariant of $S \otimes \beta$ with respect to this $\mathbb{G}_m$ is zero. $\square$

Remark 3.1. For a general point $a \in V^\vee$ we can still express $H^r \tau_a^! \simeq H_0(\hat{\mathfrak{g}}, S \otimes \beta)$. But if $a \neq 0$, the action of $\hat{\mathfrak{g}}$ on S will be the sum of two actions (induced by the actions α_1 and α_2 of $\mathfrak{g}$ on $\mathcal{O}_{V^\vee} \otimes S$, as described §2 right after Corollary 2.4). Concretely, the first action is via Z , and the second is via the $\xi(b) = \sum \xi_{ij} a_i(a) b a_j^*$ for $b \in S$. Therefore, unless $a = 0$, S is not a finite dimensional $\hat{\mathfrak{g}}$ -module and is difficult to compute.

From now on, we assume that $\beta(e) \notin \mathbb{Z}_{\leq 0}$.

Let us calculate $H^{r-1}\tau_0^!$. We have

$$\begin{array}{ccccc} (V \wedge V) \otimes (\mathcal{R} \otimes \beta) & \xrightarrow{m_2} & V \otimes (\mathcal{R} \otimes \beta) & \xrightarrow{m_1} & \mathcal{R} \otimes \beta \\ \downarrow & & \downarrow & & \downarrow \\ (V \wedge V) \otimes (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k & \xrightarrow{d_2} & V \otimes (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k & \xrightarrow{d_1} & (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k. \end{array}$$

Then

$$H^{r-1}\tau_0^! = m_1^{-1}((\mathcal{R} \otimes \beta)\hat{\mathfrak{g}})/(\text{Im } m_2 + V \otimes (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}})$$

As the Koszul complex is acyclic away from degree zero, we can rewrite the above as

$$H^{r-1}\tau_0^! = (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \text{Im } m_1 / (\text{Im } m_1)\hat{\mathfrak{g}}.$$

Consider

$$0 \rightarrow (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \text{Im } m_1 / (\text{Im } m_1)\hat{\mathfrak{g}} \rightarrow (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} / (\text{Im } m_1)\hat{\mathfrak{g}} \rightarrow (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} / (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \text{Im } m_1 \rightarrow 0.$$

Note that $0 = H^r\tau_0^!$ implies that $(\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} + \text{Im } m_1 = \mathcal{R} \otimes \beta$. Therefore,

$$(\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} / (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \text{Im } m_1 = \mathcal{R} \otimes \beta / \text{Im } m_1.$$

We therefore can write

$$H^{r-1}\tau_0^! = \ker((\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} / (\text{Im } m_1)\hat{\mathfrak{g}} \rightarrow \mathcal{R} \otimes \beta / \text{Im } m_1).$$

Therefore, there is a surjective map

$$H_1(\hat{\mathfrak{g}}, S \otimes \beta) \rightarrow H^{r-1}\tau_0^!,$$

where $\hat{\mathfrak{g}}$ acts on S via Z . (So S are direct sums of finite dimensional representations of $\hat{\mathfrak{g}}$.)

Lemma 3.2. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, we have $H_1(\hat{\mathfrak{g}}, S \otimes \beta) = 0$.*

Proof. Consider the $\hat{\mathfrak{g}}$ coinvariants functor as the composition of $\mathfrak{g}$ coinvariants functor, and the $\mathbb{C}$ coinvariants functor. The E_2 terms of the Grothendieck spectral sequence contributing to $H_1(\hat{\mathfrak{g}}, S \otimes \beta)$ are $H_1(\mathbb{C}, H_0(\mathfrak{g}, S \otimes \beta))$ and $H_0(\mathbb{C}, H_1(\mathfrak{g}, S \otimes \beta))$. As $S \otimes \beta$ breaks as direct sums according to weights as a $\mathfrak{g}$ -module, and $\mathbb{C}$ acts on each given weight piece as the weight plus $\beta(e)$, it is clear that under the above assumption on $\beta(e)$, both $H_1(\mathbb{C}, H_0(\mathfrak{g}, S \otimes \beta))$ and $H_0(\mathbb{C}, H_1(\mathfrak{g}, S \otimes \beta))$ are zero. $\square$

4. THE GEOMETRY

Let X be a smooth projective variety and $\mathcal{L}$ a very ample line bundle which gives $X \rightarrow \mathbb{P}(V)$, where $V^\vee = \Gamma(X, \mathcal{L})$. Let $\hat{i} : \hat{X} \rightarrow V$ be the closed embedding of the cone of X into V . Let $\mathbb{L}$ be the totally space of $\mathcal{L}^\vee$. Then

$$i_{\mathbb{L}} : \mathbb{L} \rightarrow X \times V$$

is a rank one subbundle of the trivial vector bundle over X with fiber V . The following diagram is commutative

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{i_{\mathbb{L}}} & X \times V \\ \pi \downarrow & & \downarrow \pi \\ \hat{X} & \xrightarrow{\hat{i}} & V \end{array}$$

and the left vertical arrow realizes $\mathbb{L}$ as the blow-up of $\hat{X}$ at the origin. We denote the open immersion

$$j_{\mathbb{L}}^\circ : \mathring{\mathbb{L}} = \mathbb{L} - X \rightarrow \mathbb{L},$$

where X is regarded as the zero section of $\mathbb{L}$.

Let $\mathbb{L}^\vee$ be the dual of $\mathbb{L}$, i.e., the total space of $\mathcal{L}$, and $j_{\mathbb{L}^\vee}^\circ : \mathring{\mathbb{L}}^\vee \rightarrow \mathbb{L}^\vee$ be the open subset away from the zero section. The dual of $i_{\mathbb{L}}$ is the evaluation map

$$ev : X \times V^\vee \rightarrow \mathbb{L}^\vee$$

which sends (x, a) to $a(x) \in \mathbb{L}^\vee$.

Let $i_{\mathbb{L}^\perp} : \mathbb{L}^\perp \rightarrow X \times V^\vee$ be the orthogonal complement of $\mathbb{L}$ in $X \times V^\vee$, i.e. the kernel of ev . The projection

$$\mathbb{L}^\perp \xrightarrow{i_{\mathbb{L}^\perp}} X \times V^\vee \xrightarrow{\pi^\vee} V^\vee$$

realizes $\mathbb{L}^\perp$ as the universal family of hyperplane sections of X . We still denote this projection by $\pi^\vee$. Let $j_U : U = X \times V^\vee - \mathbb{L}^\perp \rightarrow X \times V^\vee$ be the complement. For $a \in V^\vee$, the fiber U_a of $U \rightarrow V^\vee$ over a is $X - V(f_a)$, where f_a is the section of $\mathcal{L}$ given by a and $V(f_a)$ is its divisor. Note that the following diagram is Cartesian.

$$(4.1) \quad \begin{array}{ccc} U & \xrightarrow{j_U} & X \times V^\vee \\ ev \downarrow & & \downarrow ev \\ \mathring{\mathbb{L}}^\vee & \xrightarrow{j_{\mathbb{L}^\vee}^\circ} & \mathbb{L}^\vee. \end{array}$$

5. AN EXACT SEQUENCE FOR τ

We will complete the proof of Theorem 1.9 in this section.

Let $i_0 : \{0\} \rightarrow V$ be the inclusion of the origin, and $j_0 : \mathring{V} \rightarrow V$ be the open embedding of the complement. Let $\mathring{X} = \hat{X} - \{0\}$. The open inclusion $\mathring{X} \rightarrow \hat{X}$ is still denoted by j_0 and the closed inclusion $\mathring{X} \rightarrow \mathring{V}$ is denoted by i . By specializing (A.4), we have the following important sequence for $\hat{\tau} = \mathcal{F}our(\tau)$

$$0 \rightarrow i_{0,+} H^{-1} i_0^+ \hat{\tau} \rightarrow H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) \rightarrow \hat{\tau} \rightarrow i_{0,+} H^0 i_0^+ \hat{\tau} \rightarrow 0.$$

First we make a simplification of this sequence.

Lemma 5.1. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, $i_{0,+} H^0 i_0^+ \hat{\tau} = 0$.*

Proof. Assume that $H^0 i_0^+ \hat{\tau} = k^\ell$, so that $i_{0,+} H^0 i_0^+ \hat{\tau} = \delta_0^\ell$. I.e. there is a surjective map of D-modules $\hat{\tau} \rightarrow \delta_0^\ell$ on V . Taking the Fourier transform, we therefore have a surjective map $\tau \rightarrow \mathcal{O}_{V^\vee}^\ell$. Taking the right exact functor $H^r i_0^!$, i.e., the r th cohomology of the $!$ -fibers at $0 \in V^\vee$, we have a surjective map $H^r \tau_0^! \rightarrow k^\ell$. By Corollary 3.1, $\ell = 0$. $\square$

As a result, under our assumption

$$(5.1) \quad 0 \rightarrow i_{0,+} H^{-1} i_0^+ \hat{\tau} \rightarrow H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) \rightarrow \hat{\tau} \rightarrow 0.$$

Let $d = \dim_k H^{-1} i_0^+ \hat{\tau}$. Then $i_{0,+} H^{-1} i_0^+ \hat{\tau} = \delta_0^d$. Taking the Fourier transform of this sequence, we therefore obtain

$$(5.2) \quad 0 \rightarrow \mathcal{O}_{V^\vee}^d \rightarrow \mathcal{F}our(H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}})) \rightarrow \tau \rightarrow 0.$$

We next understand $\mathcal{F}our(H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}))$. Clearly, $\hat{\tau}$ is set-theoretically supported on $\hat{X}$. The following lemma is clear.

Lemma 5.2. *For $\mathcal{L} = \omega_X^{-1}$, we have $\hat{\tau}|_{\hat{X}} = D_{\hat{X},\beta'}$, where $D_{\hat{X},\beta'}$ is the D-module on $\hat{X}$ as introduced in Lemma A.6.*

Note that $\mathring{\mathbb{L}} \simeq \mathring{X}$ and therefore $\hat{\tau}|_{\mathring{X}} \simeq D_{\mathring{X},\beta'}$ can be regarded as a D-module on $\mathring{\mathbb{L}}$, which is naturally $(\mathbb{G}_m, \beta'(e))$ -equivariant. Then

$$(5.3) \quad H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) = H^0 \pi_!(i_{\mathbb{L},!} j_{\mathbb{L},!}^* D_{\mathring{X},-\beta'}).$$

According to Lemma A.14,

$$(5.4) \quad \mathcal{F}our(H^0 \pi_!(i_{\mathbb{L},!} j_{\mathbb{L},!}^* D_{\mathring{X},-\beta'})) = H^0 \pi_!^\vee \mathcal{F}our_X(i_{\mathbb{L},!} j_{\mathbb{L},!}^* D_{\mathring{X},-\beta'}).$$

By (A.7),

$$(5.5) \quad \mathcal{F}our_X(i_{\mathbb{L},!} j_{\mathbb{L},!}^* D_{\check{X}, -\beta'}) = ev^! \mathcal{F}our_X(j_{\mathbb{L},!}^* D_{\check{X}, -\beta'})[1-r].$$

Finally,

Lemma 5.3. *There is an isomorphism*

$$\mathcal{F}our_X(j_{\mathbb{L},!}^* D_{\check{X}, -\beta'}) \simeq j_{\mathbb{L},+}^* D_{\mathbb{L}^\vee, -\beta}.$$

Lemma 5.4. *Assume that $\beta(e) \notin \mathbb{Z}_{\leq 0}$. We have $d = \dim H^{r-1} \tau_0^! = 0$.*

Proof. The second statement follows from 3.2. We need to establish the first equality. For simplicity, let us denote $\mathcal{N} := ev^!(D_{\mathbb{L}^\vee, -\beta})[1-r]$. This is a plain D-module on U .

Taking $i_0^!$ of (5.1), it is enough to show that

$$H^r i_0^! H^0 \pi_+^\vee \mathcal{N} = H^{r-1} i_0^! H^0 \pi_+^\vee \mathcal{N} = 0.$$

Consider the distinguished triangle

$$i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N} \rightarrow i_0^! \pi_+^\vee \mathcal{N} \rightarrow i_0^! H^0 \pi_+^\vee \mathcal{N} \rightarrow .$$

The long exact sequence associated to this triangle is

$$H^{r-1} i_0^! \pi_+^\vee \mathcal{N} \rightarrow H^{r-1} i_0^! H^0 \pi_+^\vee \mathcal{N} \rightarrow H^r i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N} \rightarrow H^r i_0^! \pi_+^\vee \mathcal{N} \rightarrow H^r i_0^! H^0 \pi_+^\vee \mathcal{N} \rightarrow 0.$$

Note that U does not intersect with $X \times \{0\} \subset X \times V^\vee$. Therefore, $i_0^! \pi_+^\vee \mathcal{N} = 0$. This implies that $H^r i_0^! H^0 \pi_+^\vee \mathcal{N} = 0$, and $H^{r-1} i_0^! H^0 \pi_+^\vee \mathcal{N} = H^r i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N}$. But $H^{\leq -1} \pi_+^\vee \mathcal{N}$ sits in cohomological degree ≤ -1 and $i_0^!$ has cohomological amplitude r , $H^r i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N} = 0$. $\square$

We can now complete the proof of Theorem 1.9.

Proof. Combining (5.3)-(5.5) and Lemma 5.3, we can rewrite (5.1) as

$$(5.6) \quad 0 \rightarrow \mathcal{O}_{V^\vee}^d \rightarrow H^0 \pi_+^\vee ev^!(D_{\mathbb{L}^\vee, -\beta})[1-r] \rightarrow \tau \rightarrow 0.$$

Theorem 1.9 follows immediately from Lemma 5.4 and the sequence (5.6). $\square$

6. RANK 1 POINTS OF GENERAL G/P

In this section, we will describe a rank 1 point for a general homogeneous variety X in the case of $\mathcal{L} = \omega_X^{-1}$, using the projected Richardson stratification of X studied in [18][20][8][13]².

We follow the notations in [13]. Let G be a reductive algebraic group over an algebraically closed field k of characteristics zero, B a Borel subgroup and $P \supset B$ a parabolic subgroup

²We thank T. Lam for pointing out this stratification to us.

in G . Put $B^+ = B$ and let B^- be the opposite Borel subgroup. Let $Q(W, W_P)$ be the set of equivalence classes of P -Bruhat intervals [13, §2]. Each equivalence class is uniquely specified by a pair (u, w) of elements in the Weyl group. For $(u, w) \in Q(W, W_P)$, put $X_u^w := (B^-uB/B) \cap (B^+wB/B)$, an open Richardson variety in G/B .

Proposition 6.1. [20, §7][13, §3] *There is a stratification of $X = G/P$ of the form $X = \coprod_{(u,w) \in Q(W, W_P)} \mathring{\Pi}_u^w$, where each stratum $\mathring{\Pi}_u^w$ is the isomorphic image of X_u^w under the natural projection $G/B \rightarrow G/P$.*

Let $\Pi_1, \dots, \Pi_s$ be the closures of the codimension 1 strata in X . By Lemma 5.4 [13], $\cup_i \Pi_i$ is an anticanonical divisor in X . Since $X - \cup_i \Pi_i$ is the largest stratum, it is isomorphic to an open Richardson variety X_u^w in G/B . It is well-known that (see for example [19])

$$(6.1) \quad H_c^N(X_u^w) = \text{Hom}(M_u, M_w)$$

where $N = \dim X_u^w$ and M_w denotes the Verma module of the Lie algebra of G of highest weight $-w(\rho) - \rho$.

By combining Theorems 1-4 [2], or by the Kazhdan-Lusztig conjecture, one has

$$(6.2) \quad \dim \text{Hom}(M_u, M_w) = 1$$

Therefore, by Theorem 1.4 we have

Corollary 6.2. *Let $a \in \Gamma(X, \omega_X^{-1})$ be the defining section of the anticanonical divisor $\cup_i \Pi_i$. Then a is a rank 1 point of X .*

Example 6.3. Consider the Grassmannian $X = G(d, n)$. According to [14], $\cup_i \Pi_i$ is defined by the section $a = x_{1,2,\dots,d}x_{2,3,\dots,d+1}\dots x_{n,1,\dots,d-1}$, where the $x_{i_1,\dots,i_d}$ are the Plücker coordinates of X . This generalizes a construction in [4] for $d = 2$.

7. RANK 1 POINTS OF 1-STEP FLAGS

Notation. If m is an $p \times q$ matrix, and $J \subset (1, 2, \dots, p)$ is an ordered index set, then m_J denotes the submatrix of m given by the rows labelled by J , and we also call m_J the J -block of m . We denote by x_J , $J \subset (1, 2, \dots, n)$, the Plücker coordinates of the d -plane Grassmannian $F(d, n)$. Let M be the space of rank d matrices of size $n \times d$. Then GL_d acts freely and properly on M by right multiplication and $M/GL_d \cong F(d, n)$. Under this identification, we denote the projection map of the Stiefel bundle $M \rightarrow X$ by $m \mapsto [m] := m \cdot GL_d$. Then x_J can be viewed as the function $x_J : M \rightarrow \mathbb{C}$, $m \mapsto \det(m_J)$. Given a section f of any line bundle on X , we denote by $X(f)$ the complement of $f = 0$ in X , and by $M(f)$ the preimage of $X(f)$ under $M \rightarrow X$.

Proposition 7.1. *The 1-step flag variety $X = F(d, n)$ admits a rank 1 point $f \in \Gamma(X, \omega_X^{-1})$ such that $(x_J)^k | f$ for some $J \subset (1, 2, \dots, n)$ with $|J| = d$ and $k = \min(d, n - d)$. If $n = 2d$, then $f = (x_{1, \dots, d})^d (x_{d+1, \dots, n})^d$ is a rank 1 point.*

Proof. (a) Consider the case $n \geq l + d \geq 2d$. We have

$$(7.1) \quad \begin{aligned} X_1 &:= F(d, n - l) \hookrightarrow X, \quad E \mapsto E \oplus 0_l \\ X_2 &:= F(d, l) \hookrightarrow X, \quad E \mapsto 0_{n-l} \oplus E. \end{aligned}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-l} \oplus \mathbb{C}^l$. Let f_1 be a given rank 1 point of X_1 such that $(x_{1, \dots, d})^{k_1} | f_1$, $k_1 = \min(d, n - l - d)$, and f_2 a rank 1 point of X_2 such that $(x_{n-d+1, \dots, n})^{k_2} | f_2$, $k_2 = \min(d, l - d)$. (In case $l = d$, $X_2 = \text{pt}$, we simply take $f_2 = (x_{n-d+1, \dots, n})^d$; in case $n - l = d$, $X_1 = \text{pt}$, we take $f_1 = (x_{1, \dots, d})^d$). We can view f_1, f_2 as sections of $\mathcal{O}_X(n - l)$ and $\mathcal{O}_X(l)$ respectively on $X = F(d, n)$. Then the restriction of f_1 to X_1 under (7.1) becomes a section of $\mathcal{O}_{X_1}(n - l)$. Likewise the restriction of f_2 to X_2 becomes a section of $\mathcal{O}_{X_2}(l)$. We claim that $f = f_1 f_2 \in \Gamma(X, \omega_X^{-1})$ is a rank 1 point of X . We will first construct an explicit isomorphism

$$X_1(f_1) \times X_2(f_2) \times GL_d \rightarrow X(f).$$

Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Since $x_J | f_2$, $J = (n - d + 1, \dots, n)$, each $m'_2 \in M_2(f_2)$ has a nonsingular J -block D . Define

$$M_1(f_1) \times M_2(f_2) \rightarrow M(f), \quad m'_1, m'_2 \mapsto m = \begin{bmatrix} m'_1 D \\ m'_2 \end{bmatrix}.$$

This is well-defined since

$$f(m) = f_1(m'_1 D) f_2(m'_2) = (\det D)^{n-l} f_1(m'_1) f_2(m'_2).$$

The map is a bijection with inverse $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \mapsto m_1(m_2)_J^{-1}, m_2$. Now let $h \in GL_d$ act on $M_1(f_1) \times M_2(f_2)$ by the formula $(m'_1, m'_2 h^{-1})$. Then the map is equivariant. It follows that we have an isomorphism

$$M_1(f_1) \times X_2(f_2) \rightarrow X(f).$$

Finally, since $x_{1, \dots, d} | f_1$ each $m'_1 \in M_1(f_1)$ has a nonsingular top $d \times d$ block. It follows that the principal bundle $GL_d - M_1(f_1) \rightarrow X_1(f_1)$ is trivial. In fact, it has a (unique) section of the form $[m_1] \mapsto m'_1$ where m'_1 is the unique representative in $[m_1]$ whose top $d \times d$ block is the identity matrix I_d . This proves that

$$X(f) \cong X_1(f_1) \times X_2(f_2) \times GL_d.$$

Since the $X_i(f_i)$ are affine varieties, all de Rham cohomology of degree $> \dim X_i$, vanishes. Since f_1, f_2 are rank 1 points of X_1, X_2 respectively, we have $H^{\dim X_i}(X_i(f_i)) = \mathbb{C}$ by Theorem 1.4. It follows that

$$H^{\dim X}(X(f)) \cong H^{\dim X_1}(X_1(f_1)) \otimes H^{\dim X_2}(X_2(f_2)) \otimes H^{d^2}(GL_d) \cong \mathbb{C}.$$

So f is a rank 1 point of X such that $(x_{1, \dots, d})^{k_1} (x_{n-d+1, \dots, n})^{k_2} | f$.

(b) To complete the proof of the proposition, we proceed by induction. For $X = F(1, 2) = \mathbb{P}^1$, paragraph (a) with $n = 2$ and $l = d = 1$ shows that x_1x_2 is a rank 1 point of X , and the proposition holds. Assume that it holds for up to $F(d, n-1)$, and consider the case $X = F(d, n)$. For $n < 2d$ we have $F(d, n) \cong F(n-d, n)$, in which case paragraph (a) with l, d playing the role of $d, n-d$, yields a rank 1 point f of $F(n-d, n)$ with $(x_J)^{n-d}|f$ and $|J| = n-d$. This in turn yields a rank 1 point of $F(d, n)$ divisible by $(x_{J^c})^{n-d}$ where $J^c = (1, \dots, n) - J$. For $n = 2d$, paragraph (a) with $n = l + d = 2d$ shows that $(x_{1, \dots, d})^d (x_{d+1, \dots, n})^d$ is a rank 1 point of X . For $n > 2d$, paragraph (a) with $l = d$ and our inductive hypothesis shows that X has a rank 1 point $f = f_1 \cdot (x_{n-d+1, \dots, n})^d$, where f_1 is a rank 1 point of $F(d, n-d)$. This completes the proof. $\square$

Corollary 7.2. *Let $n = l_1 + \dots + l_s$ be a partition of n with $l_p \geq d$. Let f_p be a rank 1 point of $F(d, l_p) \hookrightarrow F(d, n)$, viewed as a degree l_p polynomial in the Plücker coordinates x_J of $X = F(d, n)$ with $J \subset (l_1 + \dots + l_{p-1} + 1, \dots, l_1 + \dots + l_p)$ and $|J| = d$, such that $(x_{l_1 + \dots + l_{p-1} + 1, \dots, l_1 + \dots + l_{p-1} + d})|f_p$. Then $f = f_1 \cdots f_s$ is a rank 1 point of X . In fact, we have an isomorphism*

$$X(f) \cong X_1(f_1) \times \cdots \times X_s(f_s) \times (GL_d)^{s-1}$$

where $X_p := F(d, l_p)$.

Proof. Start with $l = l_2 + \dots + l_s$. Then paragraph (a) in the preceding proof gives

$$X(f) \cong X_1(f_1) \times X'_2(f_2 \cdots f_s) \times GL_d$$

where $X'_2 := F(d, n - l_1)$. Now the result follows by induction on s . $\square$

8. RANK 1 POINTS OF r -STEP FLAGS

Throughout this section, let $X = F(d_1, \dots, d_r, n)$ be the r -step flag variety with $r \geq 2$. We will give a recursive procedure that produces a rank 1 point of X , by assembling rank 1 points of lower step flag varieties. We begin with some notations and terminology.

Let $\mathcal{O}_i(1)$ be the standard hyperplane bundle on $F(d_i, n)$. The space of its sections is an irreducible $G = SL_n$ module of highest weight λ_{d_i} , the d_i th fundamental weight of G . We shall denote by λ_{d_i} the pullback of $\mathcal{O}_i(1)$ via the composition map

$$X \hookrightarrow F(d_1, n) \times \cdots \times F(d_r, n) \twoheadrightarrow F(d_i, n)$$

where the first is the incidence embedding and the second is the i th projection. Then $\text{Pic}(X)$ is the free abelian group generated by $\lambda_{d_1}, \dots, \lambda_{d_r}$. We also have (see [16])

$$(8.1) \quad -K_X = \omega_X^{-1} = (n - d_{r-1})\lambda_{d_r} + (d_r - d_{r-2})\lambda_{d_{r-1}} + \cdots + (d_3 - d_1)\lambda_{d_2} + d_2\lambda_{d_1}.$$

By the Borel-Weil theorem, the restriction map

$$\Gamma(F(d_1, n), \mathcal{O}_1(k_1)) \otimes \cdots \otimes \Gamma(F(d_r, n), \mathcal{O}_r(k_r)) \rightarrow \Gamma(X, \sum_i k_i \lambda_{d_i})$$

is a G -equivariant surjective map for any $k_1, \dots, k_r \in \mathbb{Z}$ (and both spaces are zero unless $k_i \geq 0$ for all i). Thus any homogeneous polynomial in the Plücker coordinates x_{J_i} with $|J_i| = d_i$, of multi-degree $(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 0}^r$, can be viewed as a section of the line bundle $\sum_i k_i \lambda_{d_i}$ on X . Conversely, any section of this line bundle on X can be expressed as such a polynomial (not necessarily unique).

Let $k < n - d_r$ and consider the embeddings

$$(8.2) \quad \begin{aligned} X_1 &:= F(d_1, \dots, d_r, n - k) \hookrightarrow X, \quad E^\bullet \mapsto E^\bullet \oplus 0_k \\ X_2 &:= F(d_1 - k, \dots, d_r - k, n - k) \hookrightarrow X, \quad E^\bullet \mapsto E^\bullet \oplus \mathbb{C}^k. \end{aligned}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-k} \oplus \mathbb{C}^k$, and X_1, X_2 are viewed as spaces consisting of r -step flags in the factor $\mathbb{C}^{n-k}$. For each Plücker coordinate $x_{J'}$ on X_1 with $J' \subset (1, 2, \dots, n - k)$, is the restriction of $x_{J'}$, regarded as a Plücker coordinate on X . Likewise, any homogeneous polynomial f_1 in the $x_{J'}$, can be viewed as the restriction of a section $\bar{f}_1$ on X involving only the same Plücker coordinates. We shall often impose certain divisibility conditions (called the hyperplane property – see below) on $\bar{f}_1$, but will state them in terms of f_1 . Similarly each Plücker coordinate $x_{J'}$ on X_2 is the restriction of $x_{J' \cup (n-k+1, \dots, n)}$ on X ; any given homogeneous polynomial f_2 in the $x_{J'}$, is the restriction of a section $\tilde{f}_2$ on X involving only the $x_{J' \cup (n-k+1, \dots, n)}$. Again, divisibility conditions imposed on $\tilde{f}_2$ will be stated in terms of f_2 .

As in the case of 1-step flags, we can view $X = M/H$, where

$$H := GL_{d_r} \times \cdots \times GL_{d_1}$$

and M is the space of r -tuple of matrices $m = (m_r, \dots, m_1)$, m_i a $d_{i+1} \times d_i$ matrix of rank d_i ($d_{r+1} \equiv n$), where $h = (h_r, \dots, h_1) \in H$ acts on M by the formula

$$(8.3) \quad m \cdot h^{-1} := (m_r h_r^{-1}, h_r m_{r-1} h_{r-1}^{-1}, \dots, h_2 m_1 h_1^{-1}).$$

Under the identification $X = M/H$, we denote the projection map $M \rightarrow X$ by $m \mapsto [m] := m \cdot H$, and call M the *Stiefel bundle* over X . We can view a Plücker coordinate x_J , $|J| = d_i$, on X as the function $x_J : M \rightarrow \mathbb{C}$, $x_J(m) = \det(m_r \cdots m_i)_J$. In particular, f_1 is a section on X_1 and $\bar{f}_1$ a section on X restricting to it as described above, then for $J = (1, \dots, n - k)$ we have

$$\bar{f}_1(m_r, \dots, m_1) = f_1((m_r)_J, m_{r-1}, \dots, m_1)$$

whenever $\text{rk } (m_r)_J = d_r$. Let $m = (m_r, \dots, m_1) \in M$ where the m_i have the form

$$m_i = \begin{bmatrix} m'_i & * \\ O & I_k \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix and O a zero block. Then $x_{J' \cup (n-k+1, \dots, n)}(m) = \det(m'_r \cdots m'_i)_{J'}$ for any $J' \subset (1, \dots, n-k)$ with $|J'| = d_i - k$. So, if f_2 is a section on X_2 and $\tilde{f}_2$ a section on X restricting to it as described above, then

$$\tilde{f}_2(m_r, \dots, m_1) = f_2(m'_r, \dots, m'_1).$$

Let f be a nonzero section of a line bundle on X , and let $X(f)$ be the complement of $f = 0$, and $M(f)$ the preimage of $X(f)$ under $M \rightarrow X$.

Definition 8.1. (Hyperplane property) We say that f has the hyperplane property if for some $J_i \subset (1, 2, \dots, n)$ with $|J_i| = d_i$, $i = 1, \dots, r$, we have $(x_{J_1} \cdots x_{J_r})|f$. In other words, the hypersurface $f = 0$ contains the union of hyperplanes $x_{J_i} = 0$.

Note that if f has the hyperplane property, we can always find a suitable permutation matrix $g \in GL_n$ such that the g -translate of f has the hyperplane property where $J_r = (n - d_r + 1, \dots, n)$. In the construction that follows, we will often arrange our section f so that this occurs. Next, we have the following elementary lemma.

Lemma 8.1. Assume f has the hyperplane property $(x_{J_1} \cdots x_{J_r})|f$. Then the principal H -bundle $M(f) \rightarrow X(f)$, has a unique section $m = (m_r, \dots, m_1)$, where the $m_r, \dots, m_1$ are matrix valued functions on $X(f)$ such that

$$(m_r \cdots m_i)_{J_i} = I_{d_i}.$$

Definition 8.2. (Special section) We call the section given in Lemma 8.1, the special section of $M(f)$ (which depends on the index sets $J_1, \dots, J_r$).

We now describe our recursive procedure that produces a rank 1 point of X with the hyperplane property.

Case 1. Assume $d_{r-1} + d_r < n$. Consider (cf. (8.2))

$$\begin{aligned} X_1 &:= F(d_1, \dots, d_{r-1}, d_r) \hookrightarrow F(d_1, \dots, d_{r-1}, n), \quad E^\bullet \mapsto 0_{n-d_r} \oplus E^\bullet \\ X_2 &:= F(d_r, n). \end{aligned}$$

Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(8.4) \quad (x_{J_1} \cdots x_{J_{r-1}})|f_1, (x_{J_r})^k|f_2$$

for some J_i with $|J_i| = d_i$, $i = 1, \dots, r$, and $J_1 = (1, \dots, d_1)$, $J_r = (n - d_r + 1, \dots, n)$, $k = \min(d_r, n - d_r) > d_{r-1}$. Such an f_2 exists by Proposition 7.1. Put

$$(8.5) \quad f = \bar{f}_1 \cdot \bar{f}_2 \cdot (x_{J_r})^{-d_{r-1}}.$$

Then we have

$$(8.6) \quad (x_{J_1} \cdots x_{J_r})|f$$

It follows easily from (8.1) that f is a section of ω_X^{-1} .

Lemma 8.2. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$\begin{aligned} M_1(f_1) \times M_2(f_2) &\rightarrow M(f) \\ (m'_{r-1}, \dots, m'_1), m'_r &\mapsto m = (m'_r, D^{-1}m'_{r-1}, m'_{r-2}, \dots, m'_1) \end{aligned}$$

where D is the J_r -block of m'_r . Therefore the map descends to an isomorphism $X_1(f_1) \times X_2(f_2) \rightarrow X(f)$.

Proof. For $m'_r \in M_2(f_2)$, its J_r -block D is a nonsingular matrix in GL_{d_2} since $(x_{J_r})^k|f_2$. Suppose $f_1(m'_{r-1}, \dots, m'_1)f_2(m'_r) \neq 0$. Then

$$f(m) = f_1((m'_r D^{-1} m'_{r-1})_{J_r}, m'_{r-2}, \dots, m'_1) f_2(m'_r) (\det(m'_r)_{J_r})^{-d_{r-1}}.$$

Since $(m'_r)_{J_r} = D$, it follows that $(m'_r D^{-1} m'_{r-1})_{J_r} = m'_{r-1}$ and we have

$$f(m) = f_1(m'_{r-1}, \dots, m'_1) f_2(m'_r) (\det D)^{-d_{r-1}} \neq 0.$$

So, the map is well-defined. Now, $h = (h_r, \dots, h_1) \in H$ acts on $M(f)$ by (8.3), and on $M_1(f_1) \times M_2(f_2)$ by the formula

$$m_r h_r^{-1}, (m_{r-1} h_{r-1}^{-1}, m_{r-1} m_{r-2} h_{r-2}^{-1}, \dots, h_2 m_1 h_1^{-1}).$$

Therefore our map is H -equivariant. Moreover, the map

$$M(f) \rightarrow M_1(f_1) \times M_2(f_2), (m_r, \dots, m_1) \mapsto ((m_r)_{J_r} m_{r-1}, m_{r-2}, \dots, m_1), m_r$$

is well-defined and is the inverse of the map above. $\square$

The lemma and Theorem 1.4 imply

Proposition 8.3. *For $d_{r-1} + d_r < n$, if any s -step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, \dots, d_r, n)$ admits one as well.*

By Proposition 7.1, for $d_1 + d_2 < n$ it follows that $F(d_1, d_2, n)$ admits a rank 1 point with the hyperplane property. This also implies that for $d_1 + d_2 > n$, then $F(d_1, d_2, n) \cong F(n - d_2, n - d_1, n)$ admits one as well.

Case 2. Assume $d_{r-1} + d_r = n$ and $r = 2$. Consider the following section of ω_X^{-1} :

$$f = (x_{1, \dots, d_1})^{d_2} (x_{d_1+1, \dots, n})^{d_2}.$$

Then by Lemma 8.1, the special section of $M(f) \rightarrow X(f)$ has the form

$$m = (m_2, m_1) = \left(\begin{bmatrix} A_2 \\ I_{d_2} \end{bmatrix}, m_1 \right) \text{ such that } A_2 m_1 = I_{d_1}.$$

Since $m_1(o)$ has rank d_1 at each point $o \in X(f)$, the second equation shows that the function

$$m_1 : X(f) \rightarrow M_1, \quad o \mapsto m_1(o)$$

is onto. Here M_1 be the Stiefel bundle over $F(d_1, d_2)$. Moreover, the level set of this function at each point is an affine space of dimension $d_1 d_2 - d_1^2$. It follows that $X(f)$ is homotopy equivalent to M_1 . Finally, the principal GL_{d_1} -bundle $M_1 \rightarrow F(d_1, d_2)$ is over a simply connected base. Thus by the Serre spectral sequence, the highest degree nonzero cohomology group of M_1 is one dimensional at degree $2d_1 d_2 - d_1^2 = \dim X$. By Theorem 1.4, we have

Proposition 8.4. *For $d_1 + d_2 = n$, $X = F(d_1, d_2, n)$ admits the rank 1 point $f = (x_1, \dots, x_{d_1})^{d_2} (x_{d_1+1}, \dots, x_n)^{d_2}$.*

Remark 8.1. The propositions in Cases 1-2 ($r = 2$) now imply that any 2-step flag variety $F(d_1, d_2, n)$ admits a rank 1 point with the hyperplane property.

Case 3. Assume $d_{r-1} + d_r = n$ and $r \geq 3$. Consider (cf. (8.2))

$$\begin{aligned} X_1 &:= F(d_1, \dots, d_{r-2}, d_{r-1}) \hookrightarrow F(d_1, \dots, d_{r-2}, n), \quad E^\bullet \mapsto E^\bullet \oplus 0_{n-d_{r-1}} \\ X_2 &:= F(d_{r-1}, d_r, n). \end{aligned}$$

Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(8.7) \quad (x_{J_1} \cdots x_{J_{r-2}})|f_1, \quad f_2 = (x_{J_{r-1}})^{d_r} (x_{J_r})^{d_r}$$

for some J_i with $|J_i| = d_i$, $i = 1, \dots, r$, and $J_1 = (1, \dots, d_{r-1})$, $J_r = (n - d_r + 1, \dots, n)$. Note that f_2 is given by Proposition 8.4. Put

$$(8.8) \quad f = \bar{f}_1 \cdot \bar{f}_2 \cdot (x_{J_{r-1}})^{-d_{r-2}} \in \Gamma(X, \omega^{-1}).$$

Then we have

$$(8.9) \quad (x_{J_1} \cdots x_{J_{r-1}} (x_{J_r})^{d_r})|f$$

Since $x_{J_{r-1}}|f_2$, the $J_{r-1} = (1, \dots, d_{r-1})$ -block D of $m'_r m'_{r-1}$ for $(m'_r, m'_{r-1}) \in M_2(f_2)$ is nonsingular.

Lemma 8.5. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$\begin{aligned} M_1(f_1) \times M_2(f_2) &\rightarrow M(f) \\ (m'_{r-2}, \dots, m'_1), (m'_r, m'_{r-1}) &\mapsto m = (m'_r, m'_{r-1}, D^{-1} m'_{r-2}, m'_{r-3}, \dots, m'_1) \end{aligned}$$

where D is the $J_{r-1} = (1, \dots, d_{r-1})$ -block of $m'_r m'_{r-1}$. Therefore the map descends to an isomorphism $X_1(f_1) \times X_2(f_2) \rightarrow X(f)$.

The proof is closely analogous to the lemma in Case 1, and will be omitted. The lemma and Theorem 1.4 imply

Proposition 8.6. *For $d_{r-1} + d_r = n$, if any s -step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, \dots, d_r, n)$ admits one such f that satisfies $(x_{J_r})^{d_r} | f$ where $J_r = (n - d_r + 1, \dots, n)$.*

Case 4. Assume $d_1 + d_2 = n$. Then $X \cong F(n - d_r, \dots, n - d_2, n - d_1, n)$, which belongs in Case 3, and the analogue of Proposition 8.6 is

Proposition 8.7. *For $d_1 + d_2 = n$, if any s -step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, \dots, d_r, n)$ admits one such f that satisfies $(x_{J_1})^{d_1} | f$ where $J_1 = (1, \dots, d_1)$.*

Case 5. Assume $d_{r-1} + d_r > n$. There exists a unique a with $r > a > 1$ such that $d_a + d_{a+1} > n \geq d_{a-1} + d_a$. Assume $n > 2d_a$ first. We will consider $n = 2d_a$ and $n = d_{a-1} + d_a$ in Cases 6-7 below separately. Consider

$$X_1 := F(d_1, \dots, d_a, n - d_a) \hookrightarrow F(d_1, \dots, d_a, n), (E_1^i) \mapsto (E_1^i \oplus 0_{d_a})$$

$$X_2 := F(d_{a+1} - d_a, \dots, d_r - d_a, n - d_a) \hookrightarrow F(d_{a+1}, \dots, d_r, n), (E_2^j) \mapsto (E_2^j \oplus \mathbb{C}^{d_a}).$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-d_a} \oplus \mathbb{C}^{d_a}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(8.10) \quad (x_{J_1} \cdots x_{J_a}) | f_1, (x_{J'_{a+1}} \cdots x_{J'_r}) | f_2$$

for some $J_i \subset (1, 2, \dots, n - d_a)$ with $|J_i| = d_i$ ($i = 1, \dots, a$) and $J_a = (n - 2d_a + 1, \dots, n - d_a)$, and for some $J'_i \subset (1, 2, \dots, n - d_a)$ with $|J'_i| = d_i - d_a$ ($i = a + 1, \dots, r$) and $J'_r = (n - d_r + 1, \dots, n - d_a)$. Put $J := (n - d_a + 1, \dots, n)$, $J_i := J'_i \cup J$, $i = a + 1, \dots, n$, and

$$(8.11) \quad f := \bar{f}_1 \cdot \tilde{f}_2 \cdot (x_J)^{d_{a+1} + d_a - n}.$$

Then f has the hyperplane property

$$(8.12) \quad (x_{J_1} \cdots \widehat{x_{J_a}} \cdots x_{J_r} x_J) | f.$$

Lemma 8.8. *The special section $m = (m_r, \dots, m_1)$ (cf. Lemma 8.1) of $M(f) \rightarrow X(f)$ has the following form:*

$$(8.13) \quad \begin{aligned} m_i &= \begin{bmatrix} m'_i & A_i \\ O & I_{d_a} \end{bmatrix}, \quad i = a + 1, \dots, r \\ m_a &= \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix} \\ m_r \cdots m_a &= \begin{bmatrix} m'_a D \\ I_{d_a} \end{bmatrix} \\ m_{a-1} &= D^{-1} m'_{a-1} \\ m_i &= m'_i, \quad i = 1, \dots, a - 2 \end{aligned}$$

where D is a GL_{d_a} -valued function, $A_a, \dots, A_r$ are matrix valued functions, and $(m'_a, \dots, m'_1)$, $(m'_r, \dots, m'_{a+1})$ are matrix valued functions taking values in the special sections of the $M_1(f_1) \rightarrow X_1(f_1)$, $M_2(f_2) \rightarrow X_2(f_2)$ respectively.

Proof. For $o \in X(f)$, we will write $m_i \equiv m_i(o)$, $m'_i \equiv m'_i(o)$, $D \equiv D(o)$, etc. Then $m = m(o) \in M(f)$ means that

$$0 \neq f(m) = \bar{f}_1(m_r \cdots m_a, m_{a-1}, \dots, m_1) \tilde{f}_2(m_r, \dots, m_{a+1}) \det(m_r)_{J_r}.$$

(a) Since $x_{J'_r}|f_2$, we have $x_{J_r}|\tilde{f}_2$, and so our m_r has the correct form, i.e. $(m_r)_{J_r} = I_{d_r}$ (hence $\det(m_r)_{J_r} = 1$), and $(m'_r)_{J'_r} = I_{d_r-d_a}$. Since $(x_{J'_{a+1}} \cdots x_{J'_r})|f_2$, we have $(x_{J_{a+1}} \cdots x_{J_r})|\tilde{f}_2$, hence $(m_r \cdots m_i)_{J_i} = I_{d_i}$. By induction on i , it is easy to see that our $m_r, \dots, m_i$ above have the correct form, so that

$$(8.14) \quad m_r \cdots m_i = \begin{bmatrix} m'_r \cdots m'_i & * \\ O & I_{d_a} \end{bmatrix}$$

and that $(m'_r \cdots m'_i)_{J'_i} = I_{d_i-d_a}$ for $i = a+1, \dots, r$. This shows that $(m'_r, \dots, m'_{a+1})$ actually lies in the special section of $M_2(f_2) \rightarrow X_2(f_2)$, as asserted.

(b) Since $x_J|f$, we have $(m_r \cdots m_a)_J = I_{d_a}$. From (8.14), it follows that $(m_a)_J = I_{d_a}$. Since $x_{J_a}|f_1$, hence $x_{J_a}|f$, it follows that $(m_r \cdots m_a)_{J_a}$ is a nonsingular matrix $D \in GL_{d_a}$. Thus m_a has the correct form as asserted, and $(m'_a)_{J_a} = I_{d_a}$. This also shows that $(m_r \cdots m_a)_{1,2,\dots,n-d_a} = m'_a D$ has rank d_a , hence

$$0 \neq \bar{f}_1(m_r \cdots m_a, m_{a-1}, \dots, m_1) = f_1(m'_a D, m_{a-1}, \dots, m_1).$$

Since f_1 is GL_{d_a} -equivariant, this is equivalent to

$$0 \neq f_1(m'_a, D^{-1}m_{a-1}, m_{a-2}, \dots, m_1).$$

This implies that

$$(m'_a, m'_{a-1}, \dots, m'_1) = (m'_a, D^{-1}m_{a-1}, m_{a-2}, \dots, m_1)$$

lies in the special section of $M_1(f_1) \rightarrow X_1(f_1)$, as asserted.

This completes the proof. □

We now use the special section $m : X(f) \rightarrow M(f)$ described in the preceding lemma to define a map

$$(8.15) \quad \begin{aligned} X(f) &\rightarrow X_1(f_1) \times X_2(f_2) \times GL_{d_a} \\ o &\mapsto [m'_a(o), \dots, m'_1(o)], [m'_r(o), \dots, m'_{a+1}(o)], D(o). \end{aligned}$$

We will prove that this is an isomorphism. We will need the following elementary lemma.

Lemma 8.9. *Let m'_2 be a $(n - d_1) \times (d - d_1)$ matrix, and A_1, A_2 be $(d - d_1) \times d_1$ and $(n - d_1) \times d_1$ matrices. Put*

$$m_2 = \begin{bmatrix} m'_2 & A_2 \\ O & I_{d_1} \end{bmatrix}, \quad m_1 = \begin{bmatrix} A_1 \\ I_{d_1} \end{bmatrix}$$

and assume that $J' \subset (1, \dots, n - d_1)$, $|J'| = d_2 - a_1$, and that the $J = J' \cup (n - d_1 + 1, \dots, n)$ -block of m_2 is I_d (which is equivalent to that $(A_2)_{J'} = O$ and $(m'_2)_{J'} = I_{d-d_1}$). Then A_1, A_2 can be uniquely expressed as polynomial functions in terms of m'_2 and $m_2 m_1$.

Lemma 8.10. *The map (8.15) is an isomorphism.*

Proof. We will explicitly construct the inverse of (8.15). It is enough to show that given a point $m' := ((m'_a, \dots, m'_1), (m'_r, \dots, m'_{a+1}), D)$ in the special section of the bundle $M_1(f_1) \times M_2(f_2) \times GL_{d_a} \rightarrow X_1(f_1) \times X_2(f_2) \times GL_{d_a}$, the relations (8.13) uniquely determine a point $m = (m_r, \dots, m_1) \in M$, expressible polynomially in terms of m' . In fact, it is enough to show that the $A_a, \dots, A_r$ can be so-expressed. Note that the relations (8.13) ensures that m lies in the special section of the bundle $M(f) \rightarrow X(f)$.

By (8.13), we have for $i = 1, \dots, a + 1$,

$$m_r \cdots m_i = \begin{bmatrix} m'_r \cdots m'_i & m'_r \cdots m'_{i+1} A_i + \cdots + m'_r A_{r-1} + A_r \\ O & I_{d_a} \end{bmatrix}.$$

Since $m_a = \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix}$, Lemma 8.9 implies that A_a and $m'_r \cdots m'_{a+2} A_{a+1} + \cdots + m'_r A_{r-1} + A_r$ can be uniquely expressed polynomially in terms of m' . It follows that the right hand block of $m_r \cdots m_{a+1}$:

$$\begin{aligned} (m_r \cdots m_{a+1})_R &= \begin{bmatrix} m'_r \cdots m'_{a+2} A_{a+1} + \cdots + m'_r A_{r-1} + A_r \\ I_{d_a} \end{bmatrix} \\ &= \begin{bmatrix} m'_r \cdots m'_{a+2} & m'_r \cdots m'_{a+1} A_{a+2} + \cdots + m'_r A_{r-1} + A_r \\ O & I_{d_a} \end{bmatrix} \begin{bmatrix} A_{a+1} \\ I_{d_a} \end{bmatrix} \\ &= m_r \cdots m_{a+2} \begin{bmatrix} A_{a+1} \\ I_{d_a} \end{bmatrix} \end{aligned}$$

can be so-expressed. By Lemma 8.9 again, the right hand block of $m_r \cdots m_{a+2}$ and A_{a+1} can also be so-expressed. Continuing this way, we see that $A_a, \dots, A_r$ all can be so-expressed. This completes the proof. $\square$

The lemma and Theorem 1.4 imply

Proposition 8.11. *For $d_a + d_{a+1} > n > 2d_a$ with $r > a > 1$, if any s -step flag variety for $s < r$ admits a rank 1 point, then $X = F(d_1, \dots, d_r, n)$ admits one as well.*

Case 6. Assume $n = 2d_a$ with $r > a > 1$. Consider

$$\begin{aligned} X_1 &= F(d_1, \dots, d_a) \equiv F(d_1, \dots, d_a, d_a) \hookrightarrow F(d_1, \dots, d_a, n), \quad (E^\bullet) \mapsto (0_{n-d_a} \oplus E^\bullet) \\ X_2 &= F(d_{a+1} - d_a, \dots, d_r - d_a, n - d_a) \hookrightarrow F(d_{a+1}, \dots, d_r, n), \quad (E^\bullet) \mapsto (E^\bullet \oplus \mathbb{C}^{d_a}) \end{aligned}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-d_a} \oplus \mathbb{C}^{d_a}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(8.16) \quad (x_{J_1} \cdots x_{J_a})|f_1, \quad (x_{J'_{a+1}} \cdots x_{J'_r})|f_2$$

for some $J_i \subset (1, 2, \dots, n - d_a)$ with $|J_i| = d_i$ ($i = 1, \dots, a-1$) and $J_{a-1} = (d_a - d_{a-1} + 1, \dots, d_a)$, and for some $J'_i \subset (1, 2, \dots, n - d_a)$ with $|J'_i| = d_i - d_a$ ($i = a+1, \dots, r$) and $J'_r = (n - d_r + 1, \dots, n - d_a)$. Put $J := (n - d_a + 1, \dots, n)$, $J_i := J'_i \cup J$, $i = a+1, \dots, n$, and

$$(8.17) \quad f := \bar{f}_1 \cdot \tilde{f}_2 \cdot (x_J)^{d_{a+1} - d_{a-1} - 1}(x_{1, \dots, d_a}).$$

Then f has the hyperplane property

$$(8.18) \quad (x_{J_1} \cdots \widehat{x_{J_a}} \cdots x_{J_r} x_J)|f.$$

Lemma 8.12. *The special section $m = (m_r, \dots, m_1)$ (cf. Lemma 8.1) of $M(f) \rightarrow X(f)$ has the following form:*

$$\begin{aligned} m_i &= \begin{bmatrix} m'_i & A_i \\ O & I_{d_a} \end{bmatrix}, \quad i = a+1, \dots, r \\ m_a &= \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix} \\ m_r \cdots m_a &= \begin{bmatrix} D \\ I_{d_a} \end{bmatrix} \\ m_{a-1} &= D^{-1} m'_{a-1} \\ (8.19) \quad m_i &= m'_i, \quad i = 1, \dots, a-2 \end{aligned}$$

where D is a GL_{d_a} -valued function, $A_a, \dots, A_r$ are matrix valued functions, and $(m'_{a-1}, \dots, m'_1)$, $(m'_r, \dots, m'_{a+1})$ are matrix valued functions taking values in the special sections of the $M_1(f_1) \rightarrow X_1(f_1)$, $M_2(f_2) \rightarrow X_2(f_2)$ respectively.

The proof is a degenerate version of the lemma in Case 5 (with m'_a missing but with $(1, \dots, d_a)$ play the role of J_a), and will be omitted. The lemma and Theorem 1.4 imply

Proposition 8.13. *For $n = 2d_a$ with $r > a > 1$, if any s -step flag variety for $s < r$ admits a rank 1 point f with the hyperplane property, then $X = F(d_1, \dots, d_r, n)$ admits one as well.*

Case 7. Assume $n = d_{a-1} + d_a$ with $r > a > 1$. If $a = 2$ then it is Case 4, so we can assume $a \geq 3$ (and $r \geq 4$). Consider

$$\begin{aligned} X_1 &:= F(d_1, \dots, d_{a-2}, d_{a-1}) \hookrightarrow F(d_1, \dots, d_{a-2}, n), \quad E^\bullet \mapsto E^\bullet \oplus 0_{n-d_{a-1}} \\ X_2 &:= F(d_{a-1}, \dots, d_r, n). \end{aligned}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{d_{a-1}} \oplus \mathbb{C}^{n-d_{a-1}}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(8.20) \quad (x_{J_1} \cdots x_{J_{a-2}})|f_1, ((x_{J_{a-1}})^{d_{a-1}} x_{J_a} \cdots x_{J_r})|f_2$$

for some $J_i \subset (1, 2, \dots, d_{a-1})$ with $|J_i| = d_i$ ($i = 1, \dots, a-2$), and for some $J_i \subset (1, 2, \dots, n)$ with $|J_i| = d_i$ ($i = a-1, \dots, r$) and $J_{a-1} = (1, \dots, d_{a-1})$. Note that such an f_2 exists by Proposition 8.7 in Case 4, if any s -step flag variety for $s < r$ admits a rank 1 point with the hyperplane property.

Put

$$(8.21) \quad f = \bar{f}_1 \cdot \bar{f}_2 \cdot (x_{J_{a-1}})^{-d_{a-2}} \in \Gamma(X, \omega_X^{-1}).$$

Then f has the hyperplane property

$$(8.22) \quad (x_{J_1} \cdots x_{J_r})|f.$$

Lemma 8.14. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$M_1(f_1) \times M_2(f_2) \rightarrow M(f)$$

$$(m'_{a-2}, \dots, m'_1), (m'_r, \dots, m'_{a-1}) \mapsto m = (m'_r, \dots, m'_{a-1}, D^{-1}m'_{a-2}, m'_{a-3}, \dots, m'_1)$$

where D is the J_{a-1} -block of $m'_r \cdots m'_{a-1}$. Hence the map descends to an isomorphism

$$X_1(f_1) \times X_2(f_2) \rightarrow X(f).$$

The proof is almost identical to the lemmas in Cases 1 and 3, and will be omitted. The lemma and Theorem 1.4 imply

Proposition 8.15. *For $d_{a-1} + d_a = n$ with $r > a > 1$, if any s -step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, \dots, d_r, n)$ admits one as well.*

Now combining the propositions in all Cases 1-7 yields a complete recursive procedure for constructing a rank 1 point with the hyperplane property for any r -step flag variety, proving Corollary 1.6.

Example 8.16. Consider $X = F(1, 2, 3, 5)$, which belongs in Case 3. Let $X_1 = F(1, 2)$ and take $f_1 = x_1 x_2$. Let $X_2 = F(2, 3, 5)$, which belongs in Case 2, and we can take $f_2 = (x_{12})^3 (x_{345})^3$ as a rank 1 point of X_2 , by Proposition 8.4. Therefore,

$$f = x_1 x_2 (x_{12})^3 (x_{345})^3 (x_{12})^{-1}$$

is rank 1 point of X according to the construction in Case 3.

Example 8.17. Consider the flag variety of SL_5 , $X = F(1, 2, 3, 4, 5)$, which belongs in Case 7 with $a = 3$. Let $X_1 = F(1, 2)$ and take $f_1 = x_1x_2$. Let $X_2 = F(2, 3, 4, 5) \cong F(1, 2, 3, 5)$, which is the preceding example. Applying this isomorphism to the rank 1 point there, we get $f_2 = x_{2345}x_{1345}(x_{345})^2(x_{12})^3$ as a rank 1 point of X_2 . Therefore,

$$f = x_1x_2x_{2345}x_{1345}(x_{345})^2(x_{12})^3(x_{12})^{-1}$$

is a rank 1 point of X according to the construction in Case 7.

APPENDIX A. THEORY OF D -MODULES

We recall the theory of *algebraic* D -modules. A standard reference is [5].

Let X be an algebraic variety over k of characteristics zero. Let $\text{Hol}(D_X)$ be the category of holonomic (left) D -modules on X . Its bounded derived category is denoted by $D_h^b(X)$.

Let $f : X \rightarrow Y$ be a morphism, there are the following pairs of adjoint (derived) functors (following the notation of Borel's book)

$$f^+ : D_h^b(Y) \rightleftharpoons D_h^b(X) : f_+, \quad f_! : D_h^b(X) \rightleftharpoons D_h^b(Y) : f^!$$

Recall the definition of f_+ in the following cases (assuming X and Y are smooth): in the case, there is an $f^{-1}D_Y \times D_X$ -bimodule $D_{Y \leftarrow X}$ on X , and

$$f_+(M) = Rf_*(D_{Y \leftarrow X} \otimes^L M).$$

Without mentioning the exact definition of this bimodule $D_{Y \leftarrow X}$, we concentrate on the following special cases. Let $d_{X,Y} = \dim X - \dim Y$.

(i) $f : X \rightarrow Y$ is smooth. Then f_+ (up to shift) is the usual construction of the Gauss-Manin connection. I.e.

$$f_+(M) = Rf_*(M \otimes \Omega_{X/Y}^\bullet[d_{X,Y}]).$$

In particular, $H^i f_+ \mathcal{O}_X$ is the D -module on Y formed by the $(i + d_{X,Y})$ th relative De Rham cohomology. In particular, if f is an open embedding, then $f_+(M) = Rf_* M$ as quasi-coherent sheaves on Y . Observe that under the this normalization of the cohomological degrees, $H^0 f_+ \mathcal{O}_X$ is the usual “middle dimension” cohomology of the family $f : X \rightarrow Y$.

Example A.1. A particular example: $j : \mathbb{G}_m = \text{Spec} k[x, x^{-1}] \rightarrow \mathbb{A}^1 = \text{Spec} k[x]$ the open embedding. Then $j_+ \mathcal{O}_{\mathbb{G}_m}$ as a D -module on $\mathbb{A}^1$ is isomorphic to $k[x, \partial_x]/(x\partial_x + 1)$.

(ii) $f : X \rightarrow Y$ is a closed embedding given by the ideal $\mathcal{I}$. Then

$$f_+(M) = f_*(D_Y/D_Y \mathcal{I} \otimes \omega_{X/Y} \otimes M)$$

where $\omega_{X/Y}$ is the relative canonical sheaf $\omega_{X/Y} = \omega_X \otimes (\omega_Y^{-1}|_X)$.

Example A.2. A particular example: let Y be $\mathbb{A}^n = \text{Spec} k[x_1, \dots, x_n]$ and $i : X \rightarrow Y$ be the inclusion of the vector space given by $x_1 = \dots = x_r = 0$. Then $x_{r+1}, \dots, x_n$ form a coordinate system on X . Let $M = \mathcal{O}_X = D_X / D_X(\partial_{r+1}, \dots, \partial_n)$. Then

$$i_+ M = D_Y / D_Y(x_1, \dots, x_r, \partial_{r+1}, \dots, \partial_n),$$

called the delta sheaf supported on X , denoted by δ_X .

Observe that there is the following exact sequence of $D_{\mathbb{A}^1}$ -modules

$$(A.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{A}^1} \rightarrow j_+ \mathcal{O}_{\mathbb{G}_m} \rightarrow \delta_{\{0\}} \rightarrow 0.$$

Next, we recall the definition of $f^!$. There is a $D_X \times f^{-1}D_Y$ -bimodule $D_{X \rightarrow Y}$ on X , and by definition

$$f^!(M) = D_{X \rightarrow Y} \otimes_{f^{-1}D_Y}^L f^{-1}M[d_{X,Y}].$$

As quasi-coherent $\mathcal{O}_X$ -modules,

$$f^!(M) = Lf^*M[d_{X,Y}].$$

Again, let us mention the following special cases.

(i) $f : X \rightarrow Y$ is smooth. In this case, $f^![-d_{X,Y}]$ is exact, and as quasi-coherent sheaves, $f^![-d_{X,Y}](M) = f^*M$. In particular, if f is an open embedding, then $f^!M = M|_X$.

(ii) $f : X \rightarrow Y$ is a closed embedding, given by the ideal sheaf $\mathcal{I}$. In this case

$$H^0 f^!(M) \otimes \omega_{X/Y} = \{m \in M \mid xm = 0 \text{ for any } x \in \mathcal{I}\}.$$

The following distinguished triangle generalizes (A.1): Let $i : X \rightarrow Y$ be a closed embedding and $j : U \rightarrow Y$ be the complement.

$$(A.2) \quad i_+ i^! M \rightarrow M \rightarrow j_+ j^! M \rightarrow .$$

Indeed, in the case $Y = \mathbb{A}^1$ and $X = \mathbb{G}_m$, $M = \mathcal{O}_{\mathbb{A}^1}$, we recover (A.1).

The following theorem (Kashiwara's lemma) is of fundamental importance,

Theorem A.3. *Let $i : X \rightarrow Y$ be a closed embedding.*

(i) *If M is a D_Y -module, set-theoretically supported on X . Then $H^i i^! M = 0$ for $i > 0$.*

(ii) *Let $D_Y\text{-Mod}_X$ be the category of D_Y -modules, set-theoretically supported on X , and $D_X\text{-Mod}$ be the category of D_X -modules. Then there is an equivalence of categories*

$$i_+ : D_X\text{-Mod} \rightleftarrows D_Y\text{-Mod}_X : H^0 i^!.$$

In the sequel, we will make use of the following notation: let $i : X \rightarrow Y$ be a locally closed embedding. If M is a D -module on Y , set-theoretically supported on $\bar{X}$, then $H^0 i^! M$ will be denoted by $M|_X$.

This finishes the discussion of the functors $f_+, f^!$. Then $f_!$ is defined to be the left adjoint of $f^!$ and f^+ is defined to be the left adjoint of f_+ . Recall that there is the duality functor $\mathbb{D}_X : D_h^b(X) \rightarrow D_h^b(X)$. We can also express $f^+ = \mathbb{D}_X f^! \mathbb{D}_Y$ and $f_! = \mathbb{D}_Y f_+ \mathbb{D}_X$. It is known that

- (i) If $f : X \rightarrow Y$ is a closed embedding (or more generally if f is proper), $f_! = f_+$.
- (ii) If $f : X \rightarrow Y$ is an open embedding, $f^! = f^+$.

Remark A.1. The definitions of $f_+, f^!$ do not require the holonomicity, and therefore they are defined on the whole category of (not necessarily holonomic) D -modules. However, as functors on the whole category of D -modules, they do not admit adjoint functors and therefore $f_!, f^+$ are not defined in general.

Example A.4. Let $j : \mathbb{G}_m \rightarrow \mathbb{A}^1$ as before. One can show that $j_! \mathcal{O}_{\mathbb{G}_m} \simeq k[x, \partial]/x\partial$.

The dual version of (A.1) is

$$(A.3) \quad 0 \rightarrow \delta_{\{0\}} \rightarrow j_! \mathcal{O}_{\mathbb{G}_m} \rightarrow \mathcal{O}_{\mathbb{A}^1} \rightarrow 0,$$

and the dual version of (A.2) is

$$(A.4) \quad j_! j^! M \rightarrow M \rightarrow i_+ i^+ M \rightarrow .$$

Now let $k = \mathbb{C}$. Let $D_{rh}^b(X)$ be the bounded derived category of holonomic D -modules with regular singularities, and let $D_c^b(X^{an})$ be the bounded derived category of constructible sheaves on X^{an} (we denote X equipped with the classical topology by X^{an}). Then Riemann-Hilbert correspondence is an equivalence

$$\mathrm{RH} : D_{rh}^b(X) \simeq D_c^b(X), \quad \mathrm{RH}(M) = \omega_{X^{an}} \otimes^L M^{an} = \Omega_{X^{an}}^\bullet \otimes M^{an}[\dim X],$$

where $\omega_{X^{an}}$ is the canonical sheaf on X^{an} , regarded as a right D -module via Lie derivative, and the derived tensor product is over $D_{X^{an}}$. This correspondence is compatible with the six operation functors. In particular,

$$\mathrm{RH} f_+ \simeq f_* \mathrm{RH}, \quad \mathrm{RH} f_! \simeq f_! \mathrm{RH}, \quad \mathrm{RH} f^! \simeq f^! \mathrm{RH}, \quad \mathrm{RH} f^+ \simeq f^* \mathrm{RH}.$$

If M is a plain D -module, then $\mathrm{RH}(M)$ is a perverse sheaf on X^{an} . While the above equivalence is covariant, sometimes one also consider the contravariant version

$$\mathrm{Sol} : D_{rh}^b(X) \simeq D_c^b(X)^{op}, \quad \mathrm{Sol}(M) = R\mathrm{Hom}_{D_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}}).$$

The relation between Sol and RH is $\text{RH} = \text{Sol } \mathbb{D}_X[\dim X]$.

Next, we discuss background materials on equivariant D-modules, most which can be found in [5][11]. Let G be a connected algebraic group and $\mathfrak{g} = \text{Lie}G$. Let us regard $\mathfrak{g}$ as right invariant vector fields on G , and for a Lie algebra homomorphism $\chi : \mathfrak{g} \rightarrow k$, we define a character D-module on G by

$$(A.5) \quad \mathcal{L}_\chi = D_G / D_G(\xi + \chi(\xi), \xi \in \mathfrak{g}).$$

This is a rank one local system on G . In particular, it is holonomic. It is called a character sheaf because if we denote by $\text{mult} : G \times G \rightarrow G$ the multiplication map of G , then there is a canonical isomorphism $\text{mult}^! \mathcal{L}_\chi \simeq \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi[\dim G]$ satisfying the cocycle condition under the further $!$ -pullback to $G \times G \times G$.

Let Z be a G -variety and $\text{act} : G \times Z \rightarrow Z$ be the action map. A (G, χ) -equivariant, or a G -monodromic against χ , D-module on Z is a D-module on Z together with an isomorphism

$$\theta : \text{act}^! M \simeq \mathcal{L}_\chi \boxtimes M[\dim G],$$

satisfying the usual cocycle condition under the further $!$ -pullback to $G \times G \times Z$.

The following lemma is well-known, which can be proved as in [5, Theorem 12.11]. See also [11, §II.5].

Lemma A.5. *Assume that there are only finitely many orbits under the action of G on Z , then any (G, χ) -equivariant D-module is holonomic. In addition, if $\mathcal{L}_\chi$ is regular singular, then any (G, χ) -equivariant D-module is regular singular.*

We will need the following lemma. Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Then χ defines a one-dimensional $U\mathfrak{g}$ -module, denoted by k_χ . Note that if Z is a G -variety, we have the corresponding infinitesimal action $da : \mathfrak{g} \rightarrow T_Z$, which extends to $U\mathfrak{g} \rightarrow D_Z$.

Lemma A.6. *The D-module*

$$D_{Z, \chi} = D_Z / D_Z(da(\xi) + \chi(\xi), \xi \in \mathfrak{g}) = (D_Z \otimes k_\chi) \otimes_{U\mathfrak{g}} k$$

is a natural (G, χ) -equivariant D-module on Z .

More generally, note that D_Z is naturally G -equivariant as $\mathcal{O}$ -modules, i.e., there is an isomorphism of $\mathcal{O}$ -modules $\theta : \text{act}^ D_Z \simeq p_Z^* D_Z$ satisfying the cocycle condition. Let $I \subset D_Z$ be a G -invariant left ideal, then*

$$D_Z / I + D_Z(da(\xi) + \chi(\xi), \xi \in \mathfrak{g})$$

is (G, χ) -equivariant.

Note that in the above lemma, we do not need to assume that G acts on Z with finitely many orbits. See [11, §II.3].

Note that if $i : H \rightarrow G$ is a connected closed subgroup, $i^! \mathcal{L}_\chi[-\dim H] = D_H/D_H(\xi + \chi(\xi), \xi \in \mathfrak{h}) = \mathcal{L}_{\chi|_{\mathfrak{h}}}$. We have the following simple observation.

Lemma A.7. *Let $Z = G/H$ be a homogeneous G -variety. Let $\chi : \mathfrak{g} \rightarrow k$ be a Lie algebra homomorphism and $\mathcal{L}_\chi$ be the rank character D -module on G as in (A.5). Then if $\mathcal{L}_{\chi|_{\mathfrak{h}}} \neq \mathcal{O}_{H^\circ}$, where H° is the neutral connected component of H , there is no D -module on Z , equivariant with respect to G against χ .*

Proof. Let M be a non-zero (G, χ) -equivariant D -modules on Z . Let $i : H^\circ \rightarrow G$ be the inclusion, and $i_e : eH \rightarrow Z$ be the inclusion of the identity coset. Consider the diagram

$$\begin{array}{ccc} H^\circ \times eH & \longrightarrow & eH \\ i \times i_e \downarrow & & \downarrow i_e \\ G \times Z & \longrightarrow & Z. \end{array}$$

Then $i^! \mathcal{L}_\chi \otimes i_e^! M = (i \times i_e)^! \text{act}^! M = \mathcal{O}_H \otimes i_e^! M[\dim Z]$. Therefore, $\mathcal{L}_{\chi|_{\mathfrak{h}}} = \mathcal{O}_H$. $\square$

Example A.8. Let $\lambda \in k^\times$, and let $\mathcal{L}_\lambda$ be the D -module on $\mathbb{G}_m$ given by $x\partial + \lambda$. I.e. $\mathcal{L}_\lambda$ is the local system on $\mathbb{G}_m$ with monodromy $\exp(-2\pi\sqrt{-1}\lambda)$ (via the Riemann-Hilbert correspondence if $k = \mathbb{C}$). This is a character D -module on G with $\chi(x\partial) = \lambda$. If $\lambda \in \mathbb{Z}$, then $\mathcal{L}_\lambda \simeq \mathcal{O}_{\mathbb{G}_m}$. Let $j : \mathbb{G}_m \rightarrow \mathbb{A}^1$ be the open embedding. Then both $j_+ \mathcal{L}_\lambda$ and $j_! \mathcal{L}_\lambda$ are (G, λ) -equivariant D -modules on $\mathbb{G}_m$. If λ is not an integer, then $j_! \mathcal{L}_\lambda \simeq j_+ \mathcal{L}_\lambda$. In this case, this D -module is irreducible on $\mathbb{A}^1$.

Our last topic is the Fourier transform. Let “ e^x ” be the character D -module on $\mathbb{A}^1$ defined by $\partial - 1$. Let V be a vector space and $V^\vee$ be its dual. We have the natural pairing

$$m : V \times V^\vee \rightarrow \mathbb{A}^1.$$

The pullback of e^x along m is still denoted by e^x . Let $p_V, p_{V^\vee}$ be the projections of $V \times V^\vee$ to the two factors. The Fourier transform is defined as

$$\mathcal{F}our(M) = p_{V^\vee, +}(p_V^!(M) \otimes e^x)[\dim V],$$

Fourier transform $\mathcal{F}our$ is an exact functor, and can be described in the following simple way. Let M be a D -module on V , and therefore is identified with a module over the Weyl algebra $k[a_1, \dots, a_n, \partial_{a_1}, \dots, \partial_{a_n}]$. Then $\mathcal{F}our(M)$ as a vector space is identified with M , and the D -module structure is given by $a_i^* m = \partial_{a_i} m$ and $\partial_{a_i}^* = -a_i m$. In other words, if we denote the ring homomorphism

$$(A.6) \quad \widehat{\cdot} : D_V \rightarrow D_{V^\vee}, \quad \widehat{a_i} = -\partial_{a_i}^*, \quad \widehat{\partial_{a_i}} = a_i^*,$$

then $\mathcal{F}our(M) = D_{V^\vee} \otimes_{D_V} M$. See [6, p85].

Example A.9. Let $W \subset V$ be a vector subspace, and $W^\perp$ be the orthogonal complement of W in $V^\vee$. Then $\mathcal{F}our(\delta_W) = \delta_{W^\perp}$.

Example A.10. More generally, let $i : W \subset V$ be a vector subspace, and $0 \rightarrow W^\perp \rightarrow V^\vee \xrightarrow{p} W^\vee \rightarrow 0$ be the dual sequence. Let M be a D-module on W . Then

$$\mathcal{F}our(i_+M) = p^!\mathcal{F}our(M)[\dim W - \dim V].$$

Example A.11. Let $V = \mathbb{A}^1$ and we identify $V^\vee = \mathbb{A}^1$ via the natural multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Then under the Fourier transform, the exact sequence (A.1) becomes (A.3).

Example A.12. Recall the character D-module $\mathcal{L}_\lambda$ on $\mathbb{G}_m$. Let $j : \mathbb{G}_m \rightarrow \mathbb{A}^1$ be the open immersion. Then

$$\mathcal{F}our(j_+\mathcal{L}_\lambda) = j_!\mathcal{L}_{-\lambda+1}.$$

Fourier transform preserves holonomicity. If M is holonomic, then we can also write

$$\mathcal{F}our(M) = p_{V^\vee,!}(p_V^*(M) \otimes e^x)[- \dim V].$$

However, Fourier transform does not necessarily preserve the regular singularity. For example, the Fourier transform of the delta sheaf on $\mathbb{A}^1$ supported at $1 \in \mathbb{A}^1(k)$ is e^x . However, under certain circumstance, one can show that $\mathcal{F}our(M)$ is regular singular. Let $\mathbb{G}_m$ act on V via homotheties, i.e. $\text{mult} : \mathbb{G}_m \times V \rightarrow V, \text{mult}(a, v) = av$. Let $\lambda : \text{Lie}\mathbb{G}_m \rightarrow k$ be a map. Recall the notion of $(\mathbb{G}_m, \lambda)$ -equivariant D-modules. We say a holonomic D-module on V to be $\mathbb{G}_m$ monodromic if each of its irreducible constituents is $(\mathbb{G}_m, \lambda)$ -equivariant for some λ . Observe that e^x is not $\mathbb{G}_m$ -monodromic.

Let $D_{rh,m}^b(V)$ be the full subcategory of $D_{rh}^b(V)$ whose cohomology sheaves are regular holonomic and $\mathbb{G}_m$ -monodromic.

Lemma A.13. *The Fourier transform restricts to an equivalence*

$$\mathcal{F}our : D_{rh,m}^b(V) \simeq D_{rh,m}^b(V^\vee).$$

Proof. [6, Theorems 7.4, 7.24]. □

Fourier transform can be generalized to family versions. Let X be a base variety, and $\mathbb{V}$ a vector bundle over X , $\mathbb{V}^\vee$ the dual bundle, so there is

$$m : \mathbb{V} \times_X \mathbb{V}^\vee \rightarrow \mathbb{A}^1.$$

Then one can define

$$\mathcal{F}our_X(M) = p_{\mathbb{V}^\vee,+}(p_{\mathbb{V}}^!(M) \otimes e^x)[\text{rk } \mathbb{V}].$$

Note that the family version of Example A.10 still holds. More precisely, let $i : \mathbb{W} \subset \mathbb{V}$ be a subbundle on $p : \mathbb{V}^\vee \rightarrow \mathbb{W}^\vee$ be the dual map. Then

$$(A.7) \quad \mathcal{F}our_X(i_+ M) = p^! \mathcal{F}our_X(M) [\text{rk } \mathbb{W} - \text{rk } \mathbb{V}].$$

Note that family version of Fourier transforms commute with base change. Namely, if $f : Y \rightarrow X$ is a map. Then $\mathcal{F}our_Y(f^! M) = f^! \mathcal{F}our_X(M)$.

Let us consider the family version of Example A.12. So we assume that $\mathbb{V} = \mathbb{L}$ is a line bundle, on which $\mathbb{G}_m$ acts by homotheties. Let $\mathring{\mathbb{L}} = \mathbb{L} - X$, where X is regarded as the zero section of $\mathbb{L}$. Let $\mathbb{L}^\vee$ be the dual vector bundle of $\mathbb{L}$ and $\mathring{\mathbb{L}}^\vee$ is defined similarly. Let M be a $(\mathbb{G}_m, \lambda)$ -equivariant D-module on $\mathring{\mathbb{L}}$.

The following lemma is useful.

Lemma A.14. *Let X be proper and $\mathbb{V} = X \times V$ be the trivial bundle over X . Let $\pi : X \times V \rightarrow V$ and $\pi^\vee : X \times V^\vee \rightarrow V^\vee$ be the projections. Then*

$$\mathcal{F}our \circ \pi_! \simeq \pi_!^\vee \circ \mathcal{F}our_X.$$

Proof. This follows from the base change theorem for D-modules (cf. [5, VI, §8]). Namely, as X is proper, $\pi_+ = \pi_!$, etc. We have the following commutative diagrams with both squares Cartesian.

$$\begin{array}{ccccc}
 & & X \times V \times V^\vee & & \\
 & \swarrow p_V & \downarrow \pi_{V, V^\vee} & \searrow p_{V^\vee} & \\
 X \times V & & V \times V^\vee & & X \times V^\vee \\
 \downarrow \pi & \swarrow p_V & & \searrow p_{V^\vee} & \downarrow \pi^\vee \\
 V & & & & V^\vee
 \end{array}$$

Then

$$\begin{aligned}
 \mathcal{F}our(\pi_!(M)) &= p_{V^\vee, +}(p_V^! \pi_+^\vee(M) \otimes e^x) \\
 &= p_{V^\vee, +}(\pi_+^{V, V^\vee} p_V^!(M) \otimes e^x) \\
 &= p_{V^\vee, +} \pi_+^{V, V^\vee} (p_V^!(M) \otimes e^x) \\
 &= \pi_+^\vee \mathcal{F}our_X(M)
 \end{aligned}$$

□

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