THE TWO-DIMENSIONAL CONTOU-CARRÈRE SYMBOL AND RECIPROCITY LAWS

DENIS OSIPOV, XINWEN ZHU

Abstract. We define a two-dimensional Contou-Carrère symbol, which is a deformation of the two-dimensional tame symbol and is a natural generalization of the (usual) one-dimensional Contou-Carrère symbol. We give several constructions of this symbol and investigate its properties. Using higher categorical methods, we prove reciprocity laws on algebraic surfaces for this symbol. We also relate the two-dimensional Contou-Carrère symbol to the two-dimensional class field theory.

1. Introduction

This paper is a continuation of our previous paper [OsZh] and we refer to the introduction of that paper for the general background. In that paper, we developed some categorical constructions such as categorical central extensions and generalized commutators of these central extensions and applied them to the construction of the two-dimensional tame symbol and to the proof of reciprocity laws on algebraic surfaces. We also systematically used the adeles on algebraic surfaces and the categories of 1-Tate and 2-Tate vector spaces and their graded-determinantal theories introduced by M. Kapranov in [Kap].

The main goal of this paper is to extend constructions and theorems from a ground field k (as in [OsZh]) to a ground commutative ring R, by applying the categorical constructions developed there to the category of Tate R-modules, which was introduced and studied by V. Drinfeld in [Dr]. The generalized commutator now gives us some new tri-multiplicative anti-symmetric map:

$$R((u))((t))^* \times R((u))((t))^* \times R((u))((t))^* \to R^*,$$

which we call the two-dimensional Contou-Carrère symbol. This symbol coincides with the two-dimensional tame symbol when $R = k$ is a field. Using adelic complexes on an algebraic surface we prove reciprocity laws along a curve and around a point for this symbol when R is an artinian ring, see Theorem 6.1.

An analogous deformation of the (usual) one-dimensional tame symbol is known as the Contou-Carrère symbol, see [Del] and [CC1]. The reciprocity laws for the one-dimensional Contou-Carrère symbol on an algebraic curve were proved by using the (usual) commutators of central extensions of groups, see [BBE] and [AP]. The one-dimensional Contou-Carrère symbol can also be explicitly expressed by formulas. When $\mathbf{Q} \subset R$ and f and g are appropriate elements from $R((t))^*$, then (see

D. O. partially supported by Russian Foundation for Basic Research (grants no. 14-01-00178-a, no. 13-01-12420 ofi_m2, and no. 12-01-33024 mol_a_ved) and by the Programme for the Support of Leading Scientific Schools of the Russian Federation (grant no. NSh-2998.2014.1).

X. Z. partially supported by NSF grant DMS-1001280/1313894 and DMS-1303296 and by AMS Centennial Fellowship.

formula (2.4))

$$(f,g) = \exp \operatorname{res}(\log f \frac{dg}{g}).$$

Another formula (2.7) applicable to any ring R expresses (f,g) as certain finite product.

For the applications of our reciprocity laws (e.g. see Section 8), we need various explicit formulas for the two-dimensional Contou-Carrère symbol. When $\mathbf{Q} \subset R$, we introduce in Section 3.3 an obvious generalization of the previous one-dimensional formula. Namely, for appropriate elements f, g and h from $R((u))((t))^*$ our formula looks as

$$(f,g,h) = \exp \operatorname{Res}(\log f \frac{dg}{g} \wedge \frac{dh}{h}),$$

where Res is the two-dimensional residue. Then we prove in Theorem 5.9 that this explicit formula coincides with the two-dimensional Contou-Carrère symbol defined by the generalized commutator as above. We remark that the proof of this fact in the two-dimensional situation is much more difficult than in the one-dimensional situation, because we need to deal with some infinite sums (and products) here, while in the one-dimensional situation only finite sums (and products) are involved.

Basically because of the same reason, we do not have an explicit formula applicable to general commutative rings. In other words, unlike the one-dimensional situation, we do not know how to define the two-dimensional Contou-Carrère symbol for general commutative rings in an elementary way. (Our definition uses categorical central extensions! Another approach via algebraic K-theory, is outlined in Section 7.) But if R is Noetherian (or more generally, if the nil-radical of R is an nilpotent ideal), we obtain in Section 3.4 an explicit formula for the two-dimensional Contou-Carrère symbol as certain finite product, which generalizes the similar formula in the one-dimensional case.

We also investigate in Section 4 various properties of the two-dimensional Contou-Carrère symbol by elementary methods. When the nil-radical of R is a nilpotent ideal, we prove that this symbol satisfies the Steinberg property (and later in Corollary 7.5 we remove this assumption). We also show that if $R = k[\epsilon]/\epsilon^4$ where k is a field, then for any elements f, g and h from $k((u))((t))$ there is an identity:

$$(1+\epsilon f, 1+\epsilon g, 1+\epsilon h) = 1 + \epsilon^3 \operatorname{Res} f\, dg \wedge dh.$$

Thus, we obtain the two-dimensional residue (for two-dimensional local fields). From Section 5.3 it follows that the two-dimensional Contou-Carrère symbol is invariant under the change of local parameters u and t in $R((u))((t))$.

In Section 7, we outline how to obtain the Contou-Carrère symbols via algebraic K-theory, developing some ideas suggested to us by one of the editors. It is widely believed that the Contou-Carrére symbols can be obtained from certain boundary maps in algebraic K-theory (e.g. [KV, Remark 4.3.7]). However, it seems a detailed comparison did not exist in literature before.

Finally, in Section 8 we relate the two-dimensional Contou-Carrère symbol to the two-dimensional class field theory. For $R = \mathbf{F}_q[s]/s^{n+1}$, where $\mathbf{F}_q$ is a finite field, we derive from the two-dimensional Contou-Carrère symbol the two-dimensional generalization of the Witt symbol introduced and studied by A. N. Parshin. Our reciprocity laws for the two-dimensional Contou-Carrère symbol imply the reciprocity laws for the generalization of the Witt symbol. We interpret the reciprocity laws for the two-dimensional tame symbol (studied in [OsZh]) and for the Witt symbol as indication on some subgroup in the kernel of the global reciprocity map in two-dimensional class field theory. We also relate the reciprocity laws for the generalization of the Witt symbol with the reciprocity laws obtained earlier by K. Kato and S. Saito.

Notation. By **Aff** we denote the category of affine schemes, i.e. the category opposite to the category of commutative rings. We equip **Aff** with the flat topology.

By an ind-scheme we always mean an object "$\varinjlim_{i \in I}$" $X_i \in \mathrm{Ind}\,\mathbf{Aff}$, where all the structure morphisms $X_i \to X_j$ are closed embeddings of affine schemes, and I is a directed set.

For any functor $\mathcal{F}$ from the category **Aff** to the category of sets we denote by $L\mathcal{F}$ the loop space functor which assigns the set $L\mathcal{F}(R) = \mathcal{F}(R((t)))$ to every ring R.

We introduce the formal scheme $\mathcal{N} = \mathrm{Spf}\,\mathbf{Z}[[T]]$. In other words, for $R \in \mathbf{Aff}$, $\mathcal{N}(R)$ is the set of all nilpotent elements of R, i.e. the nil-radical of R. (Sometimes $\mathcal{N}(R)$ is denoted by $\mathcal{N}R$ for simplicity.)

2. One-dimensional Contou-Carrère symbol

In this section we recall some known facts about the one-dimensional Contou-Carrère symbol.

First recall the following statement, see, e.g., [CC1, Lemme(1.3)], [CC2, §0]. Let R be a commutative ring. Then for any invertible element $s = \sum_{i > -\infty}^{\infty} a_i t^i \in R((t))^*$ there is a decomposition of R into finite product of rings R_i:

$$R = \bigoplus_{i=1}^{N} R_i \tag{2.1}$$

such that if $s = \bigoplus_i s_i$ under the induced decomposition

$$R((t)) = \bigoplus_{i=1}^{N} R_i((t)), \tag{2.2}$$

then every $r = s_i$ can be uniquely decomposed into the following product in $R_i((t))$:

$$r = r_{-1} \cdot r_0 \cdot t^{\nu(r)} \cdot r_1, \tag{2.3}$$

where $r_{-1} \in 1 + t^{-1} \cdot \mathcal{N}R_i[t^{-1}]$, $r_0 \in R_i^*$, $r_1 \in 1 + t \cdot R_i[[t]]$. In addition, such a decomposition (2.1) is unique if we require $\nu(s_i) \neq \nu(s_j)$ for any $i \neq j$.

Let us rephrase decompositions (2.1)–(2.3). Let $L\mathbb{G}_m$ be the loop group of the multiplicative group $\mathbb{G}_m$, where $\mathbb{G}_m(R) = R^*$ for any commutative ring R. Let $\mathbb{W}$, $\widehat{\mathbb{W}}$ be contravariant functors from the category **Aff** to the category of abelian groups (or covariant functors from the category of commutative rings) defined in the following way: for any commutative ring R

$$\mathbb{W}(R) = \left\{ 1 + \sum_{i=1}^{\infty} b_i t^i \mid \quad b_i \in R \right\},$$

$$\widehat{\mathbb{W}}(R) = \left\{ 1 + \sum_{i=-1}^{-n} c_i t^i \mid \quad n \in \mathbf{Z}_{\geq 0}, c_i \in \mathcal{N}R \right\}.$$

Note that the functor $\mathbb{W}$ with its group structure is usually called the additive (big) Witt vectors. Then $\mathbb{W}$ is represented by an affine scheme

$$\mathbb{W} = \operatorname{Spec} \mathbf{Z}[b_1, b_2, \ldots],$$

while the functor $\widehat{\mathbb{W}}$ is represented by an ind-scheme

$$\widehat{\mathbb{W}} = \varinjlim_{\{\epsilon_i\}} \operatorname{Spec} \mathbf{Z}[c_{-1}, c_{-2}, \ldots]/I_{\{\epsilon_i\}},$$

where the limit is taken over all the sequences $\{\epsilon_i\} = (\epsilon_{-1}, \epsilon_{-2}, \ldots)$ of non-negative integers such that all but finite many ϵ_i equal 0, and the ideal $I_{\{\epsilon_i\}}$ is generated by elements $c_i^{\epsilon_i+1}$ for all $i < 0$.

Denote by $\mathbb{Z}$ the constant group scheme over $\operatorname{Spec} \mathbf{Z}$ with the fiber equal to the additive group of integers $\mathbf{Z}$. In other words, the group scheme $\mathbb{Z} = \coprod_{i \in \mathbf{Z}} (\operatorname{Spec} \mathbf{Z})_i$, and $\mathbb{Z}(R)$ is the group of locally constant functions on $\operatorname{Spec} R$ with values in $\mathbf{Z}$. The group $\mathbb{Z}(R)$ naturally embeds into the group $R((t))^*$ in the following way. Any $f \in \mathbb{Z}(R)$ determines decomposition (2.1) and the set of integers n_i, where $1 \leq i \leq N$. Then $f \mapsto \bigoplus_i t^{n_i} \in R((t))^*$ under decomposition (2.2). We can now summarize decompositions (2.1)–(2.3) and above reasonings in the following lemma.

Lemma 2.1. *There is a canonical isomorphism of group ind-schemes*

$$L\mathbb{G}_m \simeq \widehat{\mathbb{W}} \times \mathbb{Z} \times \mathbb{G}_m \times \mathbb{W}.$$

Let $\nu : L\mathbb{G}_m \to \mathbb{Z}$ be the projection to the $\mathbb{Z}$ factor under the above decomposition.

Now we recall the one-dimensional Contou-Carrère symbol (or simply Contou-Carrère symbol) (see [Del, §2.9], [CC1]).

Lemma-Definition 2.2. *The Contou-Carrère symbol is the unique bimultiplicative, anti-symmetric map*

$$(\cdot, \cdot) \; : \; L\mathbb{G}_m \times L\mathbb{G}_m \longrightarrow \mathbb{G}_m$$

such that if $\mathbf{Q} \subset R$ and $f, g \in L\mathbb{G}_m(R) = R((t))^*$, then

$$(f, g) = \exp \operatorname{res}(\log f \cdot \frac{dg}{g}) \quad \textit{when} \quad f \in \widehat{\mathbb{W}}(R) \times \mathbb{W}(R) \quad (\textit{see Lemma } 2.1), \tag{2.4}$$

$$(a, g) = a^{\nu(g)} \quad \textit{when} \quad a \in R^*, \tag{2.5}$$

$$(t, t) = (-1, t) = -1. \tag{2.6}$$

We note that formula (2.4) *is well-defined, since* $\operatorname{res}(\log f \cdot \frac{dg}{g}) \in \mathcal{N}R$.

Remark 2.1. The expression $a^{\nu(g)}$ in (2.5) is defined in the following way. The element $\nu(g) \in \mathbb{Z}(R)$ determines decomposition (2.1) and the set of integers n_i, where $1 \leq i \leq N$. Let $a = \bigoplus_i a_i$ with respect to this decomposition, then $a^{\nu(g)} = \bigoplus_i a_i^{n_i}$. Further we will also use the following expression $t^{\nu(g)} = \bigoplus_i t^{n_i} \in R((t))^*$.

Remark 2.2. Our expression for (f, g) is inverse to the corresponding expression from [Del].

We recall the uniqueness. First, note that that if $\mathbf{Q} \subset R$, then the symbol is uniquely defined by the conditions given above. In general, if X, Y are two flat $\mathbf{Z}$-schemes and $f_{\mathbf{Q}} : X_{\mathbf{Q}} \to Y_{\mathbf{Q}}$ is a morphism, then there is at most one morphism $f : X \to Y$ that extends $f_{\mathbf{Q}}$. Now the uniqueness follows from the fact that $L\mathbb{G}_m$ is represented by an inductive limit of schemes which are flat over $\mathbf{Z}$.

Next we recall the existence. Note that the symbol (f, g) defined by the formula (2.4) can be expressed as a formal series on coefficients of $f, g \in R((t))^*$. To extend this definition to arbitrary ring R amounts to show that this formal series is defined over $\mathbf{Z}$. In formula (2.3) the element r_{-1} can be uniquely decomposed as

$$r_{-1} = \prod_{i<0}^{i>-\infty} (1 - d_i t^i), \text{ where } \quad d_i \in \mathcal{N}R,$$

and the element r_1 can be uniquely decomposed as

$$r_1 = \prod_{i>0}^{\infty} (1 - c_i t^i), \text{ where } \quad c_i \in R.$$

Using the bimultiplicativity of (f, g) and the continuity of expression (2.4) with respect to the natural topology on $\mathbb{W}$ given there by the congruent subgroups $1 + t^n R[[t]]$ it is enough to show that $(1 + g_1 t^{i_1}, 1 + g_2 t^{i_2})$ is a formal series on g_1, g_2 defined over $\mathbf{Z}$, where $1 + g_k t^k \in R((t))^*$ and $i_1, i_2 \in (\mathbf{Z} \setminus 0)^2$. This leads us to the following explicit formula for the Contou-Carrère symbol. Let $f, g \in R((t))^*$, and decompose

$$f = \prod_{i<0}^{i>-\infty} (1 - a_i t^i) \cdot a_0 \cdot t^{\nu(f)} \cdot \prod_{i>0}^{\infty} (1 - a_i t^i),$$

$$g = \prod_{i<0}^{i>-\infty} (1 - b_i t^i) \cdot b_0 \cdot t^{\nu(f)} \cdot \prod_{i>0}^{\infty} (1 - b_i t^i),$$

then

$$(f, g) = (-1)^{\nu(f)\nu(g)} \frac{a_0^{\nu(g)} \prod_{i>0}^{\infty} \prod_{j>0}^{j<\infty} (1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)}}{b_0^{\nu(f)} \prod_{i>0}^{i<\infty} \prod_{j>0}^{\infty} (1 - a_{-i}^{j/(i,j)} b_j^{i/(i,j)})^{(i,j)}}. \tag{2.7}$$

This proves the existence.

Now we see that the Contou-Carrère symbol is bimultiplicative. This is clear from expression (2.4) if $\mathbf{Q} \subset R$. We consider two maps from $X = L\mathbb{G}_m \times L\mathbb{G}_m \times L\mathbb{G}_m$ to $\mathbb{G}_m$. Let $x, y, z \in L\mathbb{G}_m(R)$, then the first map is given as (xy, z), and the second map is given as $(x, z)(y, z)$. As these two maps coincide when $\mathbf{Q} \subset R$ and X is represented by ind-flat schemes over $\mathbf{Z}$, these two maps coincide for any ring R. Therefore $(\cdot, \cdot)$ is bimultiplicative with respect to the first argument. By the same argument, it is also bimultipicative with respect to the second argument.

Proposition 2.3 (Steinberg property)**.** *Let $f, 1 - f \in L\mathbb{G}_m(R) = R((t))^*$. Then $(f, 1 - f) = 1$.*

Proof. Using Lemma (2.1) we will prove that condition $x+y=1$ for $x \times y \in R((t))^* \times R((t))^*$ defines an ind-scheme, which is an inductive limit of flat schemes over $\mathbf{Z}$. Indeed, it is clear that it is enough to prove it by restriction to every connected component of $L\mathbb{G}_m \times L\mathbb{G}_m$ which is uniquely defined by the pair of integers $(\nu(x), \nu(y))$. If we fix such a pair, then we can write $x = \sum_{i>-\infty}^{\infty} e_i t^i$ and $y = \sum_{i>-\infty}^{\infty} g_i t^i$, where $e_{\nu(x)}, g_{\nu(y)} \in R^*$, and $e_i, g_j \in \mathcal{N}R$ for any $i < \nu(x), j < \nu(y)$. The condition $x+y=1$ is equivalent to: $e_0 + g_0 = 1$ and $e_i + g_i = 0$ for every integer $i \neq 0$. It is enough to study the subscheme defined by these equations in the ind-scheme $V_1 \times V_2$, where

$$V_1 = \varinjlim_{i \to -\infty} \left(\prod_{n=i}^{\nu(x)} V_{1,n} \right) \times \operatorname{Spec} \mathbf{Z}[e_{\nu(x)+1}, e_{\nu(x)+2}, \ldots],$$

and $V_{1,\nu(x)} = \operatorname{Spec} \mathbf{Z}[e_{\nu(x)}, e_{\nu(x)}^{-1}] = \mathbb{G}_m$, $V_{1,m} = \operatorname{Spf} \mathbf{Z}[[e_m]] = \mathcal{N}$ for $m < \nu(x)$;

$$V_2 = \varinjlim_{i \to -\infty} \left(\prod_{n=i}^{\nu(y)} V_{2,n} \right) \times \operatorname{Spec} \mathbf{Z}[g_{\nu(y)+1}, g_{\nu(y)+2}, \ldots],$$

and $V_{1,\nu(y)} = \operatorname{Spec} \mathbf{Z}[g_{\nu(y)}, g_{\nu(y)}^{-1}] = \mathbb{G}_m$, $V_{2,m} = \operatorname{Spf} \mathbf{Z}[[g_m]] = \mathcal{N}$ for $m < \nu(y)$. Fix some integer $k > \max(\nu(x), \nu(y), 0) + 1$, and write

$$\operatorname{Spec} \mathbf{Z}[e_{\nu(x)+1}, e_{\nu(x)+2}, \ldots] = \left(\prod_{n=\nu(x)+1}^{k-1} V_{1,n} \right) \times \operatorname{Spec} \mathbf{Z}[e_k, e_{k+1}, \ldots],$$

$$\operatorname{Spec} \mathbf{Z}[g_{\nu(y)+1}, e_{\nu(y)+2}, \ldots] = \left(\prod_{n=\nu(y)+1}^{k-1} V_{2,n} \right) \times \operatorname{Spec} \mathbf{Z}[g_k, g_{k+1}, \ldots],$$

where $V_{1,n} = \operatorname{Spec} \mathbf{Z}[e_n] = \mathbb{G}_a$ and $V_{2,n} = \operatorname{Spec} \mathbf{Z}[g_n] = \mathbb{G}_a$. Hence it follows

$$\begin{aligned} V_1 \times V_2/(x+y=1) = \varinjlim_{i \to -\infty} \left(\prod_{n=i}^{k-1} V_{1,n} \times V_{2,n}/(e_n + g_n = \delta_{n,0}) \right) \times \\ \times \operatorname{Spec} \mathbf{Z}[e_k, e_{k+1}, \ldots, g_k, g_{k+1}, \ldots]/(e_k = g_k, e_{k+1} = g_{k+1}, \ldots), \end{aligned} \tag{2.8}$$

where $V_{j,n}$ (for $j=0$ or $j=1$) is one of the following schemes: $\mathcal{N}$, $\mathbb{G}_a$, $\mathbb{G}_m$. The scheme in expression (2.8) is an affine space. By analyzing all possible cases for $V_{j,n}$, it is easy to see that $V_{1,n} \times V_{2,n}/(e_n + g_n = \delta_{n,0})$ is also (ind)-flat over $\mathbf{Z}$ (or possibly empty). Therefore, $V_1 \times V_2/(x+y=1)$ is ind-flat over $\mathbf{Z}$ (or possibly empty).

Therefore it is enough to check the Steinberg property only for the case $\mathbf{Q} \subset R$. In this case this property follows from Lemma-Definition (2.2) by explicit calculations with series in formula (2.4). Indeed, we just need to consider three cases: $\nu(f) > 0$, $\nu(f) = 0$, $\nu(f) < 0$, where we can for simplicity assume that $N = 1$ in decomposition (2.1). See the analogous but more difficult analysis for the two-dimensional Contou-Carrère symbol in Proposition 4.2. $\square$

The following reciprocity law for the Contou-Carrère symbol was proved in [AP], in [BBE, §3.4][1] and later by another methods in [Pal]. A proof via K-theory, as suggested by one of the editors, is outlined in Remark 7.1 later.

Theorem 2.4. *Let C be a smooth projective algebraic curve over an algebraically closed field k. Let R be a finite local k-algebra. Let $K = k(C) \otimes_k R$. Let $f, g \in K^*$. Then the following product in R^* contains only finitely many non-equal to 1 terms and*

$$\prod_{p \in C} (f,g)_p = 1,$$

where $(\cdot,\cdot)_p$ is the Contou-Carrère symbol on $K_p \otimes_k R$, $K_p = k((t_p))$ is the completion field of the point p on C, and $K \hookrightarrow K_p \otimes_k R$ for any $p \in C$.

The authors of [AP] and [BBE] realized the Contou-Carrère symbol as the commutator of some central extension. A consequence is that this symbol is invariant under the change of the local parameter t in the following sense. Any $t' \in R((t))^*$ with $\nu(t') = 1$ defines a well-defined continuous automorphism $\phi_{t'}$ of the ring $R((t))$ by the rule: $\sum a_i t^i \mapsto \sum a_i {t'}^i$, see e.g. [Mor, § 1] . Then for any $f, g \in R((t))^*$ we have that $(f,g) = (\phi_{t'}(f), \phi_{t'}(g))$.

Remark 2.3. One can regard the Contou-Carrère symbol as some deformation of the usual tame symbol. Indeed, if $R = k$ is a field, then $(\cdot,\cdot)$ is the tame symbol. If $R = k[\epsilon]/\epsilon^3$, then for any $f, g \in k((t))$ we have that $(1+\epsilon f, 1+\epsilon g) = 1 + \epsilon^2 \mathrm{res} f dg$.

3. Definition of the two-dimensional Contou-Carrère symbol

3.1. **Double loop group of $\mathbb{G}_m$.** Let us define the double loop group of $\mathbb{G}_m$ as $L^2\mathbb{G}_m = L(L\mathbb{G}_m)$, i.e. $L^2\mathbb{G}_m(R) = R((u))((t))^*$ for every commutative ring R. Let us show that

Proposition 3.1. *$L^2\mathbb{G}_m$ is represented by an ind-(affine) scheme over* $\mathbf{Z}$.

Proof. We have from Lemma 2.1 that

$$L^2\mathbb{G}_m \simeq L\widehat{\mathbb{W}} \times L\mathbb{Z} \times L\mathbb{G}_m \times L\mathbb{W}. \tag{3.1}$$

The proposition then is the consequence of the following Lemmas 3.2-3.4. □

Lemma 3.2. *The natural map $\mathbb{Z} \to L\mathbb{Z}$ is an isomorphism of functors.*

Concretely, let R be a commutative ring, then any decomposition of the ring $R((t))$ into the product of two rings:

$$R((t)) = L_1 \oplus L_2$$

is induced by the decomposition of the ring R into the product of two rings:

$$R = R_1 \oplus R_2,$$

such that $L_1 = R_i((t))$, $L_2 = R_j((t))$, where $i, j \in \{1,2\}$, $i \neq j$.

[1]The statement in [BBE, §3.4] is more general than we formulate here. In *loc.cit* the authors proved the reciprocity law for a smooth projective family of curves over any base ring R.

Proof. Every non-trivial idempotent e in a commutative ring A, i.e, $e^2 = e$, $e \neq 0$, $e \neq 1$, defines the non-trivial decomposition: $A = Ae \oplus A(1-e)$. Therefore the study of decompositions of rings is the same as the study of idempotents in this ring.

Let $R_{\mathrm{red}} = R/\mathcal{N}R$. As the topological spaces of $\operatorname{Spec} R$ and $\operatorname{Spec} R_{\mathrm{red}}$ coincide. Therefore idempotents of the ring R are in one-to-one correspondence with idempotents of the ring R_{red} under the natural map $R \to R_{\mathrm{red}}$.

Suppose that there is an element $f \in R((t))$ such that $f^2 = f$ and $f \neq 1, f \neq 0$. Let $\tilde{f}$ be the image of the element f in $R_{\mathrm{red}}((t))$. Then it is easy to see that $\tilde{f}^2 = \tilde{f}$ implies $\tilde{f} = \sum_{n \geq 0} a_n t^n$, where $a_0^2 = a_0$, $a_0 \in R_{\mathrm{red}}$. If $a_0 = 0$, then $\tilde{f} = 0$. If $a_0 = 1$, then $\tilde{f} = 1$, since $\tilde{f}$ is invertible in this case. If $a_0 \neq 0$, $a_0 \neq 1$, then consider $R_{\mathrm{red}} = B_1 \oplus B_2$, where $B_1 = R_{\mathrm{red}} a_0$, $B_2 = R_{\mathrm{red}}(1 - a_0)$. It is clear that the image of $\tilde{f}$ in $B_1((t))$ is equal 1, and the image of $\tilde{f}$ in $B_2((t))$ is equal 0. Hence we have $\tilde{f} = a_0$.

If $\tilde{f} = a_0 = 0$, then $f \in (\mathcal{N}R)((t))$ and $f^2 = f$. Hence we have $f(1-f) = 0$ and $1 - f \in R((t))^*$. The latter is possible only if $f = 0$.

Suppose that $\tilde{f} = a_0 = 1$. Then we have $f = 1 + f'$, where $f' \in (\mathcal{N}R)((t))$. We will prove that f is invertible in $R((t))$. Together with $f^2 = f$ it would imply that $f = 1$. So, we have to prove that $f = 1 + f'$ is invertible in $R((t))$. Let $f' = \sum_{n > -\infty}^{\infty} f_n t^n$, where $f_n \in \mathcal{N}R$ for any $n \in \mathbf{Z}$. The elements f and $(1 + f_0)^{-1} f$ are both invertible or not invertible in $R((t))$. Therefore, after replacement f by $(1 + f_0)^{-1} f$, we can assume that $f_0 = 0$. Let I' be the ideal in $R((t))$ which is generated by all elements f_n with $n < 0$. Then I' is a nilpotent ideal. After replacement f by $(1 + \sum_{n>0}^{\infty} f_n t^n)^{-1} f$ we can assume that $f = 1 + f'$, where $f' \in I'((t))$. Let $f' = \sum_{n > \infty}^{\infty} f_n t^n$. Again, as above, we can assume that $f_0 = 0$. After replacement f by $(1 + \sum_{n<0}^{n>-\infty} f_n t^n)^{-1} (1 + \sum_{n>0}^{\infty} f_n t^n)^{-1} f$ we can assume that $f \in I'^2((t))$. Repeating this process a sufficient number of times we will obtain that the element f is invertible.

Thus we have $\tilde{f} = a_0$, where $a_0 \neq 0$, $a_0 \neq 1$, as $f \neq 0$, $f \neq 1$. Let $b_0 \in R$ be the only idempotent in the ring R such that the image of b_0 in the ring R_{red} is a_0. Consider $R = C_1 \oplus C_2$, where $C_1 = Rb_0$, $C_2 = R(1 - b_0)$. By the cases considered above we know that the image of f in $C_2((t))$ is 0, and the image of f in $C_1((t))$ is 1. Therefore, $f = b_0$. □

Lemma 3.3. *The loop group of* $\mathbb{W}$, *denoted by* $L\mathbb{W}$ *is represented by an ind-(affine) scheme, which is an "inductive limit" of infinite-dimensional affine spaces over* $\mathbf{Z}$.

Proof. Let $I = \prod_{i>0} \mathbf{Z}$ be the index set with a partial order given as $(k_1, k_2, \ldots) \leq (l_1, l_2, \ldots)$ if $k_i \leq l_i$ for all i.

Fix $\underline{k} = (k_1, \ldots) \in I$, consider the functor $(L\mathbb{W})_{\underline{k}}$ which represents

$$(L\mathbb{W})_{\underline{k}}(R) = \left\{ 1 + f \mid f = \sum_{i>0, j \geq k_i} f_{ij} u^j t^i, \ f_{ij} \in R \right\}.$$

Clearly, $(L\mathbb{W})_{\underline{k}} = \operatorname{Spec} \mathbf{Z}[f_{ij} \mid i \geq 0, j \geq k_i]$ is an affine space, and $L\mathbb{W} = \varinjlim_{\underline{n} \in I} (L\mathbb{W})_{\underline{n}}$. □

Lemma 3.4. *The loop group of* $\widehat{\mathbb{W}}$, *denoted by* $L\widehat{\mathbb{W}}$ *is represented by an ind-(affine) scheme.*

Proof. Let us consider the loop space $L\mathcal{N}$ of the formal scheme $\mathcal{N}$, i.e. $L\mathcal{N}(R) = \mathcal{N}(R((t)))$. We have $\mathcal{N} = \varinjlim_n \mathcal{N}_n$, where $\mathcal{N}_n = \operatorname{Spec} \mathbf{Z}[T]/T^n$. Since $L\mathcal{N}_n$ is an ind-affine scheme over $\mathbf{Z}$, we obtain that $L\mathcal{N} = \varinjlim_n L\mathcal{N}_n$ is also an ind-affine scheme.

Observe that for any commutative ring R, elements in $L\widehat{\mathbb{W}}(R)$ can by written as $f = \sum_{i<0}^{>-\infty} f_i t^i$, with $f_i \in \mathcal{N}(R((u)))$. Therefore,

$$L\widehat{\mathbb{W}} = \varinjlim_n (L\mathcal{N})^n. \tag{3.2}$$

□

Observe that unlike the case of $L\mathbb{G}_m$, the functor $L^2\mathbb{G}_m$ is not represented as an inductive limit of flat schemes over $\mathbf{Z}$ when we use decomposition (3.1) and formula (3.2). Indeed, the ind-scheme $L\mathcal{N}_n$ is not flat. Observe that we can write

$$L\mathcal{N}_n = \varinjlim_m L_m\mathcal{N}_n, \quad L_m\mathcal{N}_n(R) = \left\{ f = a_m t^m + a_{m+1}t^{m+1} + \cdots \mid a_i \in R \ , \ f^n = 0 \right\}. \tag{3.3}$$

Therefore,

$$L_m\mathcal{N}_n = \operatorname{Spec} A_{m,n}, \quad \text{where } A_{m,n} = \mathbf{Z}[a_m, a_{m+1}, \ldots]/(a_m^n, na_m^{n-1}a_{m+1}, \ldots).$$

In particular, $L_m\mathcal{N}_n$ is not flat over $\mathbf{Z}$, since $a_m^{n-1}a_{m+1}$ is a torsion element. However, we observe that $I_{m,n} = (a_m^n, na_m^{n-1}a_{m+1}, \ldots)$ is a homogeneous ideal generated by elements of degree n. In particular, we observe that, for any nonzero element $a \in A_{m,n}$, there exists some $n' \gg n$ such that for any $m' < m$, all preimages of a under the surjective map $A_{m',n'} \to A_{m,n}$ are not torsion.

We consider the following class $\mathcal{EF}$ of ind-affine schemes over $\mathbf{Z}$.

Definition 3.1. An affine ind-scheme $X/\mathbf{Z}$ belongs to $\mathcal{EF}$ if we can write $X = \varinjlim_{i\in J} \operatorname{Spec} R_i$ (where J is a directed set) such that for any $i \in J$, and any nonzero element $a \in R_i$, there exists some $j > i$ such that all preimages of a under the map $R_j \to R_i$ are not torsion elements.

It is clear that for any integer m we have $\varinjlim_n L_m\mathcal{N}_n \in \mathcal{EF}$. Hence, $L\mathcal{N} = \varinjlim_{m,n} L_m\mathcal{N}_n \in \mathcal{EF}$.

We have the following easy lemma.

Lemma 3.5. *For any positive integer k, $(L^2\mathbb{G}_m)^k \in \mathcal{EF}$.*

Proof. By reasonings similar to the above reasonings, for any positive integer l we have $(L\mathcal{N})^l \in \mathcal{EF}$. Observe that if $X \in \mathcal{EF}$ and a ring B is a free $\mathbf{Z}$-module, then $X \times \operatorname{Spec} B \in \mathcal{EF}$. Therefore for an affine ind-scheme $Y = \varinjlim_{s\in S} \operatorname{Spec} B_s$ such that for any $s \in S$ the ring B_s is a free $\mathbf{Z}$-module we have $X \times Y \in \mathcal{EF}$. Hence, $(L^2\mathbb{G}_m)^k \in \mathcal{EF}$. □

It will be convenient to introduce the following group ind-schemes

$$\mathbb{M} = \mathbb{W} \times L\mathbb{W}, \quad \mathbb{P} = \widehat{\mathbb{W}} \times L\widehat{\mathbb{W}}. \tag{3.4}$$

Therefore, from the proof of Proposition 3.1 and Lemma 2.1, we have

$$L^2\mathbb{G}_m \simeq \mathbb{P} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{G}_m \times \mathbb{M}. \tag{3.5}$$

Let us distinguish these two $\mathbb{Z}$s. Recall that we have the group homomorphism $\nu : L\mathbb{G}_m \to \mathbb{Z}$, which induces $L\nu : L^2\mathbb{G}_m \to L\mathbb{Z} \simeq \mathbb{Z}$. We denote this map by ν_1. On the other hand, we have the group homomorphism

$$L^2\mathbb{G}_m \simeq L\widehat{\mathbb{W}} \times L\mathbb{Z} \times L\mathbb{G}_m \times L\mathbb{W} \to L\mathbb{G}_m \xrightarrow{\nu} \mathbb{Z},$$

denoted by ν_2. Therefore, for any commutative ring R and any $f \in R((u))((t))^*$, we have the following unique decomposition

$$f = f_{-1} \cdot f_0 \cdot u^{\nu_2(f)} t^{\nu_1(f)} \cdot f_1, \tag{3.6}$$

where $f_{-1} \in (\widehat{\mathbb{W}} \times L\widehat{\mathbb{W}})(R)$, $f_0 \in R^*$, $f_1 \in (\mathbb{W} \times L\mathbb{W})(R)$, $\nu_2(f), \nu_1(f) \in \mathbb{Z}(R)$[2].

For the later purpose, let us also introduce the following group functors.

Definition 3.2. Let $\mathfrak{m}$ and $\mathfrak{p}$ be a group functors from the category **Aff** given by

$$\mathfrak{m}(R) = \{f \mid f \in u \cdot R[[u]] + t \cdot R((u))[[t]]\}$$
$$\mathfrak{p}(R) = \left\{f \mid f \in u^{-1} \cdot R[u^{-1}] + t^{-1} \cdot R((u))[t^{-1}] \text{ is nilpotent } \right\}$$

with the usual addition, where R is any commutative ring.

By the same proofs of the ind-representability of $\mathbb{M}$ and $\mathbb{P}$, one can show that $\mathfrak{m}$ and $\mathfrak{p}$ are also represented by ind-schemes. (For more details, see the proof of Propositions 3.6 and 3.7.)

3.2. **Natural topologies.** Recall that there is a natural linear topology on $R((u))((t))$ which comes from the topology of inductive and projective limits: a basis of open neighborhoods of $0 \in R((u))((t))$ consists of the R submodules

$$W_{m,\{m_i\}} = t^m R((u))[[t]] + \sum_{i \in \mathbf{Z}} u^{m_i} t^i R[[u]], \tag{3.7}$$

for some $m \in \mathbf{Z}$, and the set of integers $\{m_i\}$. In other words, the R-module $R((u))((t))$ is a topological R-module when we put the discrete topology on the ring R.

We consider the induced topologies on the R-modules $\mathfrak{m}(R)$ and $\mathfrak{p}(R)$. The R-module $\mathfrak{m}(R)$ is a topological R-module such that the base of neighborhoods of $0 \in \mathfrak{m}(R)$ consists of R-submodules

$$\mathfrak{u}_{i,j}(R) = u^j R[[u,t]] + t^i R((u))[[t]] \tag{3.8}$$

with $i \in \mathbf{N}$, $j \in \mathbf{N}$. Likewise, $\mathfrak{p}(R)$ has a topology with the base of neighborhoods of 0 given by

$$\mathfrak{u}_{\{n_i\}}(R) = \left\{f \mid f \in \sum t^{-i} u^{n_i} R[[u]] \text{ is nilpotent}\right\}, \tag{3.9}$$

with $i \in \mathbf{N}$, $n_i \in \mathbf{Z}$.

Definition 3.3. We denote by $\mathcal{B}$ the full subcategory of the category of commutative rings consisting of those R such that

$$\mathcal{N}R \quad \text{is a nilpotent ideal in} \quad R.$$

[2]See Remark 2.1 about the meaning of expressions $u^{\nu_2(f)}$ and $t^{\nu_1(f)}$.

Note that Noetherian rings belong to $\mathcal{B}$.

Later on we will need the following proposition.

Proposition 3.6. *Let R be any commutative ring. We consider the discrete topology on the group R^*.*

1. *Let $\phi : \mathfrak{m}_R \to \mathbb{G}_{mR}$ be any homomorphism of group ind-R-schemes. Then ϕ restricted to R-points is a continuous map from $\mathfrak{m}(R)$ to R^*.*
2. *Let $R \in \mathcal{B}$ and $\psi : \mathfrak{p}_R \to \mathbb{G}_{mR}$ be any homomorphism of group ind-R-schemes. Then ψ restricted to R-points is a continuous map from $\mathfrak{p}(R)$ to R^*.*

Proof. (1) We prove the continuous property for the map ϕ. It is enough to find an open subgroup $\mathfrak{u}_{i,j}(R) \subset \mathfrak{m}(R)$ such that $\phi(\mathfrak{u}_{i,j}(R)) = 1$. We assume the opposite, i.e., that for any $l \in \mathbf{N}$ there is an element $f_l \in \mathfrak{u}_{l,l}(R)$ such that $\phi(f_l) \neq 1$. It is clear that there is an index $\underline{k} \in I$ (where I is defined in the proof of Lemma 3.3) such that for any $l \in \mathbf{N}$, $f_l \in \mathfrak{m}_{\underline{k}}(R)$, where

$$\mathfrak{m}_{\underline{k}}(R) = \left\{ f = \sum_{j>0} f_{0j} u^j + \sum_{i>0, j>k_i} f_{ij} u^j t^i \mid f_{ij} \in R \right\},$$

is a subgroup of $\mathfrak{m}$ represented by an infinite dimensional affine space, and $\mathfrak{m} = \varinjlim_{\underline{k} \in I} \mathfrak{m}_{\underline{k}}$.

We consider the restriction $\phi_{\underline{k}}$ of the morphism ϕ to $\mathfrak{m}_{\underline{k}R}$, which in turn is given by a map of R-algebras:

$$\phi_{\underline{k}}^* \ : \ R[T, T^{-1}] \longrightarrow R[\{f_{ij}\}].$$

Since only finite many of variables f_{ij} appear in the polynomial $\phi_{\underline{k}}^*(T)$, the morphism $\phi_{\underline{k}}$ factors through a finite-dimensional quotient of the group scheme $\mathfrak{m}_{\underline{k}R}$. Therefore there is $p \in \mathbf{N}$ such that $\phi(\mathfrak{m}_{\underline{k}}(R) \cap \mathfrak{u}_{p,p}(R)) = 1$. It contradicts to the assumption $\phi(f_p) \neq 1$. Thus, the map ϕ is continuous.

(2) We prove the continuous property for the map ψ. Let

$$\mathfrak{p}_0 = \{ f \in u^{-1}R[u^{-1}] \mid f \text{ is nilpotent } \}$$

and

$$\mathfrak{p}_- = \{ f \in t^{-1}R((u))[t^{-1}] \mid f \text{ is nilpotent } \}.$$

Then the natural map $\mathfrak{p}_0 \times \mathfrak{p}_- \to \mathfrak{p}$ is an isomorphism of groups. Let us fix $k \in \mathbf{N}$ and $n \in \mathbf{Z}$, and let

$$\mathfrak{p}_k = \{ f \in \sum_{i=1}^{k} R((u)) t^{-i} \mid f \text{ is nilpotent } \},$$

$$\mathfrak{p}_{k,n} = \{ f \in \sum_{i=1}^{k} R[[u]] t^{-i} u^n \mid f \text{ is nilpotent } \}.$$

Then $\mathfrak{p}_- = \varinjlim_{k \in \mathbf{N}} \mathfrak{p}_k$. In addition, it is enough to show that if $R \in \mathcal{B}$, then for every k, there is n_k such that $\psi(\mathfrak{p}_{k,n_k}(R)) = 1$.

Since $R \in \mathcal{B}$, there is $q \in \mathbf{N}$ such that $(\mathcal{N}R)^q = 0$. Hence for any $k \in \mathbf{N}$ we have an equality

$$\mathfrak{p}_{k,n}(R) = \mathfrak{p}_{k,n,q}(R),$$

where

$$\mathfrak{p}_{k,n,q} = \{f \in \mathfrak{p}_{k,n} \mid f^q = 0\}.$$

Therefore, it is enough to construct for every k, an integer n_k such that $\psi|_{\mathfrak{p}_{k,n_k,q}} = 1$. We consider the restriction $\psi_{k,0,q}$ of the morphism ψ on the affine R-scheme $\mathfrak{p}_{k,0,q\,R}$. This restriction is given by a map of R-algebras:

$$\psi^*_{k,0,q} \ : \ R[T,T^{-1}] \longrightarrow A_{k,0,q},$$

where $A_{k,0,q} = R[a_{ij} \mid 1 \le i \le k, j \ge 0]/I_q$ such that $\mathfrak{p}_{k,0,q} = \operatorname{Spec} A_{k,0,q}$. It is clear that there is $n_k \in \mathbf{N}$ such that the image of the element $\psi^*_{k,0,q}(T)$ in the quotient-ring

$$\frac{R[\{a_{ij}\} \mid 1 \le i \le k, j \ge 0]}{I_q + \{a_{ij}\}_{j \le n_k - 1}}$$

is equal to some $c \in R$. Then

$$\psi(\mathfrak{p}_{k,n_k,q}) = c.$$

Since $\psi(0) = 1$, we obtain that $c = 1$. □

If we do not assume that $R \in \mathcal{B}$, we can obtain a weaker statement which is sufficient for many purposes.

Proposition 3.7. *Let R be a commutative ring.*

(1) *Let $\phi : \mathfrak{m}_R \to \mathbb{G}_{mR}$ be a homomorphism of group ind-R-schemes. If for every commutative R-algebra R', and every $r \in R'$, $i > 0, j \in \mathbf{Z}$ or $i = 0, j > 0$, $\phi(ru^jt^i) = 1$, then $\phi = 1$ is the trivial homomorphism.*
(2) *Let $\psi : \mathfrak{p}_R \to \mathbb{G}_{mR}$ be a homomorphism of group ind-R-schemes. If for every commutative R-algebra R', and every $r \in \mathcal{N}R'$, $i < 0, j \in \mathbf{Z}$ or $i = 0, j < 0$, $\psi(ru^jt^i) = 1$, then $\psi = 1$ is the trivial homomorphism.*

Proof. Part (1) clearly follows from part (1) of Proposition 3.6. We consider part (2). Similar to the proof of Proposition 3.6, it is enough to show that $\psi = 1$ on $\mathfrak{p}_{k,n}$. Let $\mathfrak{q} = \{f \in R[[u]] \mid f \text{ is nilpotent }\}$. Then

$$\mathfrak{q}^k \to \mathfrak{p}_{k,n} \quad (f_1, ...f_k) \mapsto u^n \sum f_i t^i$$

is an isomorphism. Therefore, it is enough to prove a similar statement for homomorphisms $\psi : \mathfrak{q}_R \to \mathbb{G}_{mR}$.

Let $\mathfrak{q}_s = \{f \in \mathfrak{q} \mid f^s = 0\}$ for $s \in \mathbf{N}$. Then the affine R-scheme $\mathfrak{q}_{s\,R} = \operatorname{Spec} A_s$, where $A_s = R[a_i]/I_s$, and I_s is the ideal generated by $(a_0^s, sa_0^{s-1}a_1, \ldots)$. We equip the ring $R[a_i]$ with a grading such that $\deg a_i = i$. Then I_s is a homogeneous ideal generated by elements $(g_0 = a_0^s, g_1 = sa_0^{s-1}a_1, \ldots)$ with $\deg g_i = i$. We claim that

$$I_s = \bigcap_{k \ge 0} (I_s + (a_k, a_{k+1}, \ldots)).$$

Indeed, let $f \in \cap_k (I_s + (a_k, a_{k+1}, \ldots))$. We can assume that each monomial appearing in f has the degree less than l for some $l \in \mathbf{N}$. We write $f = f_1 + f_2$, where $f_1 \in I_s$ and $f_2 \in (a_l, a_{l+1}, \ldots)$. Then $f_1 = f - f_2 \in I_s$. Since I_s is a homogeneous ideal, each homogeneous component of f_1 belongs to the ideal I_s. Since all the monomials appearing in f_2 have the degree greater than $l - 1$, we must have $f \in I_s$.

Now, let $\psi : \mathfrak{q}_R \to \mathbb{G}_{mR}$ be as in the assumption of part (2) of this proposition. If we write $\mathbb{G}_{mR} = \operatorname{Spec} R[T, T^{-1}]$, then ψ induces for every $s \in \mathbf{N}$ the function $\psi_s = \psi^*(T) \in R[a_i]/I_s$. By the assumption, $\psi = 1$ on the ind-subgroup $\{f =$

$a_0 + a_1 u + \cdots + a_k u^k\} \subset \mathfrak{q}$ for every $k \geq 0$. Therefore, $\psi_s = 1 \mod (a_{k+1}, a_{k+2}, \ldots)$ for every k and s. Therefore, $\psi_s = 1$ by the above claim. This implies that $\psi = 1$. □

If $\mathbf{Q} \subset R$, then it is clear that the formal series exp and log applied to elements from $\mathfrak{p}(R)$ and from $\mathbb{P}(R)$ are well-defined[3] and induce mutually inverse isomorphisms of group ind-schemes:

$$\exp : \mathfrak{p}_{\mathbf{Q}} \longrightarrow \mathbb{P}_{\mathbf{Q}} \quad \text{and} \quad \log : \mathbb{P}_{\mathbf{Q}} \longrightarrow \mathfrak{p}_{\mathbf{Q}}. \tag{3.10}$$

To have a similar statement for the group ind-schemes $\mathfrak{m}_{\mathbf{Q}}$ and $\mathbb{M}_{\mathbf{Q}}$ we introduce a topology on the group $\mathbb{M}(R)$.

Let R be any ring. The group $\mathbb{M}(R)$ is a topological group such that the base of neighborhoods of $1 \in \mathbb{M}(R)$ consists of subgroups

$$U_{i,j}(R) = 1 + \mathfrak{u}_{i,j}(R) \tag{3.11}$$

with $i \in \mathbf{N}$, $j \in \mathbf{N}$.

If $\mathbf{Q} \subset R$, then the maps exp and log given by formal series are continuous maps on the topological groups $\mathfrak{m}(R)$ and $\mathbb{M}(R)$. Moreover, we have equalities

$$\exp(\mathfrak{u}_{i,j}(R)) = U_{i,j}(R) \quad \text{and} \quad \log(U_{i,j}(R)) = \mathfrak{u}_{i,j}(R). \tag{3.12}$$

Hence we obtain mutually inverse isomorphisms of group ind-schemes (because of the corresponding mutually inverse isomorphisms on R-points of these schemes):

$$\exp : \mathfrak{m}_{\mathbf{Q}} \longrightarrow \mathbb{M}_{\mathbf{Q}} \quad \text{and} \quad \log : \mathbb{M}_{\mathbf{Q}} \longrightarrow \mathfrak{m}_{\mathbf{Q}}. \tag{3.13}$$

It is possible to introduce topology on the group $\mathbb{P}(R)$ (we will need this topology later on.) Let R be any ring. The group $\mathbb{P}(R)$ is a topological group with the base of neighborhoods of 1 given by the following subsets (which are not subgroups in general):

$$U_{\{n_i\}}(R) = 1 + \mathfrak{u}_{\{n_i\}}(R), \tag{3.14}$$

with $i \in \mathbf{N}$, $n_i \in \mathbf{Z}$.

We do not have any formula analogous to (3.12) for the subsets $\mathfrak{u}_{\{n_i\}}(R)$ and $U_{\{n_i\}}(R)$. Instead we will prove the following lemma.

Lemma 3.8. *If a ring $R \in \mathcal{B}$ and $\mathbf{Q} \subset R$, then the maps* exp *and* log *are continuous maps on the topological groups $\mathfrak{p}(R)$ and $\mathbb{P}(R)$.*

Proof. We fix an integer q such that $(\mathcal{N}R)^q = 0$. We will prove that the map exp is continuous. We consider arbitrary subset $U_{\{n_i\}}(R) \subset \mathbb{P}(R)$ where $i \in \mathbf{N}$, $n_i \in \mathbf{Z}$. Without loss of generality we can assume that all $n_i \in \mathbf{N}$ and $n_1 \leq n_2 \leq n_3 \leq \ldots$. Let $l_i = n_{iq}$ for any $i \in \mathbf{N}$. We consider a subset $\mathfrak{u}_{\{l_i\}}(R) \subset \mathfrak{p}(R)$ where $i \in \mathbf{N}$, $l_i \in \mathbf{Z}$. We have $\exp(\mathfrak{u}_{\{l_i\}}(R)) \subset U_{\{n_i\}}(R)$. Therefore the map exp is continuous. The proof for the map log is analogous. □

[3] These series will contain only finite number non-zero terms.

3.3. **Two-dimensional symbol on** $(L^2\mathbb{G}_{m\mathbf{Q}})^3$**.** We define the following map $\nu : L^2\mathbb{G}_m \times L^2\mathbb{G}_m \longrightarrow \mathbb{Z}$ as

$$\nu(f,g) = \begin{vmatrix} \nu_2(f) & \nu_2(g) \\ \nu_1(f) & \nu_1(g) \end{vmatrix}.$$

From this definition we have the following proposition.

Proposition 3.9. *The map $\nu(\cdot,\cdot)$ is a bimultiplicative (with respect to the additive structure on $\mathbb{Z}$) map and satisfies the Steinberg relation: $\nu(f, 1-f) = 1$ for any $f, 1-f \in L^2\mathbb{G}_m(R)$ where R is a commutative ring.*

One of our main definitions is the following.

Definition 3.4. *The two-dimensional Contou-Carrère symbol* $(\cdot,\cdot,\cdot)$ is the unique tri-multiplicative, anti-symmetric map from $L^2\mathbb{G}_{m\mathbf{Q}} \times L^2\mathbb{G}_{m\mathbf{Q}} \times L^2\mathbb{G}_{m\mathbf{Q}}$ to $\mathbb{G}_{m\mathbf{Q}}$ such that for any $\mathbf{Q}$-algebra R, and $f, g, h \in L^2\mathbb{G}_m(R)$,

(3.15)
$$(f,g,h) = \exp \operatorname{Res}(\log f \cdot \frac{dg}{g} \wedge \frac{dh}{h}) \text{ when } (\nu_1,\nu_2)(f) = (0,0),\ f_0 = 1 \text{ (see (3.6))},$$

$$(a,g,h) = a^{\nu(g,h)} \quad \text{when} \quad a \in R^*, \tag{3.16}$$

$$(f,g,h) = (-1)^A \quad \text{when} \quad f = u^{j_1}t^{i_1}, \quad g = u^{j_2}t^{i_2}, \quad h = u^{j_3}t^{i_3}, \tag{3.17}$$

where

$$A = \nu(f,g)\nu(f,h) + \nu(g,h)\nu(g,f) + \nu(h,f)\nu(h,g) + \nu(f,g)\nu(f,h)\nu(g,h). \tag{3.18}$$

Here we set $df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial t}dt$, $df \wedge dg = (\frac{\partial f}{\partial u}\frac{\partial g}{\partial t} - \frac{\partial g}{\partial u}\frac{\partial g}{\partial t})du \wedge dt$, and

$$\operatorname{Res}(\sum a_{i,j}u^j t^i du \wedge dt) = a_{-1,-1}.$$

Remark 3.1. To calculate df, where f is an infinite series, we used the free two-dimensional R-module of continuous differentials $\tilde{\Omega}^1_{K/R}$, which is a quotient module of the infinite-dimensional R-module $\Omega^1_{K/R}$. See analogous constructions in [O1, prop. 2].

Note that the symbol given in Definition 3.4 for the $\mathbf{Q}$-algebra R is well-defined. First note that in formula (3.15) we have

$$\operatorname{Res}(\log f \cdot \frac{dg}{g} \wedge \frac{dh}{h}) \in \mathcal{N}R. \tag{3.19}$$

Indeed, using decomposition (3.6) and tri-multiplicativity of expression from (3.19) (with respect to additive structure on R): $\log f = \log f_{-1} + \log f_1$ and so on, it is enough to verify (3.19) for elements which appear from decomposition (3.6), which is clear. Besides, expression from (3.15) is anti-symmetric. This is obvious for the permutation of g and h from the definition of the wedge product. And, for example, for the permutation of f and g (when g is also equal $g_{-1}g_1$ by (3.6)) it follows from the following equalities:

$$0 = \operatorname{Res} d(\log f \cdot \log g \cdot \frac{dh}{h}) = \operatorname{Res}(\log f \cdot d\log g \wedge \frac{dh}{h}) + \operatorname{Res}(\log g \cdot d\log f \wedge \frac{dh}{h}).$$

Finally, expressions given by formulas (3.15)-(3.17) are tri-mutltiplicative. (It is obvious for (3.15) and (3.16). For (3.17) it can be proved by using calculations modulo 2 with direct definitions of A and $\nu(\cdot,\cdot)$.)

Remark 3.2. The formula (3.15) is very similar to an explicit formula from [BM, th. 3.6], which is an analytic expression for the two-dimensional tame symbol and coincides with the holonomy of some gerbe with connective structure and curving constructed by three meromorphic functions on a complex algebraic surface.

One of the main problems is to extend the definition of the two-dimensional Contou-Carrère symbol to any ring R. If such an extension exists, then it is unique by the following lemma.

Lemma 3.10. *There is at most one tri-multiplicative, anti-symmetric map from* $(L^2\mathbb{G}_m)^3$ *to* $\mathbb{G}_m$ *such that this map restricted to* $(L^2\mathbb{G}_{m\mathbf{Q}})^3$ *satisfies properties* (3.15)-(3.18).

Proof. Clearly, such map over $\mathbf{Q}$ is unique. We want to extend the uniqueness to $\mathbf{Z}$. Observe that the argument as in the 1-dimensional case can not be applied directly. The problem, as we explained, is that $L^2\mathbb{G}_m$ is not inductive limit of flat $\mathbf{Z}$-schemes. However, recall that from Lemma 3.5, $(L^2\mathbb{G}_m)^3 \in \mathcal{EF}$. Then the following general fact will imply the lemma.

Let $X = \varinjlim_{i\in J} \operatorname{Spec} R_i \in \mathcal{EF}$ and $f_{\mathbf{Q}} : X_{\mathbf{Q}} \to Y_{\mathbf{Q}}$ be any morphism, where $Y = \operatorname{Spec} R$ is a $\mathbf{Z}$-scheme. We prove that there is at most one extension $f : X \to Y$. Let f_1, f_2 be two such extensions. Then f_l $(l = 1, 2)$ is determined by a compatible family $\{f_{l,i} \mid i \in J\}$ of the ring homomorphisms $f_{l,i} : R \to R_i$. By assumption, for any $r \in R$, $f_{1,i}(r) - f_{2,i}(r)$ is a torsion element in R_i. This implies $f_{1,i} = f_{2,i}$ for all $i \in J$, since $X \in \mathcal{EF}$. □

Let us turn to the question of extension of the definition of two-dimensional Contou-Carrère symbol. Let us indicate here that the argument as in the 1-dimensional case does not admit an obvious generalization. Namely, it is tempting to calculate $(1 - au^jt^i, 1 - bu^lt^k, 1 - cu^nt^m)$ and to show that the result does make sense for any commutative ring (we will do such calculation in Section 3.4 and show that this is indeed the case). However, unlike the 1-dimensional case, if we write f_{-1} (as in the decomposition (3.6)) as $\prod(1 - a_{i,j}u^jt^i)$, there will be infinite many terms in the product. Therefore, formula (2.7) in current setting will be an infinite product a priori, and it is not obvious at all whether such expression makes sense (we need to restrict ourself to the rings from the category $\mathcal{B}$, see Section 3.4). Our strategy in Section 5.3 will be to construct a map $(L^2\mathbb{G}_m)^3 \to \mathbb{G}_m$ directly using some general categorical formalism developed in [OsZh], and show that this map satisfies properties (3.15)-(3.18). This categorical construction will help to prove the reciprocity laws for the symbol on an algebraic surface and to obtain some new non-trivial properties of the symbol such as, for example, the invariance under the change of local parameters.

3.4. **Explicit formulas.** In this subsection we will extend the definition of the two-dimensional Contou-Carrère symbol to commutative rings from the category $\mathcal{B}$ by means of some explicit functorial (with respect to R) formulas.

By Definition 3.4 we know an explicit formula for the two-dimensional Contou-Carrère symbol when $\mathbf{Q} \subset R$. We want to extend the definition of two-dimensional

Contou-Carrère symbol to the case $\mathbf{Q} \not\subseteq R$, $R \in \mathcal{B}$ also by some explicit formulas. Similarly to the one-dimensional case it would be enough to consider the formal series for (f,g,h). This series appears from formula (3.15) and depends on coefficients of $f,g,h \in L^2\mathbb{G}_m(R) = R((u))((t))^*$ (see decomposition (3.26)). We should prove that the coefficients of the series, which are a priori from $\mathbf{Q}$, are from $\mathbf{Z}$, i.e. they does not contain the denominators. We will not prove it in a direct way. We will use some infinite product decomposition of elements from $L^2\mathbb{G}_m(R)$. To achieve this goal we will need some restrictions on the ring R (sometimes we will assume that $R \in \mathcal{B}$). But, first, we have the following lemmas.

Lemma 3.11. *Let $\mathbf{Q} \subset R$. Let $f,g,h \in L^2\mathbb{G}_m(R)$ such that $f = 1 - a_{i,j}u^jt^i$, $g = 1 - b_{k,l}u^lt^k$, $h = 1 - c_{m,n}u^nt^m$, where $i,j,k,l,m,n \in \mathbf{Z}$ and $a_{i,j}, b_{k,l}, c_{m,n} \in R$. Let*

$$p_0 = \begin{vmatrix} l & n \\ k & m \end{vmatrix}, \qquad q_0 = \begin{vmatrix} n & j \\ m & i \end{vmatrix}, \qquad r_0 = \begin{vmatrix} j & l \\ i & k \end{vmatrix}.$$

Let (p_0,q_0,r_0) be the greatest common divisor of p_0,q_0,r_0, where we set $(p_0,q_0,r_0) > 0$ if $p_0,q_0,r_0 > 0$, and $(p_0,q_0,r_0) < 0$ if $p_0,q_0,r_0 < 0$. Then

$$(f,g,h) = T(a_{i,j}, b_{k,l}, c_{m,n}) = \left(1 - a_{i,j}^{\frac{p_0}{(p_0,q_0,r_0)}} \cdot b_{k,l}^{\frac{q_0}{(p_0,q_0,r_0)}} \cdot c_{m,n}^{\frac{r_0}{(p_0,q_0,r_0)}}\right)^{(p_0,q_0,r_0)} \tag{3.20}$$

iff $p_0,q_0,r_0 > 0$ or $p_0,q_0,r_0 < 0$. In other cases of signs of p_0,q_0,r_0 we have $(f,g,h) = T(a_{i,j}, b_{k,l}, c_{m,n}) = 1$.

Proof. The proof is by direct calculation with formula (3.15) and explicit series for log and exp. We have to calculate the following expression:

$$\exp \operatorname{Res} \log(1 - a_{i,j}u^jt^i) \cdot d\log(1 - b_{k,l}u^lt^k) \wedge d\log(1 - c_{m,n}u^nt^m). \tag{3.21}$$

We obtain

$$\log(1 - a_{i,j}u^jt^i) = -\sum_{p\geq 1} \frac{a_{i,j}^p}{p} u^{jp}t^{ip}. \tag{3.22}$$

Besides

$$d\log(1 - b_{k,l}u^lt^k) = -\sum_{q\geq 1} (l b_{k,l}^q u^{lq-1}t^{kq}du + k b_{k,l}^q u^{lq}t^{kq-1}dt).$$

Now we have

$$d\log(1-b_{k,l}u^lt^k) \wedge d\log(1-c_{m,n}u^nt^m) = \sum_{\substack{q\geq 1\\ r\geq 1}} \begin{vmatrix} l & n \\ k & m \end{vmatrix} \cdot b_{k,l}^q \cdot c_{n,m}^r \cdot u^{lq+nr-1}t^{kq+mr-1} du \wedge dt. \tag{3.23}$$

Using the anti-symmetric property of $(\cdot,\cdot,\cdot)$ and formula (3.23) we obtain from expression (3.21) that $(f,g,h) = 1$ if $p_0 = 0$, or $q_0 = 0$, or $r_0 = 0$. Therefore further we assume that $p_0 \cdot q_0 \cdot r_0 \neq 0$.

To calculate Res from formula (3.21) we can use formulas (3.22) and (3.23) and need to find all integers $p \geq 1$, $q \geq 1$, $r \geq 1$ which solve the following system of equations:

$$\left\{ \begin{array}{rcl} jp + lq + nr &=& 0 \\ ip + kq + mr &=& 0. \end{array} \right.$$

It is clear that all the solutions of this system of equations are given by the following formula:

$$\{p,q,r\} = \{s\frac{p_0}{(p_0,q_0,r_0)},\ s\frac{q_0}{(p_0,q_0,r_0)},\ s\frac{r_0}{(p_0,q_0,r_0)}\},$$

where we take any integer $s \geq 1$, and the following condition has to be satisfied: $p_0, q_0, r_0 > 0$ or $p_0, q_0, r_0 < 0$.

Now since

$$-\sum_{s\geq 1} \frac{a_{i,j}^{\frac{sp_0}{(p_0,q_0,r_0)}}}{\frac{sp_0}{(p_0,q_0,r_0)}} \cdot p_0 \cdot b_{k,l}^{\frac{sq_0}{(p_0,q_0,r_0)}} \cdot c_{m,n}^{\frac{sr_0}{(p_0,q_0,r_0)}} =$$

$$= \log\left(\left(1 - a_{i,j}^{\frac{p_0}{(p_0,q_0,r_0)}} \cdot b_{k,l}^{\frac{q_0}{(p_0,q_0,r_0)}} \cdot c_{m,n}^{\frac{r_0}{(p_0,q_0,r_0)}}\right)^{(p_0,q_0,r_0)}\right),$$

we obtain formula (3.20). □

Remark 3.3. We note that the same expression as expression (3.20) appeared recently (after appearence of the first e-print version of our paper in arXiv) in [HL].

Lemma 3.12. *Let* $\mathbf{Q} \subset R$. *Let* $f, g \in L^2\mathbb{G}_m(R)$ *such that* $f = 1 - a_{i,j}u^jt^i$, $g = 1 - b_{k,l}u^lt^k$, *where* $i,j,k,l \in \mathbf{Z}$ *and* $a_{i,j}, b_{k,l} \in R$.

(1) *If at least one of the following conditions is satisfied:*

$$\begin{vmatrix} j & l \\ i & k \end{vmatrix} \neq 0, \qquad jl + ik \geq 0, \qquad jl = 0, \tag{3.24}$$

then $(f,g,t) = S(a_{i,j}, b_{k,l}) = 1$.

(2) *If none of conditions in* (3.24) *is satisfied, then*

$$(f,g,t) = S(a_{i,j}, b_{k,l}) = \left(1 - a_{i,j}^{\left|\frac{l}{(j,l)}\right|} b_{k,l}^{\left|\frac{j}{(j,l)}\right|}\right)^{-\frac{l\cdot|(j,l)|}{|l|}},$$

where (j,l) *is the greatest common divisor of two integers with any sign.*

Lemma 3.13. *Let* $\mathbf{Q} \subset R$. *Let* $f, g \in L^2\mathbb{G}_m(R)$ *such that* $f = 1 - a_{i,j}u^jt^i$, $g = 1 - b_{k,l}u^lt^k$, *where* $i,j,k,l \in \mathbf{Z}$ *and* $a_{i,j}, b_{k,l} \in R$.

(1) *If at least one of the following conditions is satisfied:*

$$\begin{vmatrix} j & l \\ i & k \end{vmatrix} \neq 0, \qquad jl + ik \geq 0, \qquad ik = 0, \tag{3.25}$$

then $(f,g,u) = Q(a_{i,j}, b_{k,l}) = 1$.

(2) *If none of conditions* (3.25) *is satisfied, then*

$$(f,g,u) = Q(a_{i,j}, b_{k,l}) = \left(1 - a_{i,j}^{\left|\frac{k}{(i,k)}\right|} b_{k,l}^{\left|\frac{i}{(i,k)}\right|}\right)^{\frac{k\cdot|(i,k)|}{|k|}},$$

where (i,k) *is the greatest common divisor of two integers with any sign.*

Proof of Lemmas 3.12 and 3.13 follows from the following formulas (see (3.15)):

$$(f,g,t) = \exp \operatorname{Res} \log f d\log g \wedge \frac{dt}{t} = \exp \operatorname{Res} \log f \frac{\partial \log g}{\partial u} du \wedge \frac{dt}{t},$$

$$(f,g,u) = \exp \operatorname{Res} \log f d \log g \wedge \frac{du}{u} = (\exp \operatorname{Res} \log f \frac{\partial \log g}{\partial t} \frac{du}{u} \wedge dt)^{-1}$$

and explicit calculations with formal log-series similarly to the proof of Lemma 3.11.

If $\mathbf{Q} \subset R$, $f = 1 - a_{i,j}u^j t^i$ and g, h are any from the set $\{u, t\}$, then it follows from formula (3.15) that $(f,g,h) = 1$.

Recall that we introduced in § 3.1 two subgroups of the group $L^2\mathbb{G}_m(R) = R((u))((t))^*$: $\mathbb{P}(R) = 1 + \mathfrak{p}(R)$ and $\mathbb{M}(R) = 1 + \mathfrak{m}(R)$. We have considered in § 3.2 the topology on $\mathbb{M}(R)$ such that the base of neighbourhoods of 1 consists of subgroups $U_{i,j}(R) = 1 + \mathfrak{u}_{i,j}(R)$, where $i, j \in \mathbf{N}$ (see formulas (3.8) and (3.11)). We have considered also $\mathbb{P}(R)$ as a topological group, where the base of neighbourhoods of 1 consists of subsets $U_{\{n_i\}}(R) = 1 + \mathfrak{u}_{\{n_i\}}(R)$, where $i \in \mathbf{N}$, $n_i \in \mathbf{N}$ (see formulas(3.9) and (3.14)). We will speak about infinite products in $\mathbb{M}(R)$ or in $\mathbb{P}(R)$, which will converge in these topologies.

Note that every element $f \in R((u))((t))$ can canonically be written as

$$f = \sum_{\substack{(i,j) \in \mathbf{Z}^2 \\ (i,j) \geq (i_f, j_f)}} a_{i,j} u^j t^i, \tag{3.26}$$

where $a_{i,j} \in R$, $a_{i_f,j_f} \neq 0$, and we consider on $\mathbf{Z}^2$ the following lexicographical order: $(i_1, j_1) > (i_2, j_2)$ iff either $i_1 > i_2$ or $i_1 = i_2, j_1 > j_2$.

Proposition 3.14. (1) *Let R be any ring. Then every $f \in \mathbb{M}(R)$ can be uniquely decomposed into the following infinite product*

$$f = \prod_{\substack{(i,j) \in \mathbf{Z}^2 \\ (i,j) \geq (0,1)}} (1 - b_{i,j} u^j t^i), \ \textit{where} \quad b_{i,j} \in R. \tag{3.27}$$

(2) *Let $R \in \mathcal{B}$. Then every $g \in \mathbb{P}(R)$ can be uniquely decomposed into the following infinite product*

$$g = \prod_{\substack{(i,j) \leq (0,-1) \\ i > n_g, j > m_g}} (1 - c_{i,j} u^j t^i), \ \textit{where} \quad c_{i,j} \in \mathcal{N}R, \quad n_g < 0. \tag{3.28}$$

Proof. Uniqueness of both decompositions can be easily verified.

We explain now how to obtain decomposition (3.27). Let

$$f_0 = f = 1 + \sum_{(i,j) \geq (0,1)} a_{i,j} u^j t^i.$$

We can define $h_0 = 1 + \sum\limits_{j \geq 1} a_{0,j} u^j = \prod\limits_{j \geq 1} (1 - b_{0,j} u^j)$. Then it is clear that

$$f_1 = f_0 h_0^{-1} \in 1 + tR((u))[[t]].$$

Let $f_1 = 1 + \sum\limits_{(i,j) \geq (1, j_{f_1})} d_{i,j} u^j t^i$. We define $h_1 = \prod\limits_{j \geq j_{f_1}} (1 + d_{1,j} u^j t)$. Then we have $f_2 = f_1 h_1^{-1} \in 1 + t^2 R((u))[[t]]$. Repeating this procedure we will obtain that $f_n \to 1$ when $n \to \infty$. Thus we have decomposition (3.27).

We explain now how to obtain decomposition (3.28). Let

$$g_0 = g = 1 + \sum_{\substack{(i,j) \le (0,-1) \\ i > k_g, j \ge l_g}} a_{i,j} u^i t^j,$$

where $a_{i,j} \in \mathcal{N}R$, $k_g < 0$. We can define $e_0 = 1 + \sum_{j \ge l_g}^{j \le -1} a_{0,j} u^j = \prod_{j \le -1} (1 - c_{0,j} u^j)$, where the last product contains only a finite number of multipliers. Then we have $g_1 = g_0 e_0^{-1} \in 1 + t^{-1} R((u))[t^{-1}]$. Let $g_1 = 1 + \sum_{i > k_{g_1}, j \ge l_{g_1}} d_{i,j} u^j t^i$, where $d_{i,j} \in \mathcal{N}R$. We define $e_1 = \prod_{j \ge l_{g_1}} (1 + d_{-1,j} u^j t^{-1})$. The element e_1 is a well-defined element from $R((u))((t))^*$, since $\mathcal{N}R$ is a nilpotent ideal. We have that $g_2 = g_1 e_1^{-1} \in 1 + t^{-2} R((u))[t^{-1}]$. Repeating this procedure we will obtain that $g_{-k_{g_1}} \in 1 + t^{-k_{g_1}} \cdot (\mathcal{N}R)^2((u))[t^{-1}]$. Since $\mathcal{N}R$ is a nilpotent ideal, we will obtain decomposition (3.28) after repeating some times all this procedure. □

Now using decompositions (3.6), (3.27)-(3.28) and Lemmas 3.11-3.13 we obtain the following definition.

Definition 3.5 (Explicit formula)**.** Let $R \in \mathcal{B}$. Let $f, g, h \in L^2\mathbb{G}_m(R)$. Let

$$f = f_0 \cdot u^{\nu_2(f)} t^{\nu_1(f)} \cdot \prod_{(i,j) \in \mathbf{Z}^2 \setminus (0,0)} (1 - a_{i,j} u^j t^i), \text{ where } \quad f_0 \in R^*, \quad a_{i,j} \in R, \tag{3.29}$$

$$g = g_0 \cdot u^{\nu_2(g)} t^{\nu_1(g)} \cdot \prod_{(k,l) \in \mathbf{Z}^2 \setminus (0,0)} (1 - b_{k,l} u^l t^k), \text{ where } \quad g_0 \in R^*, \quad b_{k,l} \in R, \tag{3.30}$$

$$h = h_0 \cdot u^{\nu_2(h)} t^{\nu_1(h)} \cdot \prod_{(m,n) \in \mathbf{Z}^2 \setminus (0,0)} (1 - c_{m,n} u^n t^m), \text{ where } \quad h_0 \in R^*, \quad c_{m,n} \in R, \tag{3.31}$$

then *the two-dimensional Contou-Carrère symbol* from $L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R)$ to R^* is given as

$$\begin{aligned}
(f,g,h) = (-1)^A f_0^{\nu(g,h)} g_0^{\nu(h,f)} h_0^{\nu(f,g)} \cdot \prod_{\substack{(i,j) \in \mathbf{Z}^2 \setminus (0,0) \\ (k,l) \in \mathbf{Z}^2 \setminus (0,0) \\ (m,n) \in \mathbf{Z}^2 \setminus (0,0)}} T(a_{i,j}, b_{k,l}, c_{m,n}) \times \\
\times \prod_{\substack{(i,j) \in \mathbf{Z}^2 \setminus (0,0) \\ (k,l) \in \mathbf{Z}^2 \setminus (0,0)}} S(a_{i,j}, b_{k,l})^{\nu_1(h)} \cdot \prod_{\substack{(i,j) \in \mathbf{Z}^2 \setminus (0,0) \\ (k,l) \in \mathbf{Z}^2 \setminus (0,0)}} Q(a_{i,j}, b_{k,l})^{\nu_2(h)} \times \\
\times \prod_{\substack{(i,j) \in \mathbf{Z}^2 \setminus (0,0) \\ (m,n) \in \mathbf{Z}^2 \setminus (0,0)}} S(c_{m,n}, a_{i,j})^{\nu_1(g)} \cdot \prod_{\substack{(i,j) \in \mathbf{Z}^2 \setminus (0,0) \\ (m,n) \in \mathbf{Z}^2 \setminus (0,0)}} Q(c_{m,n}, a_{i,j})^{\nu_2(g)} \times \\
\times \prod_{\substack{(k,l) \in \mathbf{Z}^2 \setminus (0,0) \\ (m,n) \in \mathbf{Z}^2 \setminus (0,0)}} S(b_{k,l}, c_{m,n})^{\nu_1(f)} \cdot \prod_{\substack{(k,l) \in \mathbf{Z}^2 \setminus (0,0) \\ (m,n) \in \mathbf{Z}^2 \setminus (0,0)}} Q(b_{k,l}, c_{m,n})^{\nu_2(f)} ,
\end{aligned} \tag{3.32}$$

where $A \in \mathbb{Z}(R)$ is given by formula (3.18).

To verify that this definition is well-defined, we have to check that the infinite products in (3.32) contain only a finite number non-equal to 1 terms. We will explain it, first, when $\mathbf{Q} \subset R$. It follows from the following general property of continuity of expression (3.15). Let $d, \{d_i\}_{i\in\mathbf{Z}}$ and $e, \{e_i\}_{i\in\mathbf{Z}}$ be the collections of integers. Then there is some open subset $U_{\{n_i\}}(R) = 1 + \mathfrak{u}_{\{n_i\}} \subset \mathbb{P}(R)$ and some open subgroup $U_{i,j}(R) = 1 + \mathfrak{u}_{i,j}(R) \subset \mathbb{M}(R)$ (which depend on collections $d, \{d_i\}$ and $e, \{e_i\}$) such that for any $f_1 \in U_{\{n_i\}}(R)$, for any $f_2 \in U_{i,j}(R)$, for any $g = \sum\limits_{\substack{i>d \\ (i,j)>(i,d_i)}} g_{i,j}u^j t^i$, for any $h = \sum\limits_{\substack{i>e \\ (i,j)>(i,e_i)}} h_{i,j}u^j t^i$ we have

$$\operatorname{Res}\left(\log f_i \cdot \frac{dg}{g} \wedge \frac{dh}{h}\right) = 0, \quad i = 1, 2. \tag{3.33}$$

(This formula is equivalent to the fact that the resulting series does not contain the non-zero coefficient at $u^{-1}t^{-1}du \wedge dt$, and therefore it is easy to construct the corresponding open subset $U_{\{n_i\}}(R) \subset \mathbb{P}(R)$ and the open subgroup $U_{i,j}(R) \subset \mathbb{M}(R)$ such that formula (3.33) is satisfied. To construct f_2 one can use also formula (3.12). To construct f_1 it is important that $R \in \mathcal{B}$, see Lemma 3.8.) Now using this and Lemmas 3.11-3.13 we obtain that only a finite number of $a_{i,j}$ give non-trivial contributions to products which contain $T(\cdot,\cdot,\cdot)$, $S(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ in formula (3.32). Now using the obvious anti-symmetric properties of $T(\cdot,\cdot,\cdot)$, $S(\cdot,\cdot)$ and $Q(\cdot,\cdot)$, we obtain the same for $b_{k,l}$, and then for $c_{m,n}$.

Now we verify that Definition 3.5 is well-defined in the general case. We reduce this case to the previous one. We will find rings $S_1, S_2 \in \mathcal{B}$ and elements $\tilde{f}, \tilde{g}, \tilde{h} \in S_1((u))((t))^*$ such that there is a map of rings $S_1 \to R$, $S_1 \subset S_2$, $\mathbf{Q} \subset S_2$, and the elements $\tilde{f}, \tilde{g}, \tilde{h}$ go to the elements f, g, h under the natural map $S_1((u))((t))^* \to R((u))((t))^*$. For elements $\tilde{f}, \tilde{g}, \tilde{h}$ we have decompositions (3.6), (3.27)-(3.28). Since these decompositions are uniquely defined, they are functorial and go to decompositions (3.29)-(3.31) for f, g, h under the natural map $S_1((u))((t))^* \to R((u))((t))^*$. For $\tilde{f}, \tilde{g}, \tilde{h}$ the corresponding analogous formula (3.32) contains only a finite number of multipliers by the previous case, because $S_1((u))((t))^* \subset S_2((u))((t))^*$ and $\mathbf{Q} \subset S_2$. Therefore formula (3.32) is well-defined for f, g, h. Now the existence of such S_1, S_2 and $\tilde{f}, \tilde{g}, \tilde{h}$ follows from the following lemma.

Lemma 3.15. *Let $R \in \mathcal{B}$. Let $f_1, \ldots, f_n \in R((u))((t))^*$. Then there are rings $S_1, S_2 \in \mathcal{B}$ and elements $\tilde{f}_1, \ldots, \tilde{f}_n \in S_1((u))((t))^*$ such that there is a map of rings $S_1 \to R$, $S_1 \subset S_2$, $\mathbf{Q} \subset S_2$, and the elements $\tilde{f}_1, \ldots, \tilde{f}_n$ go to the elements $f_1, \ldots, f_n$ under the natural map $S_1((u))((t))^* \to R((u))((t))^*$.*

Proof. Without loss of generality we can assume that $\nu_1(f_l) \in \mathbf{Z} \subset \mathbb{Z}(R)$, $1 \le l \le n$ and $\nu_2(f_l) \in \mathbf{Z} \subset \mathbb{Z}(R)$. (Otherwise we can find a ring decomposition $R = R_1 \oplus \ldots \oplus R_m$ such that the previous condition is satisfied for every R_k and work then separately with every R_k.) Let $f_l = \sum\limits_{(i,j)\in\mathbf{Z}^2} a_{l,i,j}u^j t^i$, where $1 \le l \le n$ and $a_{l,i,j} \in R$. According to formula (3.6), for any $1 \le l \le n$ we have that $a_{l,i,j} \in \mathcal{N}R$ when $(i,j) < (\nu_1(f_l), \nu_2(f_l))$, and $a_{l,\nu_1(f_l),\nu_2(f_l)} \in R^*$. Let $(\mathcal{N}R)^m = 0$. We define the ring

$$S_1 = P^{-1}\mathbf{Z}\left[\{A_{l,i,j}\}\right]/I^m,$$

where the set of variables $\{A_{l,i,j}\}$ depends on the indices $1 \le l \le n$ and $(i,j) \in \mathbf{Z}^2$ such that $a_{l,i,j} \neq 0$, P is a set which is multiplicatively generated by elements $A_{l,\nu_1(f_l),\nu_2(f_l)}$, where $1 \le l \le n$, and I is an ideal generated by all $A_{l,i,j}$ such that $(i,j) < (\nu_1(f_l), \nu_2(f_l))$. We define a map $S_1 \to R$ which is given on variables as $A_{l,i,j} \mapsto a_{l,i,j}$. Now for any $1 \le l \le n$ we define $\tilde{f}_l = \sum\limits_{(i,j)\in \mathbf{Z}^2} A_{l,i,j} u^j t^i$. It is clear that $\tilde{f}_l \in S_1((u))((t))^*$. We define the ring

$$S_2 = P^{-1}\mathbf{Q}\,[\,\{A_{l,i,j}\}\,]/I^m,$$

where $\{A_{l,i,j}\}$, P and I are as above. The proof is finished. □

Now we have the following proposition.

Proposition 3.16. *Let $R \in \mathcal{B}$. The two-dimensional Contou-Carrère symbol $(\cdot,\cdot,\cdot)$, constructed by the explicit formula from Definition 3.5, is an anti-symmetric, functorial with respect to R map. If $\mathbf{Q} \subset R$, then values of $(\cdot,\cdot,\cdot)$ calculated by Definition 3.4 and Definition 3.5 coincide.*

Proof. The anti-symmetric property of $(\cdot,\cdot,\cdot)$ easily follows from formula (3.32), because $T(\cdot,\cdot,\cdot)$, $S(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ satisfy the anti-symmetric property. The functoriality follows from the uniqueness properties of decompositions (3.6) and (3.27)-(3.28). If $\mathbf{Q} \subset R$, then two definitions give the same values because of Lemmas (3.11)-(3.13), the anti-symmetric property of $(\cdot,\cdot,\cdot)$, the tri-multiplicativity of $(\cdot,\cdot,\cdot)$ given by Definition 3.4 and the continuity of expression (3.15) (see reasonings before formula (3.33)). □

Remark 3.4. Starting from expression

$$\exp \operatorname{Res}\, \log(f_1 \cdot \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}}),$$

where $f_1, \ldots, f_{n+1} \in R((t_n))\ldots((t_1))^*$ and where $\log f_1$ is well-defined, one could similarly define and develop the theory of the n-dimensional Contou-Carrère symbol. We restricted ourself to the case $n = 2$, since further we will prove for the two-dimensional Contou-Carrère symbol the reciprocity laws on algebraic surfaces.

4. Properties of the two-dimensional Contou-Carrère symbol

4.1. **Case $\mathbf{Q} \subset R$.** We assume in this subsection that R is a ring such that $\mathbf{Q} \subset R$. We will need the following lemma.

Lemma 4.1. *Let $f, g \in L^2\mathbb{G}_m(R)$. Then*

$$\nu(f,g) = \operatorname{Res}(\frac{df}{f} \wedge \frac{dg}{g}). \tag{4.1}$$

Proof. It follows by direct calculations using decomposition (3.6) and bimultiplicativity of both parts of (4.1). We note that if $(\nu_1,\nu_2)(f) = (0,0)$ and $f_0 = 1$, then $\operatorname{Res}(\frac{df}{f} \wedge \frac{dg}{g}) = \operatorname{Res}(d\log f \wedge \frac{dg}{g}) = \operatorname{Res} d(\log f \frac{dg}{g}) = 0$. □

We have the following proposition.

Proposition 4.2. *The two-dimensional Contou-Carrère symbol satisfies the Steinberg properties, i.e. $(f, 1-f, g) = 1$ for any $f, 1-f, g \in L^2\mathbb{G}_m(R)$ (and other analogous equalities are satisfied from the anti-symmetric property of $(\cdot,\cdot,\cdot)$).*

Proof. If $R = R_1 \oplus R_2$, where R_i is a ring, $i = 1, 2$, then we can prove the Steinberg property separately for elements restricted to R_1 and to R_2. Therefore without loss of generality we can assume that $\nu_1(f) \in \mathbf{Z} \subset \mathbb{Z}(R)$ and $\nu_2(f) \in \mathbf{Z} \subset \mathbb{Z}(R)$ (and similar conditions for $1-f$). We consider several cases.

- Let $(\nu_1, \nu_2)(f) > (0,0)$. Then for $f' = 1 - f$ we have $(\nu_1, \nu_2)(f') = (0,0)$ and $1 - f'_0 \in \mathcal{N}R$ (see decomposition (3.6) for the definition of f'_0). Therefore using anti-symmetric property it is enough to prove $(f', f, g) = 1$. From Lemma 4.1 it is easy to see that we can correctly apply formula (3.15) to calculate (f', f, g). Now this case follows from

$$\log(1-f)\frac{df}{f} \wedge \frac{dg}{g} = -(f + \frac{f^2}{2} + \ldots)\frac{df}{f} \wedge \frac{dg}{g} = -(1 + \frac{f}{2} + \ldots)df \wedge \frac{dg}{g} =$$
$$= -d(f + \frac{f^2}{4} + \ldots) \wedge \frac{dg}{g} = d(-(f + \frac{f^2}{4} + \ldots)\frac{dg}{g}),$$

since $\operatorname{Res} d(\cdots) = 0$.

- Let $(\nu_1, \nu_2)(f) = (0,0)$, then $(\nu_1, \nu_2)(1-f) \geq (0,0)$. If $(\nu_1, \nu_2)(1-f) > (0,0)$, then interchanging f and $1-f$ we reduce this case to the previous one. So, we have to consider the case when

$$(\nu_1, \nu_2)(f) = (\nu_1, \nu_2)(1-f) = (0,0). \tag{4.2}$$

Let $f = f_0 h$, where $f_0 \in R^*$, $h = f_{-1} f_1$ (see decomposition (3.6)). Then

$$(f_0 h, 1 - f_0 h, g) = (f_0, 1 - f_0 h, g)(h, 1 - f_0 h, g).$$

We have that $(f_0, 1 - f_0 h, g) = f_0^{\nu(1-f, g)} = 1$, since $(\nu_1, \nu_2)(1-f) = (0,0)$. From (4.2) we have that $1 - f_0 \in R^*$. Let $h = 1 - e$. Then

$$(h, 1 - f_0 h, g) = (1 - e, 1 - f_0 + f_0 e, g) =$$
$$= \exp \operatorname{Res} (-e - \frac{e^2}{2} - \ldots) \cdot \frac{f_0}{1 - f_0} \cdot \frac{de}{1 + \frac{f_0}{1 - f_0} e} \wedge \frac{dg}{g} =$$
$$= \exp \operatorname{Res} \frac{f_0}{1 - f_0} \cdot (-e - \frac{e^2}{2} - \ldots) \cdot (1 - \frac{f_0}{1 - f_0} e + (\frac{f_0}{1 - f_0} e)^2 - \ldots) \cdot de \wedge \frac{dg}{g} =$$
$$= \exp \operatorname{Res} \Phi(e) de \wedge \frac{dg}{g} = \exp \operatorname{Res} d\Psi(e) \wedge \frac{dg}{g} = \exp \operatorname{Res} d(\Psi(e) \wedge \frac{dg}{g}) = 1,$$

where Φ and Ψ are some formal series from $\mathbf{Q}[[x]]$.

- Let $(\nu_1, \nu_2)(f) < (0,0)$. Then $(\nu_1, \nu_2)(f^{-1}) > 0$. From the tri-multiplicativity we have

$$(f, 1 - f, g) = (f, f, g)(f, -1 + f^{-1}, g).$$

We claim that $(f, f, g) = (-1)^{\nu(f,g)}$ for any $f, g \in L^2\mathbb{G}_m(R)$. Indeed, from the anti-symmetric and tri-multiplicative property of $(\cdot, \cdot, \cdot)$ it follows that one has to verify this equality only when f and g are multipliers from decomposition (3.6). This can be easily done.

We have also

$$(f, -1 + f^{-1}, g) = (f, -1, g)(f, 1 - f^{-1}, g) = (-1)^{\nu(f,g)}(f, 1 - f^{-1}, g),$$
$$(f, 1 - f^{-1}, g) = (f^{-1}, 1 - f^{-1}, g)^{-1} = 1,$$

where the last equality follows from the first case applied to f^{-1}.

Thus in this case we obtained

$$(f, 1-f, g) = (-1)^{\nu(f,g)} \cdot (-1)^{\nu(f,g)} = 1.$$

□

Remark 4.1. By means of tri-multiplicativity, anti-symmetric property and property $(f, f, g) = (-1, f, g)$ for $f, g \in L^2\mathbb{G}_m(R)$ (which follow from Steinberg relations) one can reduce formula (3.17) to formula (3.16).

4.2. **Case $R \in \mathcal{B}$.** We assume in this subsection that R is any ring from $\mathcal{B}$.

Proposition 4.3. *Let k be a field.*

1. *Let $R = k$. Then the two-dimensional Contou-Carrère symbol coincides with the two-dimensional tame symbol.*
2. *Let $R = k[\epsilon]/\epsilon^4$. Then for any $f, g, h \in k((u))((t))$*

$$(1+\epsilon f, 1+\epsilon g, 1+\epsilon h) = 1 + \epsilon^3 \mathrm{Res}\, f dg \wedge dh. \tag{4.3}$$

Proof. Assertion 1 follows from the explicit formula for the two-dimensional tame symbol, see [Pa1], [OsZh, § 4A].

Assertion 2 follows from the following calculaton when $\mathbf{Q} \subset R$ (see formula (3.15))

$$\begin{aligned}(1+\epsilon f, 1+\epsilon g, 1+\epsilon h) &= \exp \mathrm{Res}\, \log(1+\epsilon f) d \log(1+\epsilon g) \wedge d\log(1+\epsilon h) = \\ &= \exp \mathrm{Res}\, (\epsilon f - \frac{\epsilon^2 f^2}{2} + \ldots) d(\epsilon g - \frac{\epsilon^2 g^2}{2} + \ldots) \wedge d(\epsilon h - \frac{\epsilon^2 h^2}{2} + \ldots) = \\ &= \exp(\epsilon^3 \mathrm{Res} f dg \wedge dh) = 1 + \epsilon^3 \mathrm{Res}\, f dg \wedge dh \mod \epsilon^4.\end{aligned}$$

If $\mathbf{Q} \not\subseteq R$, then we use the continuity of left and right hand sides of (4.3). Therefore it is enough to verify this equality for elements f, g, h of type $1 + \epsilon a_{i,j} u^j t^i$. To achieve this goal we use Lemma 3.11. □

Proposition 4.4. *The two-dimensional Contou-Carrère symbol constructed by the explicit formula from Definition 3.5 is a tri-multiplicative map from $L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R)$ to R^* and it satisfies the Steinberg relations.*

Proof. First we explain the tri-multiplicativity. Let $f_1, f_2, g, h \in L^2\mathbb{G}_m(R)$. We want to prove, for example, that $(f_1 f_2, g, h) = (f_1, g, h)(f_2, g, h)$. Using Lemma 3.15 we will find the rings $S_1 \subset S_2$ such that $\mathbf{Q} \subset S_2$ and elements $\tilde{f_1}, \tilde{f_2}, \tilde{g}, \tilde{h} \in S_1((u))((t))^*$ which are mapped to the elements f_1, f_2, g, h. Then, by Definition 3.4 and Proposition 3.16, $(\tilde{f_1}\tilde{f_2}, \tilde{g}, \tilde{h}) = (\tilde{f_1}, \tilde{g}, \tilde{h})(\tilde{f_2}, \tilde{g}, \tilde{h})$, since $\mathbf{Q} \subset S_2$. Hence we obtain the tri-multiplicativity.

Now we explain the Steinberg property. Let $f, 1-f, h \in L^2\mathbb{G}_m(R)$. We want to prove that $(f, 1-f, h) = 1$. Without loss of generality we can assume that $\nu_1(f) \in \mathbf{Z} \subset \mathbb{Z}(R)$ and $\nu_2(f) \in \mathbf{Z} \subset \mathbb{Z}(R)$ (and similar conditions for $1-f$). We consider several cases.

If $(\nu_1, \nu_2)(f) > (0, 0)$, then we consider by Lemma 3.15 the rings $S_1 \subset S_2$ and an element $\tilde{f} \in S_1((u))((t))^*$ which is mapped to the element f. Since $(\nu_1, \nu_2)(\tilde{f}) > (0, 0)$, we have that $1 - \tilde{f} \in S_1((u))((t))^*$. Therefore we can apply Proposition 4.2 to the ring $S_2((u))((t))^*$ and elements $\tilde{f}, 1-\tilde{f}$. Hence this case follows.

If $(\nu_1, \nu_2)(f) = (0, 0)$, then $(\nu_1, \nu_2)(1-f) \geq (0, 0)$. If $(\nu_1, \nu_2)(1-f) > (0, 0)$, then interchanging f and $1-f$ we reduce this case to the previous one. So, we have to consider the case when $(\nu_1,\nu_2)(f) = (\nu_1,\nu_2)(1-f) = (0,0)$. By Lemma 3.15 we have the rings $S_1 \subset S_2$ such that $\mathbf{Q} \subset S_2$, and $\tilde{f} \in S_1((u))((t))^*$. Let T be a subset of S_1 which is mutiplicatively generated by $1 - A_{0,0}$ (see the proof of Lemma 3.15 for the definition of the element $A_{0,0}$, or more exactly $A_{1,0,0}$.) We define the rings $S_1' = T^{-1}S_1$, $S_2' = T^{-1}S_2$. Then S_1' is mapped to R, $S_1' \subset S_2'$ and $\mathbf{Q} \subset S_2'$. Moreover, $\tilde{f}, 1-\tilde{f} \in S_1'((u))((t))^*$. Therefore we can apply Proposition 4.2 to the ring $S_2'((u))((t))^*$. Hence this case follows.

If $(\nu_1,\nu_2)(f) < (0,0)$, then this case can be reduced to the first case by the same method as in Proposition 4.2. □

Remark 4.2. The tri-multiplicativity of two-dimensional Contou-Carrère symbol will follow also from Corollary 5.14 (of Theorem 5.9) later, because we know the tri-multiplicativity for the commutator (the map C_3) in a categorical central extension. The Steinberg relations will follow also from Corollary 7.5 (of Theorem 7.2) later, where these relations will be obtained from the product structure in algebraic K-theory. In this subsection we used elementary methods to prove these properties.

5. Categorical central extensions

5.1. **Central extensions.** In [OsZh], we defined the notion of a central extension of a group by a Picard groupoid. We need to extend such formalism to any topos. This is straightforward if we adapt Grothendieck's point of view of functor of points. Let us briefly indicate how to do this.

Let $\mathcal{T}$ be any topos. Let $\mathcal{P}$ be a sheaf (a.k.a a stack) of Picard groupoids in $\mathcal{T}$, then it makes sense to talk about a $\mathcal{P}$-torsor (see [Br1]). Namely, a $\mathcal{P}$-torsor $\mathcal{L}$ is a sheaf (a.k.a a stack) of groupoids with an action given by some (Cartesian) bifunctor $\mathcal{P} \times_{\mathcal{T}} \mathcal{L} \to \mathcal{L}$ which satisfies certain axioms. In particular, for any $U \in \mathcal{T}$ it gives rise to a bifunctor $\mathcal{P}(U) \times \mathcal{L}(U) \to \mathcal{L}(U)$. Moreover, for any $U \in \mathcal{T}$, $\mathcal{L}(U)$ is either empty or a $\mathcal{P}(U)$-torsor (see [OsZh, § 2C]), and for any U, there is a covering $V \to U$ such that $\mathcal{L}(V)$ is a $\mathcal{P}(V)$-torsor. All $\mathcal{P}$-torsors form naturally a Picard 2-stack in 2-groupoids in $\mathcal{T}$, see [Br2, ch. 8]. If A is a sheaf of abelian groups in $\mathcal{T}$, and BA is the Picard groupoid of A-torsors, then an BA-torsor has another name as an A-gerbe. (More precisely, there is a canonical equivalence between the 2-stack of BA-torsors and the 2-stack of abelian A-gerbes in $\mathcal{T}$, see [Br2, Prop. 2.14].)

Let G be a group in $\mathcal{T}$ and $\mathcal{P}$ be a sheaf of Picard groupoids in $\mathcal{T}$. Then a central extension of G by $\mathcal{P}$ is a rule to assign to every $U \in \mathcal{T}$ and to every $g \in G(U)$ a $\mathcal{P}_U$-torsor $\mathcal{L}_g$ over $\mathcal{T}/U$ such that: 1) for any $V \to U$ the $\mathcal{P}_V$-torsor $\mathcal{L}_g|_V$ corresponds to $g|_V \in G(V)$, 2) for any $U \in \mathcal{T}$ the $\mathcal{P}_U$-torsors $\mathcal{L}_g$ satisfy the properties as in [OsZh, § 2E] which are compatible with restrictions. If A is a sheaf of abelian groups in $\mathcal{T}$, and BA is the Picard groupoid of A-torsors, then a central extension of G by BA was described in [Del, § 5.5].

Now let $\mathcal{L}$ be a central extension of G by $\mathcal{P}$, and assume that G is abelian. Recall that in [OsZh], we constructed certain maps C_2 and C_3 (generalized commutators), which have obvious generalization to the sheaf theoretical contents. It means that we have a bimultiplicative and anti-symmetric morphism

$$C_2^{\mathcal{L}} : G \times G \to \mathcal{P} \tag{5.1}$$

and a tri-multiplicative and anti-symmetric morphism

$$C_3^{\mathcal{L}} : G \times G \times G \to \pi_1(\mathcal{P}). \tag{5.2}$$

5.2. **The central extension of** $\mathrm{GL}_{\infty,\infty}$**.** Now we specify the central extensions that we are considering in the paper.

(i) Let $\mathcal{P}ic^{\mathbb{Z}}$ be the Picard groupoid of graded lines. It means that for any commutative ring R, $\mathcal{P}ic^{\mathbb{Z}}(R)$ is the category of graded line bundles over $\operatorname{Spec} R$, together with the natural Picard structure. It forms a sheaf (with respect to the flat topology) of Picard groupoids over **Aff**. Proposition 3.6 of [OsZh] remains true in this sheaf version.

(ii) Let $\mathbb{V}$ be a 2-Tate vector space over a field k. In [OsZh] we studied the central extension of $\mathrm{GL}(\mathbb{V})$ by $\mathcal{P}ic^{\mathbb{Z}}$. In the current setting, we need a sheaf version of this construction.

The first goal to endow the group $\mathrm{GL}(\mathbb{V})$ a structure as a sheaf of groups over **Aff**. This would be clear if we can make sense of family of 2-Tate vector spaces (or 2-Tate R-modules). I.e., for any k-algebra R, some certain "complete" tensor product $\mathbb{V}\hat{\otimes}_k R$ should be a 2-Tate R-module and $\mathrm{GL}(\mathbb{V})(R)$ should be the group of automorphisms of this 2-Tate R-module. However, to keep the size of the paper, we will not try to develop a full theory of 2-Tate R-modules here. Instead, let us specialize to the case $\mathbb{V} = k((u))((t))$. Then $\mathbb{V}\hat{\otimes}_k R$ should be just $R((u))((t))$, and we follow the idea of [FZ] to define a group that acts on $R((u))((t))$. More precisely, let

$$\mathrm{GL}_{\infty,\infty}(R) = \mathfrak{gl}_\infty(\mathfrak{gl}_\infty(R))^*.$$

Here for any ring A (not necessarily commutative nor unital), $\mathfrak{gl}_\infty(A)$ is the algebra of continuous endomorphisms of $A((t))$ as a *right* A-module, where we consider A as a discrete topological space.

The group $\mathrm{GL}_{\infty,\infty}$ is a sheaf of groups over **Aff**, which acts on $R((u))((t))$. Explicitly, the action can be described as follows.

If we give $A((t))$ the topological basis $\{t^i\}$, then elements in $\mathfrak{gl}_\infty(A)$ could be regarded as $\infty \times \infty$-matrices $X = (X_{ij})_{i,j\in\mathbf{Z}}$ which act on $A((t))$ by the formula

$$X(t^j) = \sum_{i\in\mathbf{Z}} X_{ij} t^i.$$

It is easy to see that

$$\mathfrak{gl}_\infty(A) = \left\{ \begin{array}{l} (X_{ij})_{i,j\in\mathbf{Z}},\, X_{ij} \in A \;\mid\; \forall m \in \mathbf{Z},\, \exists\, n \in \mathbf{Z}, \\ \text{such that whenever } i < m, j > n,\, X_{ij} = 0 \end{array} \right\}.$$

Therefore,

$$\mathfrak{gl}_{\infty,\infty}(R) = \left\{ \begin{array}{l} (X_{ij})_{i,j\in\mathbf{Z}},\, X_{ij} \in \mathfrak{gl}_\infty(R) \;\mid\; \forall m \in \mathbf{Z}, \exists\, n \in \mathbf{Z}, \\ \text{such that whenever } i < m, j > n,\, X_{ij} = 0 \end{array} \right\}.$$

From this presentation, it is clear that $\mathfrak{gl}_{\infty,\infty}(R)$ acts on $R((u))((t))$ by the following formula. If we represent an element in $\mathfrak{gl}_{\infty,\infty}(R)$ by $X = (X_{ij})_{i,j\in\mathbf{Z}}$ and $X_{ij} = (X_{ij,mn})_{m,n\in\mathbf{Z}}$. Then

$$X(u^n t^j) = \sum_{m,n\in\mathbf{Z}} X_{ij,mn} u^m t^i.$$

Observe that $\mathrm{GL}_{\infty,\infty}(R)$ acts on $R((u))((t))$ by the same formula as above.

Remark 5.1. It is possible to give a more invariant definition of the R-ring $\mathfrak{gl}_{\infty,\infty}(R)$ and the group $\mathrm{GL}_{\infty,\infty}(R) = \mathfrak{gl}_{\infty,\infty}(R)^*$.

We define for any integer n an R-submodule of $R((u))((t))$

$$\mathcal{O}_n = t^n R((u))[[t]].$$

If $m < n$, then the R-module $\mathcal{O}_m/\mathcal{O}_n$ is a free $R((t))$-module, and therefore it is a topological R-module with the topology induced by open subspaces

$$E_l = u^l t^m R[[u,t]]/u^l t^n R[[u,t]] \subset \mathcal{O}_m/\mathcal{O}_n.$$

We say that an R-linear map $F : R((u))((t)) \to R((u))((t))$ belongs to $\mathfrak{gl}_{\infty,\infty}(R)$, if the following conditions hold

(1) for any integer n there exists an integer m such that $F\mathcal{O}_n \subset \mathcal{O}_m$,
(2) for any integer m there exists an integer n such that $F\mathcal{O}_n \subset \mathcal{O}_m$,
(3) for any integer $n_1 < n_2$ and $m_1 < m_2$ such that $F\mathcal{O}_{n_1} \subset \mathcal{O}_{m_1}$ and $F\mathcal{O}_{n_2} \subset \mathcal{O}_{m_2}$ we have that the induced R-linear map

$$\bar{F} \quad : \quad \mathcal{O}_{n_1}/\mathcal{O}_{n_2} \longrightarrow \mathcal{O}_{m_1}/\mathcal{O}_{m_2}$$

belongs to $\mathrm{Hom}_{cont}(\mathcal{O}_{n_1}/\mathcal{O}_{n_2}, \mathcal{O}_{m_1}/\mathcal{O}_{m_2})$.

If $R = k$ is a field, then $\mathfrak{gl}_{\infty,\infty}(R)$ and $\mathrm{GL}_{\infty,\infty}(R)$ were studied also in [O2] and [O3].

Lemma 5.1. *Let $R((u))((t))^*$ act on $R((u))((t))$ by the multiplication. Then this action induces an embedding $L^2\mathbb{G}_m \subset \mathrm{GL}_{\infty,\infty}$.*

We recall that there is a natural linear topology on the R-module $R((u))((t))$, see formula (3.7).

Lemma 5.2. *The natural action of $\mathrm{GL}_{\infty,\infty}(R)$ on $R((u))((t))$ is continuous.*

Proof is the same as [O2, Lemma 2].

However, Remark 5.1 indicates that $R((u))((t))$ has some more delicate structures than just a topological R-module, and $\mathrm{GL}_{\infty,\infty}(R)$ preserves this more delicate structure. To make it precise, we first recall that there is a notion of Tate R-modules as in [Dr, §3].

Definition 5.1. (Drinfeld) Let R be a commutative ring. An elementary Tate R-module is a topological R-module which is isomorphic to $P \oplus Q^*$, where P, Q are discrete projective R-modules, and $Q^* = \mathrm{Hom}(Q, R)$ with the natural topology[4]. A Tate R-module is a topological R-module which is a topological direct summand of an elementary Tate R-module.

Definition 5.2. A lattice $\mathbb{L} \subset R((u))((t))$ is an R-submodule such that there exists some $N \gg 0$ with the properties: $t^N R((u))[[t]] \subset \mathbb{L} \subset t^{-N} R((u))[[t]]$ and $t^{-N} R((u))[[t]]/\mathbb{L}$ (with the induced topology) is a Tate R-module.

Note that since a Tate R-module is always Hausdorff, a lattice $\mathbb{L}$ is a closed R-submodule in $R((u))((t))$. Examples of lattices in $R((u))((t))$ include $t^N R((u))[[t]]$. If $\mathbb{L}$ is a lattice, then for any N such that $\mathbb{L} \subset t^{-N} R((u))[[t]]$, $t^{-N} R((u))[[t]]/\mathbb{L}$ is a Tate R-module (as it follows from the following lemma).

Lemma 5.3. *Any open continuous surjection[5] $f : M_1 \to M_2$ between Tate R-modules is splittable when M_1 has a countable basis of open neighborhoods of 0.*

[4] The basis of open neighborhoods of $0 \in Q^*$ consists of annihilators of finite subsets in Q.

[5] We note that $M_1 \to M_2$ is an open continuous surjection if and only if the quotient module M_2 is endowed with the quotient topology.

Proof. By definition, we can assume that M_2 is a topological direct summand of a topological R-module $N = N_1 \oplus N_2$, where $N_1 = \oplus_{i \in I} Re_i$ is a discrete free R-module, and $N_2 = \prod_{j \in J} Re_j$ is endowed with product topology of discrete R-modules. Let $p : N \to M_2$ be the projection. It is enough to construct a continuous R-module map $g : N \to M_1$ such that $p = fg$. Moreover, it is enough to consider two cases: to construct the map g restricted to N_1 and restricted to N_2. The first case is obvious since N_1 is a discrete free R-module. For the second case we note that f is an open map. Now let $U_1 \supset U_2 \supset \dots$ be a countable basis of open neighborhoods of 0 from M_1. For any $l \in \mathbf{N}$ there is the minimal finite (or empty) subset K_l of the set J such that $p(e_j) \subset f(U_l)$ for any $j \in J \setminus K_l$. For any $j \in K_{l+1} \setminus K_l$ we define an element $g(e_j) \subset U_l$ with the property $fg(e_j) = p(e_j)$. For any $j \in J \setminus \bigcup_l K_l$ we have that $p(e_j)$ belongs to $\bigcap_l f(U_l)$ which is a zero submodule, since $f(U_l)$, where l runs over $\mathbf{N}$, is a basis of open neighborhoods of 0 in M_2. Therefore we put $g(e_j) = 0$. We note that R-modules which we consider are complete with the topology. Hence we obtained a well-defined continuous R-module map $g : N_2 \to M_1$ such that $fg = p$. □

We note that from Lemma 5.3 we have that if $\mathbb{L}$ is a lattice in $R((u))((t))$, then $\mathbb{L}/t^m R((u))[[t]]$ is a Tate R-module for any integer m such that $t^m R((u))[[t]] \subset \mathbb{L}$.

Proposition 5.4. *Let $\mathbb{L}$ be a lattice in $R((u))((t))$ and $g \in \mathrm{GL}_{\infty,\infty}(R)$. Then $g\mathbb{L}$ is also a lattice in $R((u))((t))$.*

Proof. As before, for simplicity, we denote by $\mathcal{O}_n = t^n R((u))[[t]]$. Let us also denote $R((u))((t))$ by K. By Remark 5.1, there is some integer $N > 0$ such that

$$\mathcal{O}_N \subset g\mathbb{L} \subset \mathcal{O}_{-N}.$$

We have the following exact sequence of R-modules:

$$0 \longrightarrow \frac{g\mathbb{L}}{\mathcal{O}_N} \longrightarrow \frac{\mathcal{O}_{-N}}{\mathcal{O}_N} \longrightarrow \frac{\mathcal{O}_{-N}}{g\mathbb{L}} \longrightarrow 0 \tag{5.3}$$

Clearly, $\frac{\mathcal{O}_{-N}}{\mathcal{O}_N}$ with the induced topology is an elementary Tate R-module and therefore it is enough to show that there is a splitting of (5.3) as topological R-modules. We consider the following Cartesian square

$$\begin{array}{ccc} \frac{\mathcal{O}_{-N}}{\mathcal{O}_N} & \longrightarrow & \frac{\mathcal{O}_{-N}}{g\mathbb{L}} \\ \big\downarrow & & \big\downarrow \\ \frac{K}{\mathcal{O}_N} & \longrightarrow & \frac{K}{g\mathbb{L}} \end{array}$$

Therefore, it is enough to show the bottom row is splittable as topological R-modules. But this will follow if we can show that $K \to \frac{K}{g\mathbb{L}}$ admits a splitting $\frac{K}{g\mathbb{L}} \to K$ (then $\frac{K}{g\mathbb{L}} \to K \to \frac{K}{\mathcal{O}_N}$ splits of the bottom row). Finally, to see that $K \to \frac{K}{g\mathbb{L}}$ admits a splitting, we note that, by Lemma 5.2, g is a continuous automorphism of K and therefore it is enough to show that $K \to \frac{K}{\mathbb{L}}$ admits a splitting.

Now we reverse the above reasoning. We choose an integer $M > 0$ such that $\mathcal{O}_M \subset \mathbb{L} \subset \mathcal{O}_{-M}$. Then there is an obvious splitting of $K \to \frac{K}{\mathcal{O}_M}$. Therefore it is enough to find a splitting of $\frac{K}{\mathcal{O}_M} \to \frac{K}{\mathbb{L}}$. As $\mathbb{L}$ is a lattice, we can find a splitting of $\frac{\mathcal{O}_{-M}}{\mathcal{O}_M} \to \frac{\mathcal{O}_{-M}}{\mathbb{L}}$ by Lemma 5.3, which can be obviously extended to a splitting of $\frac{K}{\mathcal{O}_M} \to \frac{K}{\mathbb{L}}$. □

Now if $\mathbb{L}, \mathbb{L}'$ are two lattices in $R((u))((t))$, we can define as usual the $\mathcal{P}ic^{\mathbb{Z}}_R$-torsor $\mathcal{D}et(\mathbb{L}|\mathbb{L}')$ over $\operatorname{Spec} R$ (locally in the Nisnevich topology), and therefore we obtain a central extension of $\mathrm{GL}_{\infty,\infty}$ by $\mathcal{P}ic^{\mathbb{Z}}$ similarly to the central extension from [OsZh, § 4C]. To do this, let m be minimal such that

$$t^m R((u))[[t]] \subset \mathbb{L} \qquad \text{and} \qquad t^m R((u))[[t]] \subset \mathbb{L}'.$$

Then $\mathbb{L}/t^m R((u))[[t]]$ and $\mathbb{L}'/t^m R((u))[[t]]$ are Tate R-modules. For a Tate R-module M, let $\mathcal{D}et(M)$ be the $\mathcal{P}ic^{\mathbb{Z}}_R$-torsor of all graded-determinantal theories[6] on M. (By the theorem of Drinfeld, see [Dr, Th. 3.4] and [BBE, §2.12], there is a Nisnevich covering $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that the category $\mathcal{D}et(M)(R')$ is not empty.) Then we define[7]

$$\mathcal{D}et(\mathbb{L}|\mathbb{L}') = \mathcal{D}et(\mathbb{L}'/t^m R((u))[[t]]) - \mathcal{D}et(\mathbb{L}/t^m R((u))[[t]]).$$

Therefore, we obtain a (categorical) central extension of $\mathrm{GL}_{\infty,\infty}$ by $\mathcal{P}ic^{\mathbb{Z}}$:

$$g \mapsto \mathcal{D}et(\mathbb{L}|g\mathbb{L}).$$

(We need to fix a lattice $\mathbb{L}$ to construct such an extension. For example $\mathbb{L} = \mathcal{O}_n$ for some integer n.) By restriction, we have a central extension of $L^2\mathbb{G}_m$ by $\mathcal{P}ic^{\mathbb{Z}}$.

5.3. **The generalized commutator and the change of local parameters.** We constructed a central extension of $L^2\mathbb{G}_m$ by $\mathcal{P}ic^{\mathbb{Z}}$ (after fixing some lattice $\mathbb{L}$). For such a central extension there are generalized commutators C_2 and C_3, see formulas (5.1) and (5.2). Therefore, we have a map

$$C_3 \ : \ L^2\mathbb{G}_m \times L^2\mathbb{G}_m \times L^2\mathbb{G}_m \longrightarrow \mathbb{G}_m. \tag{5.4}$$

Note that the map C_3 in formula (5.4) does not depend on the choice of a lattice $\mathbb{L}$, which we used to construct a central extension. Indeed, if $\mathbb{L}'$ is another lattice in $R((u))((t))$, then any object of $\mathcal{D}et(\mathbb{L}|\mathbb{L}')(R')$ (which exists after some Nisnevich covering $\operatorname{Spec} R' \to \operatorname{Spec} R$) gives an isomorphism (over $\operatorname{Spec} R'$) of categorical central extensions constructed by $\mathbb{L}$ and by $\mathbb{L}'$. Therefore the corresponding maps C_3 coincide, see [OsZh, Cor. 2.19].

We recall the definition of the group functor Aut[8]. This is the group functor which associates with every ring R the group of continuous automorphisms of the R-algebra $R((t))$. Then

$$\mathrm{Aut}(R) = \{t' = \sum a_i t^i \in R((t))^* \mid a_1 \in R^*, a_i \in \mathcal{N}R \text{ if } i < 0\},$$

and the action of $\mathrm{Aut}(R)$ on $R((t))$ is given by $t' \in \mathrm{Aut}(R) \mapsto \phi_{t'} : R((t)) \to R((t))$, where

$$\phi_{t'}(\sum a_i t^i) = \sum a_i t'^i.$$

This is called the change of a local parameter in $R((t))$.

Now we return to the two-dimensional story. There is a question: what is a well-defined change of local parameters in $R((u))((t))$? In this case of a two-dimensional

[6]Like in our previous paper [OsZh], we use the notion of "a graded-determinantal theory" and notation $\mathcal{D}et(M)$ for the $\mathcal{P}ic^{\mathbb{Z}}_R$-torsor of all graded-determinant theories on M. In [Dr, § 5.2] this notion was under the name "a determinant theory" and the notation for the $\mathcal{P}ic^{\mathbb{Z}}_R$-torsor of all determinant theories on M was $\mathcal{D}et_M$.

[7]As in [OsZh] we use the additive notation for the multiplication of two $\mathcal{P}ic^{\mathbb{Z}}_R$-torsors.

[8]The central extension of this group is usually called the algebraic Virasoro group.

local field $k((u))((t))$, where k is a field, we can define the change of local parameters as a map: $t \mapsto t'$, $u \mapsto u'$,

$$\phi_{t',u'} \ : \ \sum a_{i,j} u^j t^i \mapsto \sum a_{i,j} {u'}^j {t'}^i,$$

where $u', t' \in L^2\mathbb{G}_m(R)$, $\nu_1(t') = 1$, $(\nu_1, \nu_2)(u') = (0,1)$, $t' \in tR[[t]]$, $u' \in \mathfrak{m}(R)$. By induction, similarly to the case of two-dimensional local fields (see also the proof of Lemma 5.5 below), one can prove that $\phi_{t',u'}$ is a well-defined automorphism of the R-algebra $R((u))((t))$. Moreover, it easy to see, that the set of these automorphisms is a subgroup in the group of continuous automorphisms of the R-algebra $R((u))((t))$, i.e, the composition of two change of local parameters of above kind and the inverse will be again the change of local parameters of above kind. But this is only the analogy with two-dimensional local fields, and we did not use that a ring R may contain nilpotent elements. Therefore we have the following lemma.

Lemma 5.5. *Let $u', t' \in L^2\mathbb{G}_m(R)$, $\nu_1(t') = 1$, $(\nu_1, \nu_2)(u') = (0,1)$, $u' \in R((u))[[t]]$. Then a map*

$$\phi_{t',u'} \ : \ \sum a_{i,j} u^j t^i \mapsto \sum a_{i,j} {u'}^j {t'}^i$$

is a well-defined continuous automorphism of the R-algebra $R((u))((t))$.

Proof. We can change, first, the local parameter $u \mapsto u'$. To see that for any $f \in R((u))((t))$ the series $\phi_{t,u'}(f)$ is well-defined, i.e, it converges in the topology of $R((u))((t))$, we use a Taylor formula:

$$g(\tilde{u} + \delta, t) = \sum_{k \geq 0} (D_k(g))(\tilde{u}, t)) \delta^k, \tag{5.5}$$

where $D_0 \equiv \mathrm{id}$, and for $k \geq 1$ we have on monomials

$$D_k(u^j t^i) = \frac{1}{k!} \frac{d^k(u^j)}{du^k} t^i = \binom{j}{k} u^{j-k} t^i, \qquad \binom{j}{k} = \frac{j \cdot (j-1) \cdot \ldots \cdot (j-k+1)}{1 \cdot 2 \cdot \ldots \cdot k}.$$

The map D_k is extended to series g from $R((u))((t))$ as an R-linear continuous map. We suppose that the series $D_k(g))(\tilde{u}, t)$, $k \geq 0$ are already well-defined. We apply formula (5.5) successively in the following cases: 1) if $g = D_k(f)$, $\tilde{u} = cu$, $c \in R^*$, $\delta \in u \cdot \mathfrak{m}(R) = u^2 R[[u]] + tR((u))[[t]]$, then the right hand side of (5.5) is a converge series in topology of $R((u))((t))$, and 2) if $g = f$, $\tilde{u} = cu + a$, $c \in R^*$, $a \in u \cdot \mathfrak{m}(R)$, $\delta = (u' - \tilde{u}) \in \mathcal{N}R((u))((t))$, then the right hand side of (5.5) is a finite sum. It gives us that the series $\phi_{t,u'}(f)$ is well-defined.

To prove that $\phi_{t,u'}$ is an automorphism of the algebra $R((u))((t))$, we find a series $v \in R((u))[[t]]^*$ such that the substitution of v in u' instead of u is equal u, i.e. $u'(v,t) = u$. Using the corresponding one-dimensional result for $R((u))$, we can find $v_1 \in R((u))^*$ such that $u'(v_1, t) \equiv u \mod tR((u))[[t]]$. Now it is easy to find an element $v_2 = u + b_1(u)t$, $b_1(u) \in R((u))$ such that $u'(v_1(v_2, t), t) \equiv u \mod t^2 R((u))[[t]]$. After some steps we find $v_k = u + b_{k-1}(u)t^k$, $b_{k-1}(u) \in R((u))$ such that $u'(v_1(v_2(\ldots v_k)), t) \equiv u \mod t^k R((u))[[t]]$. Then the sequence $w_k = v_1(v_2(\ldots v_k))$ tends to v when $k \to +\infty$.

Now $\phi_{t',u'} = \phi_{t,u'} \, \phi_{t'(v,t),u}$, where $u'(v,t) = v(u',t) = u$. But $\phi_{t'(v,t),u}$ is a well-defined automorphism of the algebra $R((u))((t))$ by the one-dimensional result applied to the ring $B((t))$, where $B = R((u))$. From Remark 5.1 it is easy to see that $\phi_{t'(v,t),u}$ and $\phi_{t,u'}$ are from $\mathrm{GL}_{\infty,\infty}(R)$. Therefore, by Lemma 5.2 they are continuous automorphisms of $R((u))((t))$. $\square$

Remark 5.2. The composition of two automorphisms of $R((u))((t))$ as in Lemma 5.5 is not always the automorphism of the same type (as in Lemma 5.5), since we demanded $u' \in R((u))[[t]]$. It would be natural to withdraw this condition. Then as in the proof of Lemma 5.5, we can obtain that the operator $\phi_{t',u'}$ is a well-defined continuous homomorphism from the R-algebra $R((u))((t))$ to itself. But there is the problem to prove that $\phi_{t,u'}$ is an automorphism, because there can be infinite many nilpotent elements in the series for u' (compare with the last paragraph of Section 3.3). If $R \in \mathcal{B}$ (see Definition 3.3), then by the similar method as in the end of [Mor, § 1] it is possible to prove that there is $v \in L^2\mathbb{G}_m(R)$ such that $u'(v,t) = u$. Therefore, in this case, $\phi_{t',u'}$ is an automorphism of $R((u))((t))$ for any $u', t' \in L^2\mathbb{G}_m(R)$ with $\nu_1(t') = 1$, $(\nu_1, \nu_2)(u') = (0, 1)$.

Proposition 5.6. *The map C_3 is invariant under the change of the local parameters t and u. Namely, in conditions of Lemma 5.5, for any $f, g, h \in L^2\mathbb{G}_m(R)$ we have that*

$$C_3(f, g, h) = C_3(\phi_{t',u'}(f),\ \phi_{t',u'}(g),\ \phi_{t',u'}(h)).$$

Proof. From Remark 5.1 it is easy to see that $\phi_{t',u'} \in \mathrm{GL}_{\infty,\infty}(R)$. Therefore, by Proposition 5.4, if $\mathbb{L}$ is a lattice in $R((u))((t))$, then an R-submodule $\phi_{t',u'}(\mathbb{L})$ is again a lattice in $R((u))((t))$. Now we apply the fact that the maps C_3 coincide even if they are constructed by various lattices in $R((u))((t))$. □

We will need some lemmas that the property to be invariance under the change of local parameters can uniquely define some tri-linear and anti-symmetric map over a field. Let k be a field of characteristic zero in the following two lemmas. We recall the following well-known lemma.

Lemma 5.7. *Let $K = k((t))$. Let $\langle \cdot, \cdot \rangle$ be a pairing $K \times K \to k$ such that it is continuous*[9]*, bilinear, anti-symmetric and invariant under the change of local parameter in K: $t \mapsto t'$. Then $\langle f, g \rangle = c \cdot \mathrm{res}(f dg)$, where $c \in k$ is fixed, and f, g are any elements from K.*

Proof. It is enough to prove that $\langle t^{-n}, t^n \rangle = cn$, $\langle t^n, t^m \rangle = 0$ if $n + m \neq 0$. If $n + m \neq 0$, then we take $t' = 2t$. Now from $\langle t^n, t^m \rangle = \langle t'^n, t'^m \rangle = 2^{n+m} \langle t^n, t^m \rangle$ it follows that $\langle t^n, t^m \rangle = 0$. Hence, $\langle t^{-n}, t \rangle = 0$ when $n \geq 2$. Let $t' = t + t^n$ be a new local parameter. Then $t'^{-n} = t^{-n} - nt^{-1} + \ldots$. Now we have

$$0 = \langle t^{-n}, t \rangle = \langle t'^{-n}, t' \rangle = \langle t^{-n} - nt^{-1} + \ldots,\ t + t^n \rangle = \langle t^{-n}, t^n \rangle - n \langle t^{-1}, t \rangle.$$

(We can consider only the finite sum of $\langle \cdot, \cdot \rangle$ in the above expression due to the continuous property of $\langle \cdot, \cdot \rangle$.) Thus, $\langle t^{-n}, t^n \rangle = n \langle t^{-1}, t \rangle = nc$. □

Now we consider the case of a two-dimensional local field.

Lemma 5.8. *Let $K = k((u))((t))$. Let $\langle \cdot, \cdot, \cdot \rangle$ be a map $K \times K \times K \to k$ such that it is continuous, tri-linear, anti-symmetric and invariant under the change of local parameters in K: $t \mapsto t'$, $u \mapsto u'$. Then $\langle f, g, h \rangle = c \cdot \mathrm{Res}(f dg \wedge dh)$, where $c \in k$ is fixed, and f, g, h are any elements from k.*

[9] Here and in Lemma 5.8 a continuous map means a map which is continuous in each argument, i.e. when we fix other arguments of the map.

Proof. We will consider several cases and reduce them to the case

$$\langle u, t, u^{-1}t^{-1}\rangle = c. \tag{5.6}$$

We note that

$$\langle u^j t^i, u^l t^k, u^n t^m\rangle = 0 \quad \text{if} \quad j+l+n \neq 0 \quad \text{or} \quad i+k+m \neq 0. \tag{5.7}$$

(The proof is by twisting $t' = 2t$, $u' = 3u$ as in the 1-dimensional case.)

We have to prove that $X = \langle u^j t^i, u^l t^k, u^{-j-l}t^{-i-k}\rangle = c \cdot (jk - il)$.

We will calculate the following easy case

$$\langle u^m, u^n, u^l\rangle = 0. \tag{5.8}$$

We consider a new local parameter $t' = t + t^2$. Then $t'^{-1} = t^{-1} - 1 + t^2 + \ldots$. We have

$$0 = \langle t^{-1}u^m, u^n, u^l\rangle = \langle t'^{-1}u^m, u^n, u^l\rangle = \langle t^{-1}u^m - u^m + tu^m + \ldots, u^n, u^l\rangle =$$
$$= \langle t^{-1}u^m, u^n, u^l\rangle - \langle u^m, u^n, u^l\rangle + \langle tu^m, u^n, u^l\rangle + \ldots.$$

Hence, using the continuous property of $\langle\cdot,\cdot,\cdot\rangle$, we have $\langle u^m, u^n, u^l\rangle = 0$, since by (5.7), $\langle t^q u^m, u^n, u^l\rangle = 0$ for any $q \neq 0$.

Now we will explain, why

$$\langle u^j t^i, t, u^{-j}t^{-i-1}\rangle = j \cdot c \tag{5.9}$$

when $(i,j) > (0,1)$ (with respect to the lexicographical order in $\mathbf{Z}^2$, see (3.26)). Then $u' = u + u^j t^i$ is a well-defined change of the local parameter u. We have

$$0 = \langle u, t, u^{-j}t^{-i-1}\rangle = \langle u', t, u'^{-j}t^{-i-1}\rangle.$$

We compute $u'^{-j} = (u + u^j t^i)^{-j} = u^{-j} - ju^{-1}t^i + du^{j-2}t^{2i} + \ldots$, where $d \in \mathbf{Z}$. Therefore, we have

$$0 = \langle u + u^j t^i, t, u^{-j}t^{-i-1} - ju^{-1}t^{-1} + du^{j-2}t^{i-1} + \ldots\rangle =$$
$$= -j\langle u, t, u^{-1}t^{-1}\rangle + \langle u^j t^i, t, u^{-j}t^{-i-1}\rangle = -j \cdot c + \langle u^j t^i, t, u^{-j}t^{-i-1}\rangle.$$

Thus, we explained this case.

Analogously, we can obtain that

$$\langle u, u^l t^k, u^{-l-1}t^{-k}\rangle = k \cdot c \tag{5.10}$$

when $k \geq 0$. The case $k = 0$ we have proved above. If $k = 1$, then we consider $t' = u^{-l}t$. We obtain

$$\langle u, u^l t, u^{-l-1}t^{-1}\rangle = \langle u, u^l t', u^{-l-1}t'^{-1}\rangle = \langle u, t, u^{-1}t^{-1}\rangle = c.$$

Now if $k > 1$, then we consider a well-defined change of the local parameter $t \mapsto t' = t + u^l t^k$. We have $t'^{-k} = t^{-k} - ku^l t^{-1} + eu^{2l}t^{k-2} + \ldots$, where $e \in \mathbf{Z}$. We insert this expression into the following equality: $0 = \langle u, t, u^{-l-1}t^{-k}\rangle = \langle u, t', u^{-l-1}t'^{-k}\rangle$. Thus, using (5.7), we obtained this case.

Now we consider the case when $(k,l) \geq (0,1)$ and $(i,j) \geq (0,1)$. Without loss of generality (using the anti-symmetric property of $\langle\cdot,\cdot,\cdot\rangle$) we can assume that $(k,l) \geq (i,j)$. If $(i,j) = (0,1)$, then this is the previous case, see formula (5.10). Therefore we assume that $(i,j) > (0,1)$. If $k = 0$, this was also calculated above. If $k = 1$, then let $t' = u^{-l}t$ be a new local parameter. We have

$$\langle u^j t^i, u^l t, u^{-j-l}t^{-i-1}\rangle = \langle u^j t'^i, u^l t', u^{-j-l}t'^{-i-1}\rangle = \langle u^{j-li}t^i, t, u^{li-j}t^{-i-1}\rangle.$$

Since $(i, j-li) > (0,1)$, we have that the last expression was also calculated above. Therefore we can assume that $k > 1$. There is a well-defined change of the local parameter $t \mapsto t'$, where $t' = t + u^l t^k$. We have

$$\begin{aligned} 0 = \langle u^j t^i, t, u^{-j-l} t^{-i-k} \rangle &= \langle u^j t'^i, t', u^{-j-l} t'^{-i-k} \rangle = \\ &= \langle u^j (t+u^l t^k)^i, t + u^l t^k, u^{-j-l}(t+u^l t^k)^{-i-k} \rangle. \end{aligned} \tag{5.11}$$

We calculate now $(t+u^lt^k)^i = t^i + iu^l t^{i+k-1} + au^{2l}t^{i+2k-2} + \ldots$, where $a \in \mathbf{Z}$, and calculate $(t+u^lt^k)^{-i-k} = t^{-i-k} + (-i-k)u^l t^{-i-1} + bu^{2l}t^{k-i-2} + \ldots$, where $b \in \mathbf{Z}$. We substitute these expressions into formula (5.11). We obtain

$$\begin{gathered} 0 = \langle u^j t^i + iu^{l+j}t^{i+k-1} + \ldots, t + u^l t^k, u^{-j-l}t^{-i-k} + (-i-k)u^{-j}t^{-i-1} + \ldots \rangle = \\ = \langle u^j t^i, u^l t^k, u^{-j-l} t^{-i-k} \rangle + (-i-k)\langle u^j t^i, t, u^{-j} t^{-i-1} \rangle + i \langle u^{l+j} t^{i+k-1}, t, u^{-j-l} t^{-i-k} \rangle. \end{gathered}$$

Hence and using formula (5.9) we obtain

$$0 = X + (-i-k) \cdot j \cdot c + i \cdot (l+j) \cdot c.$$

Therefore we calculated $X = (jk - il) \cdot c$. We finished the proof of case $(k,l) \geq (0,1)$, $(i,j) \geq (0,1)$.

We calculate now

$$Z = \langle u^j, u^l t^k, u^{-j-l} t^{-k} \rangle,$$

where $k < 0$. Then $u' = u + u^{-j-l}t^{-k}$ is a well-defined change of the local parameter u, since $-k > 0$. We have $0 = \langle u^j, u^l t^k, u \rangle = \langle u'^j, u'^l t^k, u' \rangle$. We compute $u'^j = u^j + ju^{-l-1}t^{-k} + \ldots$ and $u'^l = u^l + lu^{-j-1}t^{-k} + \ldots$. Hence,

$$\begin{aligned} 0 &= \langle u^j + ju^{-l-1}t^{-k} + \ldots, u^l t^k + lu^{-j-1} + \ldots, u + u^{-j-l}t^{-k} \rangle = \\ &= Z + l \langle u^j, u^{-j-1}, u \rangle + j \langle u^{-l-1} t^{-k}, u^l t^k, u \rangle = Z + j \cdot (-k) \cdot c, \end{aligned}$$

where we used formulas (5.8), (5.10) and the anti-symmetric property of $\langle \cdot, \cdot, \cdot \rangle$. Hence $Z = jkc$.

We calculate now $Y = \langle u^j t^i, u^l t^k, u^{-j-l} t^{-i-k} \rangle$ when $(i,j) < (0,0)$ and $(k,l) < (0,0)$. Then $(-i-k, -j-l) > (0,0)$. If $i = 0$ or $k = 0$, then this is the previous case, where we calculated Z. (If $i = k = 0$, then it follows from formula (5.8).) Therefore, we assume that $-k \geq 1$ and $-i \geq 1$. We consider a well-defined change of local parameters: $t' = t + t^{-i-k}$. We have $0 = \langle u^j t^i, u^l t^k, u^{-j-l} t \rangle = \langle u^j t'^i, u^l t'^k, u^{-j-l} t' \rangle$. We compute $t'^i = t^i + it^{-k-1} + \ldots$ and $t'^k = t^k + kt^{-i-1} + \ldots$. Therefore, we have

$$\begin{aligned} 0 = \langle u^j t^i + iu^j t^{-k-1} + \ldots, u^l t^k + ku^l t^{-i-1} + \ldots, u^{-j-l} t + u^{-j-l} t^{-i-k} \rangle = \\ = Y + k \langle u^j t^i, u^l t^{-i-1}, u^{-j-l} t \rangle + i \langle u^j t^{-k-1}, u^l t^k, u^{-j-l} t \rangle. \end{aligned}$$

Since $-k-1 \geq 0$ and $-i-1 \geq 0$, we can use the previous cases and the anti-symmetric property of $\langle \cdot, \cdot, \cdot \rangle$. We obtain $Y = -k(j(i-1) - il)c - i(jk - l(-k-1))c = (kj - li)c$.

Now, using the anti-symmetric property of $\langle \cdot, \cdot, \cdot \rangle$, it is easy to see that we have considered all possible cases. □

5.4. **The generalized commutator and the symbol.** We recall that in Sections 3.3 and 3.4 we have explicitly defined the two-dimensional Contou-Carrère symbol $L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \longrightarrow \mathbb{G}_m(R)$ for a ring R when $\mathbf{Q} \subset R$ or $R \in \mathcal{B}$. Lemma 3.10, the following important theorem and the corollary from this theorem show that we can define the two-dimensional Contou-Carrère symbol for any ring R by means of the map C_3. Besides, this theorem greatly generalizes

Theorem 4.11 from [OsZh]. We recall that the map C_3 is tri-multiplicative and anti-symmetric.

Theorem 5.9. *Let R be any $\mathbf{Q}$-algebra. Then the map*

$$C_3 \ : \ L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \times L^2\mathbb{G}_m(R) \longrightarrow \mathbb{G}_m(R)$$

satisfies properties (3.15)-(3.18).

Proof. During the proof we assume that all the schemes (and ind-schemes) are defined over the field $\mathbf{Q}$. For simplicity of notations, we will omit indication on the field $\mathbf{Q}$.

We will use the following fact. Let V be an (elementary, for simplicity) Tate R-module, and let $\mathrm{GL}(V)$ be the group of automorphisms of V as a Tate R-module (i.e. $\mathrm{GL}(V)(R')$ is the group of continuous automorphisms of the Tate R'-module $V\hat{\otimes}_R R'$), which is a sheaf of groups over $\mathbf{Aff}/R$. Then we have a canonical homomorphism (explicitly defined for an elementary Tate R'-module $V\hat{\otimes}_R R'$ by the choice of a coprojective lattice[10], after a Nisnevich covering $\operatorname{Spec} R' \to \operatorname{Spec} R$), see also [Dr, § 5.6]:

$$\mathcal{D}et_V : \mathrm{GL}(V) \to \mathcal{P}ic_R^{\mathbb{Z}}.$$

Let $Z_2 \subset \mathrm{GL}(V)$ be the subsheaf of commuting elements, then we obtain the usual commutator map for $\mathcal{D}et_V$ (see Section 2D of [OsZh]):

$$\mathrm{Comm} : Z_2 \to \mathbb{G}_m.$$

Now let $f, g, h \in \mathrm{GL}_{\infty,\infty}(R)$ be commuting with each other elements. Assume that $\mathbb{L}$ is a lattice in $R((u))((t))$ fixed by f and g, and assume that $h\mathbb{L} \subset \mathbb{L}$. Then f, g induces $\pi_h(f), \pi_h(g) \in \mathrm{GL}(V)$ where $V = \mathbb{L}/h\mathbb{L}$ and

$$C_3(f, g, h) = \mathrm{Comm}(\pi_h(f), \pi_h(g))^{-1}. \tag{5.12}$$

Formula (5.12) can be proved in exactly the same way as in [OsZh] (see the proof of Theorem 4.11 in *loc. cit.*, in particular Lemma 4.12 and the commutative diagram (4-8)).

We recall (see formula (3.5)) that for the group ind-scheme $L^2\mathbb{G}_m$ we have decomposition

$$L^2\mathbb{G}_m = \mathbb{P} \times \mathbb{Z}^2 \times \mathbb{G}_m \times \mathbb{M}.$$

We will use also the following fact. Let $\mathbb{S}$ and $\mathbb{T}$ be any connected group (ind)-subschemes in the ind-scheme $L^2\mathbb{G}_m$. We show that

$$C_3(f, g, h) = 1, \tag{5.13}$$

where $f \in \mathbb{S}(R)$, $g \in \mathbb{T}(R)$, $h \in \mathbb{G}_m(R)$. Indeed, by varying $f \in \mathbb{S}(R)$ and $g \in \mathbb{T}(R)$ for any ring R, we may regard C_3 as a morphism of (ind)-schemes

$$\mathbb{S} \times \mathbb{T} \to \underline{\mathrm{Hom}}_{\mathrm{gr}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}.$$

The (ind)-scheme $\mathbb{S} \times \mathbb{T}$ is connected. Therefore the claim is clear, since $C_3(0, g, h) = 1$ for any g and h from $L^2\mathbb{G}_m(R)$. We note that by the same reason formula (5.13) remains true if we put in this formula $g = u$ or $g = t$ and f, h as above.

[10] We recall that by definition from [Dr], a lattice L in a Tate R-module V is an open R-submodule of V such that L/U is finitely generated for any open R-module $U \subset L$. A lattice L is called coprojective if V/L is a projective R-module.

The schemes $\mathbb{P}$, $\mathbb{G}_m$ and $\mathbb{M}$ are connected. Indeed, it is clear that $\mathbb{G}_m$ and $\mathbb{M}$ are connected schemes. Besides, the topological space (over $\operatorname{Spec}\mathbf{Q}$) of the scheme $\mathbb{P}$ is equal to one point, because if $f \in L_m\mathcal{N}_n(R)$ (see formula (3.3)) then $a_i^{k_{i,n}} = 0$, where $i \geq m$ and $k_{i,n}$ depends on i and n.)

We will use also that if f, g and h are elements from $L^2\mathbb{G}_m(R)$ such that they preserve a lattice in $R((u))((t))$, then $C_3(f,g,h) = 1$. Indeed, we have directly from the construction that the categorical central extension restricted to the subgroup generated by elements f, g and h is trivial.

The proof of the theorem is based on several case by case inspections. We will use the tri-multiplicative and anti-symmetric property of the map C_3.

We will check property (3.17) for the map C_3. Since u and t belong to $\mathbf{Q}((u))((t))^*$, then this property directly follows from [OsZh, Th. 4.11].

We will check now property (3.16) for the map C_3. We will consider several cases.

Let $a \in \mathbb{G}_m(R)$, g and h be any elements from $(\mathbb{P} \times \mathbb{M} \times \mathbb{G}_m)(R)$ for any ring R. Then

$$C_3(a,g,h) = C_3(g,h,a) = 1$$

by formula (5.13). It satisfies formula (3.16).

Let $a \in \mathbb{G}_m(R)$, g be any element from $(\mathbb{P} \times \mathbb{M} \times \mathbb{G}_m)(R)$ for any ring R. Then

$$C_3(a,g,t) = C_3(g,t,a) = 1$$

by an analog of formula (5.13). It satisfies formula (3.16).

The case $C_3(a,g,u) = 1$ when $a \in \mathbb{G}_m(R)$ and g is any element from $(\mathbb{P} \times \mathbb{M} \times \mathbb{G}_m)(R)$ for any ring R is analogous to the previous case. It satisfies formula (3.16).

The case $C_3(a,t,t) = C_3(a,u,u) = 1$ when $a \in \mathbb{G}_m(R)$ follow from the fact $\operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$. Therefore we check these equalities on $a \in \mathbf{Q}^* \subset \mathbb{G}_m(R)$ and this checking follows from [OsZh, Th. 4.11]. It satisfies formula (3.16).

The last case $C_3(a,u,t) = a$ when $a \in \mathbb{G}_m(R)$ follows from formula (5.12) for $V = R((u))$ and $h = t$. It satisfies also formula (3.16). We have finally checked property (3.16) for the map C_3.

We will check now property (3.15) for the map C_3. We will consider again several cases.

Let $f \in (\mathbb{P} \times \mathbb{M})(R)$, $g \in \mathbb{G}_m(R)$ and h be any from $L^2\mathbb{G}_m(R)$, where R is any ring. Using the anti-symmetric property of the map C_3 we reduce this case to the cases above when we checked the property (3.16).

Now let f, g and h be any elements from $(\mathbb{P}\times\mathbb{M})(R)$. Using the tri-multiplicativity of the map C_3, we will assume that any of these elements belongs either to $\mathbb{P}(R)$ or to $\mathbb{M}(R)$. We recall that over $\mathbf{Q}$, there are isomorphisms of group ind-schemes (see formulas (3.13) and (3.10)):

$$\exp : \mathfrak{m} \simeq \mathbb{M}, \quad \exp : \mathfrak{p} \simeq \mathbb{P}.$$

Our goal is to prove that

$$C_3(f,g,h) = (f,g,h), \tag{5.14}$$

where $(\cdot,\cdot,\cdot)$ is an expression given by formula (3.15). By Proposition 3.7, it is enough to check formula (5.14) on the following elements:

$$f = \exp(au^jt^i), \quad g = \exp(bu^lt^k), \quad h = \exp(cu^nt^m), \tag{5.15}$$

where a, b and c are from R.

We note that if $(i,j) > (0,0)$, then $f \in \mathbb{M}(R)$. If $(i,j) < (0,0)$, then $f \in \mathbb{P}(R)$ and $a \in \mathcal{N}R$. For the sequel it will be convenient for us to include also variant when $(i,j) = (0,0)$ and $a \in \mathcal{N}R$ (although in this case f is not from $(\mathbb{P} \times \mathbb{M})(R)$). The same is true for g with (k,l) and h with (m,n) (including the cases when $(k,l) = 0$ and $(m,n) = 0$).

Let us consider, for example, the variant when indices $(i,j) \le 0$, $(k,l) > 0$ and $(m,n) > 0$. We consider a map $\widehat{\mathbb{G}}_a \to \mathbb{P} \times (1 + \mathcal{N})$ which is given as $a \mapsto \exp(au^j t^i)$, where $\widehat{\mathbb{G}}_a = \operatorname{Spf} \mathbf{Q}[[T]] = \mathcal{N}$ is the additive formal group, the group ind-schemes $1 + \mathcal{N} \xleftarrow{\exp} \widehat{\mathbb{G}}_a$ are isomorphic (the group ind-scheme $1 + \mathcal{N}$ corresponds to the index $(i,j) = (0,0)$). We consider also two maps $\mathbb{G}_a \to \mathbb{M}$ which are given as $b \mapsto \exp(bu^l t^k)$ and $c \mapsto \exp(cu^n t^m)$. By composing these maps with C_3 and $(\cdot,\cdot,\cdot)$, we obtain two tri-multiplicative morphisms

$$\widehat{\mathbb{G}}_a \times \mathbb{G}_a \times \mathbb{G}_a \longrightarrow \mathbb{G}_m.$$

Analogously, when we consider other variants for indices (i,j), (k,l) and (m,n) from formula (5.15) we will also obtain two tri-multiplicative morphisms of the following type:

$$\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbb{H}_3 \longrightarrow \mathbb{G}_m, \tag{5.16}$$

where the group ind-scheme $\mathbb{H}_i$ $(1 \le i \le 3)$ is equal either to $\mathbb{G}_a$ or to $\widehat{\mathbb{G}}_a$.

Lemma 5.10. *Let ϕ be a tri-multiplicative morphism from $\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbb{H}_3$ to $\mathbb{G}_m$, where the group ind-scheme $\mathbb{H}_i$ $(1 \le i \le 3)$ is equal either to $\mathbb{G}_a$ or to $\widehat{\mathbb{G}}_a$ (over $\mathbf{Q}$). If $\mathbb{H}_i = \mathbb{G}_a$ for all i, then $\phi = 1$. In other cases the set of all such tri-multiplicative morphisms is isomorphic to the set $\mathbb{G}_a(\mathbf{Q})$ such that if $d \in \mathbf{Q}$ then the morphism ϕ_d is given by a formula:*

$$\phi_d(a,b,c) = \exp(dabc), \tag{5.17}$$

where $a \in \mathbb{H}_1(R)$, $b \in \mathbb{H}_2(R)$ and $c \in \mathbb{H}_3(R)$ (for any ring R).

Proof. We consider, first, the case when all $\mathbb{H}_i = \widehat{\mathbb{G}}_a$. Let **Tri** be the set of all tri-multiplicative morphisms: $\widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a \longrightarrow \mathbb{G}_m$. We note that $\widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}} = \operatorname{Spf} \mathbf{Q}[[x_1,x_2,x_3]]$ and $\mathbb{G}_m = \operatorname{Spec} \mathbf{Q}[T,T^{-1}]$. Then any morphism ϕ from the ind-scheme $\operatorname{Spf} \mathbf{Q}[[x_1,x_2,x_3]]$ to the scheme $\mathbb{G}_m$ is given by an invertible series

$$p = \phi^*(T) \in \mathbf{Q}[[x_1,x_2,x_3]].$$

We note that the group law $\widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_a$ is given by the map of rings $\mathbf{Q}[[y]] \to \mathbf{Q}[[x,x']]$, where $y \mapsto x + x'$. The group law $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is given by the maps of rings $\mathbf{Q}[S,S^{-1}] \to \mathbf{Q}[T,T^{-1},T',T'^{-1}]$, where $S \mapsto TT'$. Therefore, using the tri-multiplicativity condition, we obtain that the set **Tri** is described in the following way:

$$\mathbf{Tri} = \left\{ p(x_1,x_2,x_3) \in \mathbf{Q}[[x_1,x_2,x_3]] \;\middle|\; \begin{array}{c} p \in \mathbf{Q}[[x_1,x_2,x_3]]^* \\ p(x_1+x_1',x_2,x_3) = p(x_1,x_2,x_3)p(x_1',x_2,x_3) \\ p(x_1,x_2+x_2',x_3) = p(x_1,x_2,x_3)p(x_1,x_2',x_3) \\ p(x_1,x_2,x_3+x_3') = p(x_1,x_2,x_3)p(x_1,x_2,x_3'). \end{array} \right\}$$

We note that from this description we obtain that the series $p(x_1,x_2,x_3)$ has constant coefficient equal to 1. By applying the log-map to the conditions describing

$p(x_1, x_2, x_3)$, one can see[11] by induction on number of variables that

$$p(x_1, x_2, x_3) = \exp(dx_1x_2x_3)$$

for some $d \in \mathbf{Q}$. The last formula implies formula (5.17).

The other variants for the group ind-schemes $\mathbb{H}_i$ can be done analogously using that $\mathbb{G}_a \times \widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a = \operatorname{Spf} \mathbf{Q}[x_1][[x_2, x_3]]$ and $\mathbb{G}_a \times \mathbb{G}_a \times \widehat{\mathbb{G}}_a = \operatorname{Spf} \mathbf{Q}[x_1, x_2][[x_3]]$. For the case when all $\mathbb{H}_i = \mathbb{G}_a$ we note that $\mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_a = \operatorname{Spec} \mathbf{Q}[x_1, x_2, x_3]$. Therefore the expression $\exp(dx_1x_2x_3)$ belongs to the ring $\mathbf{Q}[x_1, x_2, x_3]$ if and only if $d = 0$. □

From this lemma we have the following formula for any tri-multiplicative morphism $\phi : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to \mathbb{G}_m$, where $\mathbb{U} = \mathbb{P} \times \mathbb{M} \times (1 + \mathcal{N})$:

$$\phi(\exp(au^jt^i), \exp(bu^lt^k), \exp(cu^nt^m)) = \exp(d_{(i,j),(k,l),(m,n)}abc), \tag{5.18}$$

where the group ind-schemes $(1 + \mathcal{N}) \xleftarrow{\exp} \widehat{\mathbb{G}}_a$ are isomorphic, indices (i, j), (k, l), (m, n) are from $\mathbf{Z} \times \mathbf{Z}$, a, b and c are any elements from any ring R (and a or b or c is a nilpotent element if the corresponding index (i, j) or (k, l) or (m, n) is less or equal to $(0, 0)$). Besides, the elements $d_{(i,j),(k,l),(m,n)} \in \mathbf{Q}$ depend only on ϕ and indices (i, j), (k, l), (m, n).

Therefore, to prove that two tri-multiplicative morphisms C_3 and $(\cdot, \cdot, \cdot)$ composed with exp-maps coincide as morphisms $\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbb{H}_3 \to \mathbb{G}_m$, it is enough to prove that the corresponding elements $d_{(i,j),(k,l),(m,n)} \in \mathbf{Q}$ in formula (5.18) coincide for both morphisms. This can be done by the choice of a particular ring R and particular elements a, b and c in this ring. We have a lemma.

Lemma 5.11. *Let $R = \mathbf{Q}[\varepsilon_1, \varepsilon_2, \varepsilon_3]/(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2)$. Let $f = \exp(\varepsilon_1 u^j t^i) \in L^2\mathbb{G}_m(R)$, $g = \exp(\varepsilon_2 u^l t^k) \in L^2\mathbb{G}_m(R)$ and $h = \exp(\varepsilon_3 u^n t^m) \in L^2\mathbb{G}_m(R)$, then*

$$C_3(f, g, h) = (f, g, h) = \exp((lm - nk)\delta_{i+k+m,0}\delta_{j+l+n,0}\varepsilon_1\varepsilon_2\varepsilon_3). \tag{5.19}$$

Proof. The second equality in (5.19) follows from formula (3.15) by easy calculations. We note that we have a well defined map $\langle \cdot, \cdot, \cdot \rangle : K \times K \times K \to \mathbf{Q}$, where $K = \mathbf{Q}((u))((t))$, which is defined by the following equality (this equality is a consequence of formula (5.18) and the continuous property of the map C_3 which follows from Proposition 3.6):

$$1 + \langle p, q, r\rangle \varepsilon_1\varepsilon_2\varepsilon_3 = C_3(1 + p\varepsilon_1, 1 + q\varepsilon_2, 1 + r\varepsilon_3), \tag{5.20}$$

where p, q, r are any elements from K[12]. Since C_3 is a tri-multiplicative, continuous, anti-symmetric and invariant under the change of local parameters map (see Proposition 5.6), the map $\langle \cdot, \cdot, \cdot \rangle$ is tri-linear, continuous, anti-symmetric and invariant under the change of local parameters. Now the first formula in (5.19) coincides with the last formula in (5.19) by Lemma 5.8 (from which it follows that $\langle p, q, r\rangle = c\operatorname{Res}(pdq \wedge dr)$,

[11]Indeed, it is enough to prove that if $F(x) \in S[[x]]$ and $F(x+y) = F(x)+F(y)$ for a $\mathbf{Q}$-algebra S, then $F(x) = cx$ for some $c \in S$. We consider the Taylor formula $F(x + y) = F(x) + F'(x)y + (1/2)F''(x)y^2 + \ldots$. Hence we have that $F(y) = F'(x)y + (1/2)F''(x)y^2 + \ldots$. Therefore $F'(x) = c$ for some $c \in S$. Now, to apply induction we consider the ring $S[[x_3]]$, where $S = \mathbf{Q}[[x_1, x_2]]$.

[12]We note that K is the Lie algebra of the group ind-scheme $\mathbb{U} = \mathbb{P} \times \mathbb{M} \times (1 + \mathcal{N})$. Then by any tri-multiplicative morphism $\phi : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to \mathbb{G}_m$ we can construct a continuous tri-linear map $\langle \cdot, \cdot, \cdot \rangle : K \times K \times K \to \mathbf{Q}$ by a formula analogous to formula (5.20). From the above reasonings we have the fact: the morphism ϕ is uniquely defined by the map $\langle \cdot, \cdot, \cdot \rangle$.

where $c \in \mathbf{Q}$) and from the following easy direct calculation of $\langle \cdot, \cdot, \cdot \rangle$ on particular elements p, q and r:

$$C_3(1+u\varepsilon_1, 1+t\varepsilon_2, 1+u^{-1}t^{-1}\varepsilon_3) = C_3(t+u^{-1}\varepsilon_3, 1+u\varepsilon_1, 1+t\varepsilon_2) =$$
$$= \mathrm{Comm}(\pi_{f'}(1+u\varepsilon_1), \pi_{f'}(1+t\varepsilon_2)) = 1+\varepsilon_1\varepsilon_2\varepsilon_3,$$

where $f' = t + u^{-1}\varepsilon_3$ and we used formula (5.12). □

From Lemma 5.11 we have that

$$d_{(i,j),(k,l),(m,n)} = (lm - nk)\delta_{i+k+m,0}\delta_{j+l+n,0} \tag{5.21}$$

in formula (5.18) for both morphisms C_3 and $(\cdot, \cdot, \cdot)$ composed with exp-maps. Therefore we have checked[13] the property (3.15) for elements $f \in (\mathbb{P} \times \mathbb{M})(R)$, $g \in (\mathbb{P} \times \mathbb{M})(R)$ and $h \in (\mathbb{P} \times \mathbb{M})(R)$.

Now we will check property (3.15) when f, g are any elements from $(\mathbb{P} \times \mathbb{M})(R)$, and h is fixed and equal either to t or to u. As in the previous case, it will be convenient for us to consider elements f and g from the bigger group ind-scheme, i.e. $(\mathbb{P} \times \mathbb{M} \times (1+\mathcal{N}))(R)$, where the group ind-schemes $1+\mathcal{N} \xleftarrow{\exp} \widehat{\mathbb{G}}_a$ are isomorphic. As in the previous case, using Proposition 3.7, we can reduce the proof to the analysis of the following case: $f = \exp(au^jt^i)$ and $g = \exp(bu^lt^k)$, where a, b from the ring R, and $a \in \mathcal{N}R$ if $(i,j) \le 0$, $b \in \mathcal{N}R$ if $(k,l) \le 0$ since in this case they belong to $(\mathbb{P} \times \widehat{\mathbb{G}}_a)(R)$. We consider the map: $\widehat{\mathbb{G}}_a \to \mathbb{P} \times (1+\mathcal{N})$, $c \mapsto \exp(cu^mt^n)$ if $(n,m) \le 0$, and the map $\mathbb{G}_a \to \mathbb{M}$: $c \mapsto \exp(cu^mt^n)$ if $(n,m) > 0$. Using these maps and restricting the maps C_3 and $(\cdot, \cdot, \cdot)$ to elements f and g as above (under fixed indices (i,j) and (k,l) from the set $\mathbf{Z} \times \mathbf{Z}$) and $h = u$ we obtain two bimultiplicative maps

$$\mathbb{H}_1 \times \mathbb{H}_2 \longrightarrow \mathbb{G}_m, \tag{5.22}$$

where the group ind-scheme $\mathbb{H}_i$ $(1 \le i \le 2)$ is isomorphic either to $\mathbb{G}_a$ or to $\widehat{\mathbb{G}}_a$.

Now we have the full analog of Lemma 5.10 with the analogous proof. If $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{G}_a$, then any bimultiplicative morphism of type (5.22) is equal to 1. In other cases of the group ind-schemes $\mathbb{H}_i$ the set all such bimultiplicative morphisms is isomorphic to the set $\mathbb{G}_a(\mathbf{Q})$: by an element $d \in \mathbf{Q}$ we construct the morphism ϕ_d of type (5.22) with an explicit formula

$$\phi_d(a,b) = \exp(dab), \tag{5.23}$$

where $a \in \mathbb{H}_1(R)$ and $b \in \mathbb{H}_2(R)$ for any ring R.

Now to calculate that d coincides for the maps C_3 and $(\cdot, \cdot, \cdot)$ composed with exp-maps as above, we consider the ring $R = \mathbf{Q}[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$ and R-points of group ind-schemes as in (5.22). We need a lemma.

Lemma 5.12. *Let $R = \mathbf{Q}[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. Let $f = \exp(\varepsilon_1 u^j t^i) \in L^2\mathbb{G}_m(R)$, $g = \exp(\varepsilon_2 u^l t^k) \in L^2\mathbb{G}_m(R)$, then*

$$C_3(f,g,u) = (f,g,u) = \exp(i\delta_{i+k,0}\delta_{j+l,0}\varepsilon_1\varepsilon_2). \tag{5.24}$$

[13] We note that from formula (5.21) we immediately obtain, for example, that $C_3(f,g,h) = 1$ if all elements f, g and h belong to $\mathbb{P}(R)$ or all elements f, g and h belong to $\mathbb{M}(R)$

Proof. The second equality in (5.24) follows from formula (3.15) by easy direct calculations. Now it is enough to calculate $C_3(1+u^jt^i\varepsilon_1, 1+u^lt^k\varepsilon_2, u)$. By above reasonings with bimultiplicative maps this expression has to be equal to $1+d\varepsilon_1\varepsilon_2$, where $d \in \mathbf{Q}$ depends on i,j,k,l. We will compute this d. We note that for any $a,b,c \in \mathbf{Q}$ it is true an equality (by formula (3.16) which we already checked):

$$C_3(1+bu^jt^i\varepsilon_1, 1+cu^lt^k\varepsilon_2, a) = 1.$$

We consider the change of local parameters: $t' = 2t$, $u' = 3u$. Since C_3 is invariant under the change of local parameters (by Proposition 5.6) and C_3 is a trimultiplicative map, we have

$$\begin{aligned}1+d\varepsilon_1\varepsilon_2 &= C_3(1+u^jt^i\varepsilon_1, 1+u^lt^k\varepsilon_2, u) = C_3(1+u'^jt'^i\varepsilon_1, 1+u'^lt'^k\varepsilon_2, u') = \\ &= C_3((1+u^jt^i\varepsilon_1)^{2^j3^i}, (1+u^lt^k\varepsilon_2)^{2^l3^k}, u) = 1+2^{j+l}3^{i+k}d\varepsilon_1\varepsilon_2.\end{aligned}$$

Hence we have that $d=0$ if $j+l \neq 0$ or $i+k \neq 0$. It means that we have to compute now $C_3(1+u^jt^i\varepsilon_1, 1+u^{-j}t^{-i}\varepsilon_2, u)$. Without loss of generality we assume that $i \le 0$. If $i=0$, then the last expression is equal to 1, since elements $u, 1+u^j, 1+u^{-j}$ preserve the lattice $R((u))[[t]]$. If $i=-1$, then we consider the change of the local parameter $t' = u^jt$. The map C_3 is invariant under this change. Therefore we obtain

$$\begin{aligned}C_3(1+u^jt^{-1}\varepsilon_1, 1+u^{-j}t\varepsilon_2, u) &= C_3(1+u^jt'^{-1}\varepsilon_1, 1+u^{-j}t'\varepsilon_2, u) = \\ &= C_3(1+t^{-1}\varepsilon_1, 1+t\varepsilon_2, u) = C_3(t+\varepsilon_1, 1+t\varepsilon_2, u) = 1-\varepsilon_1\varepsilon_2.\end{aligned}$$

(Here the last equality is easily calculated in the ring $R((u))[[t]]/(t+\varepsilon_1)R((u))[[t]]$ by formula (5.12).)

Therefore we suppose that $i < -1$. Then $t' = t + u^{-j}t^{-i}$ is a well-defined change of the local parameter t. We have

$$\begin{aligned}1 = C_3(1+u^jt^i\varepsilon_1, 1+t\varepsilon_2, u) &= C_3(1+u^jt'^i\varepsilon_1, 1+t'\varepsilon_2, u) = \\ &= C_3(1+(u^jt^i+it^{-1}+\ldots)\varepsilon_1, 1+(t+u^{-j}t^{-i})\varepsilon_2, u).\end{aligned}$$

Hence, using the continuous property of C_3 (see Proposition 3.6), we obtain that

$$\begin{aligned}C_3(1+u^jt^i\varepsilon_1, 1+u^{-j}t^{-i}\varepsilon_2, u) &= C_3(1+t^{-1}\varepsilon_1, 1+t\varepsilon_2, u)^{-i} = \\ &= C_3(t+\varepsilon_1, 1+t\varepsilon_2, u)^{-i} = 1+i\varepsilon_1\varepsilon_2.\end{aligned}$$

□

Now from formula (5.24) we see that $d = i\delta_{i+k,0}\delta_{j+l,0}$ for both maps C_3 and $(\cdot,\cdot,\cdot)$ composed with exp-maps on indices (i,j), (k,l) from the set $\mathbf{Z}\times\mathbf{Z}$. Thus, by above reasonings, we have checked formula (3.15) when elements f and g are from $(\mathbb{P}\times\mathbb{M})(R)$ and $h=u$.

Formula (3.15) with elements f and g from $(\mathbb{P}\times\mathbb{M})(R)$ and $h=t$ follows from the same arguments and a lemma.

Lemma 5.13. *Let* $R = \mathbf{Q}[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_2^2)$. *Let* $f = \exp(\varepsilon_1u^jt^i) \in L^2\mathbb{G}_m(R)$, $g = \exp(\varepsilon_2u^lt^k) \in L^2\mathbb{G}_m(R)$, *then*

$$C_3(f,g,t) = (f,g,t) = \exp(l\delta_{i+k,0}\delta_{j+l,0}\varepsilon_1\varepsilon_2).$$

Proof. The second equality follows from formula (3.15) by easy direct calculations. Now it is enough to calculate $C_3(1+u^jt^i\varepsilon_1, 1+u^lt^k\varepsilon_2, t)$. By above reasonings with bimultiplicative maps this expression has to be equal to $1+d\varepsilon_1\varepsilon_2$, where $d \in \mathbf{Q}$

depends on i,j,k,l. We will compute this d. We note that for any $a,b,c \in \mathbf{Q}$ it is true an equality:

$$C_3(1 + bu^j t^i \varepsilon_1, 1 + cu^l t^k \varepsilon_2, a) = 1.$$

We consider the change of local parameters: $t' = 2t$, $u' = 3u$. Since C_3 is an invariant under the change of local parameters and tri-multiplicative map, we have

$$1 + d\varepsilon_1\varepsilon_2 = C_3(1 + u^j t^i \varepsilon_1, 1 + u^l t^k \varepsilon_2, t) = C_3(1 + {u'}^j {t'}^i \varepsilon_1, 1 + {u'}^l {t'}^k \varepsilon_2, t') =$$
$$= C_3((1 + u^j t^i \varepsilon_1)^{2^j 3^i}, (1 + u^l t^k \varepsilon_2)^{2^l 3^k}, t) = 1 + 2^{j+l} 3^{i+k} d\varepsilon_1\varepsilon_2.$$

Hence we have that $d = 0$ if $j + l \neq 0$ or $i + k \neq 0$. It means that we have to compute now $C_3(1 + u^j t^i \varepsilon_1, 1 + u^{-j} t^{-i} \varepsilon_2, t)$. Omitting trivial cases (when by formula (5.12) the computation is done in $R((u)))$, we can suppose that $(-i,-j) > (0,1)$. Then $u' = u + u^{-j} t^{-i}$ is a well-defined change of the local parameter u. We have

$$1 = C_3(1 + u^j t^i \varepsilon_1, 1 + u\varepsilon_2, t) = C_3(1 + {u'}^j t^i \varepsilon_1, 1 + u'\varepsilon_2, t) =$$
$$= C_3(1 + (u^j t^i + ju^{-1} + \ldots)\varepsilon_1, 1 + (u + u^{-j} t^{-i})\varepsilon_2, t).$$

Hence, using the continuous property of the map C_3, we obtain that

$$C_3(1 + u^j t^i \varepsilon_1, 1 + u^{-j} t^{-i} \varepsilon_2, t) = C_3(1 + u^{-1}\varepsilon_1, 1 + u\varepsilon_2, t)^{-j} = 1 - j\varepsilon_1\varepsilon_2.$$

□

At the end we have to verify that $C_3(f,t,t) = C_3(f,u,u) = C_3(f,u,t) = 1$ when $f \in (\mathbb{P} \times \mathbb{M})(R)$. It can be done, for example, in the following way. We fix g and h from the two-element set $\{t,u\}$. Arguing as above and using the facts $\underline{\mathrm{Hom}}_{\mathrm{gr}}(\mathbb{G}_a, \mathbb{G}_m) = \widehat{\mathbb{G}}_a$ and $\underline{\mathrm{Hom}}_{\mathrm{gr}}(\widehat{\mathbb{G}}_a, \mathbb{G}_m) = \mathbb{G}_a$ we see that it is enough to prove an equality:

$$C_3(1 + u^j t^i \varepsilon, g, h) = 1.$$

when the ring $R = \mathbf{Q}[\varepsilon]/\varepsilon^2$ and $(i,j) \in \mathbf{Z} \times \mathbf{Z} \setminus (0,0)$. If $(i,j) > 0$, then this equality follows from the fact $\widehat{\mathbb{G}}_a(\mathbf{Q}) = 1$. If $(i,j) < 0$, then we have that $C_3(1 + u^j t^i \varepsilon, g, h) = 1 + d\varepsilon$ where $d \in \mathbb{G}_a(\mathbf{Q})$. We consider a change of local parameters: $u' = 2u$, $t' = 3t$. Then we obtain

$$1 + d\varepsilon = C_3(1 + u^j t^i \varepsilon, g, h) = C_3(1 + {u'}^j {t'}^i \varepsilon, g, h) = C_3(1 + 2^j 3^i u^j t^i \varepsilon, g, h) = 1 + 2^j 3^i d\varepsilon.$$

Hence we have $d = 0$. We have finished to verify the last case of the theorem. □

Corollary 5.14. *Let $R \in \mathcal{B}$. The two-dimensional Contou-Carrère symbol $(\cdot,\cdot,\cdot)$ given by the explicit formula from Definition 3.5 and the map C_3 coincide as the maps from $(L^2\mathbb{G}_m)^3(R)$ to $\mathbb{G}_m(R)$.*

Proof. The map C_3 is functorial with respect to the ring R. By Proposition 3.16, the two-dimensional Contou-Carrère symbol $(\cdot,\cdot,\cdot)$ given by an explicit formula from Definition 3.5 is also functorial with respect to the ring R. Therefore using Lemma 3.15 we can reduce the proof to the case of a $\mathbf{Q}$-algebra R from $\mathcal{B}$. Now we apply Theorem 5.9 and the end of Proposition 3.16 to show that these two maps coincide. □

6. Reciprocity laws

In this section we fix a perfect field k and a local finite k-algebra R.

Let V be a Tate R-module. We note that any projective R-module is a free. Since the ring R is an Artinian ring, any connected Nisnevich covering of $\mathrm{Spec}\, R$ is $\mathrm{Spec}\, R$ itself. Hence, using Drinfeld's theorem (see [Dr, Th. 3.4] and also an explicit exposition in [BBE, §2.12]), we obtain that V is an elementary Tate R-module, i.e. $V = P \oplus Q^*$, where P and Q are discrete free R-modules.

We will use also the following remark.

Remark 6.1. Let $\phi : M \to N$ be an open continuous surjection between Tate R-modules such that these topological R-modules have countable bases of open neighborhoods of 0. Then by Lemma 5.3 we have topological decomposition $M = \mathrm{Ker}\,\phi \oplus N$ and $\mathrm{Ker}\,\phi$ is a Tate R-module. We have also a canonical isomorphism of $\mathcal{P}ic_R^{\mathbb{Z}}$-torsors:

$$\mathcal{D}et(\mathrm{Ker}\,\phi) + \mathcal{D}et(N) \longrightarrow \mathcal{D}et(M),$$

because for any coprojective lattices L_1 from $\mathrm{Ker}\,\phi$ and L_2 from N we have that $L_1 \oplus L_2$ is a coprojective lattice in M (in other words, we can find a coprojective lattice in M such that its intersection with $\mathrm{Ker}\,\phi$ and its image in N will be coprojective lattices in $\mathrm{Ker}\,\phi$ and in N correspondingly).

Now we introduce the norm map which generalizes the usual norm map for extensions of fields. Let $L \supset K \supset k$ will be finite extensions of fields. We recall that R is a finite local k-algebra and k is a perfect field. We *define* the norm map

$$\mathrm{Nm}_{L/K} \ : \ (L \otimes_k R)^* \longrightarrow (K \otimes_k R)^*, \quad a \mapsto \prod_{i=1}^{n} \sigma_i(a), \tag{6.1}$$

where the σ_i are all the isomorphisms of L into the algebraic closure $\bar{K}$ of K fixing the elements of K, and σ_i is extended to the ring homomorphism $L \otimes_k R \to \bar{K} \otimes_k R$ by the natural rule $b \otimes x \mapsto \sigma_i(b) \otimes x$ where $b \in L$, $x \in R$. It is clear that the map $\mathrm{Nm}_{L/K}$ is well-defined[14] and this map is a multiplicative map. Besides, from the corresponding property of isomorphisms of fields we have that for any finite extension $M \supset L$ of fields the maps $\mathrm{Nm}_{L/K} \circ \mathrm{Nm}_{M/L}$ and $\mathrm{Nm}_{M/K}$ from $(M \otimes_k R)^*$ to $(K \otimes R)^*$ coincide.

Let X be a smooth connected algebraic surface over k. For any closed point $x \in X$ let $\hat{\mathcal{O}}_x$ be a completion of the local ring $\mathcal{O}_x$ at the point x. Let K_x be the localization of the ring $\hat{\mathcal{O}}_x$ with respect to the multiplicative system $\mathcal{O}_x \setminus 0$. For any irreducible curve C on X (in other words, for any integral one-dimensional subscheme C of X) let K_C be a field which is the completion of the field $k(X)$ with respect to the discrete valuation given by the curve C. By any pair $x \in C$, where x is a closed point and $C \subset X$ is an irreducible curve (which contains x) we will canonically construct the ring $K_{x,C}$ (see also details in a survey [O4]). We consider the decomposition

$$C \mid_{\mathrm{Spec}\, \hat{\mathcal{O}}_x} = \bigcup_{i=1}^{s} \mathbf{C}_i, \tag{6.2}$$

[14] To see that $\mathrm{Nm}_{L/K} \subset (K \otimes R)^*$ we note that $(\bar{K} \otimes_K R)^{\mathrm{Gal}\, \bar{K}/K} = K \otimes R$.

where every $\mathbf{C}_i$ is an integral one-dimensional subscheme in $\operatorname{Spec}\hat{\mathcal{O}}_x$. We *define*

$$K_{x,C} = \prod_{i=1}^{s} K_i, \tag{6.3}$$

where the field $K_i = K_{x,\mathbf{C}_i}$ is the completion of the field $\operatorname{Frac}\hat{\mathcal{O}}_x$ with respect to the discrete valuation given by $\mathbf{C}_i$.

We consider any one-dimensional integral subscheme $\mathbf{C} \subset \operatorname{Spec}\hat{\mathcal{O}}_x$. We consider the field $K = K_{x,\mathbf{C}}$ which is the completion of the field $\operatorname{Frac}\hat{\mathcal{O}}_x$ with respect to the discrete valuation given by $\mathbf{C}$. Let M be the residue field of the discrete valuation field K. Since $\hat{\mathcal{O}}_x = k(x)[[v, w]]$ where $k(x)$ is the residue field of the point x, by the Weierstrass preparation theorem we have that the field M is a finite extension of at least one of the fields: $k(x)((v))$ or $k(x)((w))$. Therefore M is a complete discrete valuation field with the residue field k' which is a finite extension of the field $k(x)$. Hence

$$K = M((t)) = k'((u))((t)) \tag{6.4}$$

for some u and t from $\operatorname{Frac}\hat{\mathcal{O}}_x$, i.e K is a two-dimensional local field. (We used that on K there is the natural topology of inductive and projective limits which extends the topology on $k(x)[[u, v]]$. This topology comes to the topology of two-dimensional local field under isomorphisms (6.4).)

Let F be a field such that $k \subset F \subset k(x)$. Let L be a field such that $L \supset k'$ and $L \supset F$ is a finite Galois extension. We have that

$$k' \otimes_F L = \prod_{l=1}^{m} L_{\sigma_l},$$

where $L_{\sigma_l} = L$ and the product is taken over all the isomorphisms σ_l of k' into L fixing the elements of F, and $m = [k' : k]$. We consider a scheme

$$Y = \operatorname{Spec}\hat{\mathcal{O}}_x \times_{\operatorname{Spec} F} \operatorname{Spec} L$$

with a canonical morphism p from this scheme to $\operatorname{Spec}\hat{\mathcal{O}}_x$. Let $\mathbf{T}$ be the set of all pairs $y \in D$ where D is a one-dimensional integral subscheme of Y and y is a closed point on D such that $p(y) = x$ and $p(D) = \mathbf{C}$. From the properties of complete discrete valuation fields and extensions of valuations we have isomorphisms of rings

$$\prod_{l=1}^{m} L_{\sigma_l}((u))((t)) = k'((u))((t)) \otimes_F L = K \otimes_F L = \prod_{\{y \in D\} \in \mathbf{T}} K_{y,D}, \tag{6.5}$$

where any $K_{y,D}$ is a two-dimensional local field and it is equal to $L_{\sigma_l}((u))((t))$ for some l. We have that the set $\mathbf{T}$ consists of m elements and the group $\operatorname{Gal}(L/F)$ acts by permutations of direct summands in (6.5) such that this group acts on L_{σ_l} as $\sigma(e_{\sigma_l}) = \sigma(e)_{\sigma\sigma_l}$ for $\sigma \in \operatorname{Gal}(L/F)$ and $e_{\sigma_l} \in L_{\sigma_l}$. Besides, we have that in view of isomorphisms (6.5) the embedding $K \to K \otimes_F L : x \mapsto x \otimes 1$ is given as

$$\sum_{p,q} a_{p,q} u^q t^p \longmapsto \prod_{l=1}^{m} \sum_{p,q} \sigma_l(a_{p,q}) u^q t^p \quad \text{where } a_{p,q} \in k'. \tag{6.6}$$

For any pair $x \in C$ as above, we have in formula (6.3) that $K_i = k_i((u_i))((t_i))$, where the field k_i is a finite extension of the field $k(x)$. We consider

$$K_{x,C} \otimes_k R = \prod_{i=1}^{s} (K_i \otimes_k R) = \prod_{i}^{s} (k_i \otimes_k R)((u_i))((t_i)).$$

Let the map $(\cdot,\cdot,\cdot)_i : ((k_i \otimes_k R)((u_i))((t_i))^*)^3 \to (k_i \otimes_k R)^*$ be the two-dimensional Contou-Carrère symbol. We *define* a map:

$$(\cdot,\cdot,\cdot)_{x,C} \quad : \quad (K_{x,C} \otimes_k R)^* \times (K_{x,C} \otimes_k R)^* \times (K_{x,C} \otimes_k R)^* \longrightarrow (k(x) \otimes_k R)^*$$

in the following way[15]

$$(f,g,h)_{x,C} = \prod_{i=1}^{s} \mathrm{Nm}_{k_i/k(x)} (f_i, g_i, h_i)_i \tag{6.7}$$

where $f = \prod_{i=1}^{s} f_i$, $f_i \in (K_i \otimes_k R)^*$, and the same notations we take for g and h.

We note that for any irreducible curve $C \subset X$ we have the canonical embedding $K_C \hookrightarrow K_{x,C}$ for any closed point $x \in C$. This embedding induces a map $K_C \otimes_k R \to K_{x,C} \otimes_k R$.

For any closed point $x \in X$ we have the canonical embedding $K_x \hookrightarrow K_{x,C}$ for any irreducible curve $C \subset X$ which contains the point x. This embedding induces a map $K_x \otimes_k R \to K_{x,C} \otimes_k R$.

Theorem 6.1 (Reciprocity laws for the two-dimensional Contou-Carrère symbol)**.** *Let X be a smooth connected algebraic surface over a perfect field k. Let R be a local finite k-algebra. The following reciprocity laws are satisfied.*

(1) *Let x be a closed point on X. Then for any f, g and h from $(K_x \otimes_k R)^*$ we have*

$$\prod_{C \ni x} (f,g,h)_{x,C} = 1, \tag{6.8}$$

where this product is taken over all irreducible curves C containing the point x on X and in the product only finitely many terms are distinct from 1.

(2) *Let C be a projective irreducible curve on X. Then for any f, g and h from $(K_C \otimes_k R)^*$ we have*

$$\prod_{x \in C} \mathrm{Nm}_{k(x)/k} (f,g,h)_{x,C} = 1, \tag{6.9}$$

where this product is taken over all closed points x on C and in the product only finitely many terms are distinct from 1.

Proof. Before to prove parts (1) and (2) of the theorem we will make some general remarks which will be useful for the proof of both parts.

We note that formula (6.8) depends only on the two-dimensional local regular ring $\mathcal{O}_x$.

From formula (6.5) and the above description of the field $K_{x,\mathbf{C}}$ we have that if an integral one-dimensional subscheme $\mathbf{C} \subset \operatorname{Spec} \hat{\mathcal{O}}_x$ (for some point x) splits in $\operatorname{Spec}(\hat{\mathcal{O}}_x \hat{\otimes}_F \bar{k})$ (where $k \subset F \subset k(x)$ and the field $\bar{k}$ is an algebraic closure of k), then it splits on the same irreducible components over some finite Galois extension

[15] We note that from § 5 we have that $(\cdot,\cdot,\cdot)$ does not depend on the choice of local parameters u_i and t_i.

$L \supset k$. Conversely, if $\bar{C} \subset X \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ is an irreducible curve, then $\bar{C}$ is defined over some finite extension $L \supset k$. Now every $\bar{\mathbf{C}}_i$ (see formula (6.2)) which is defined over L comes from some $\bar{\mathbf{C}} \subset \mathrm{Spec}\, \hat{\mathcal{O}}_x$ after the base change given by the field extension from $k(x)$ to L.

Using formula (6.6), the definition of the norm map given by formula (6.1) and the functoriality of $(\cdot,\cdot,\cdot)$ we reduce at once the proof of this theorem to the case of an algebraically closed ground field if we consider the scheme $\mathrm{Spec}\, \mathcal{O}_x \times_{\mathrm{Spec}\, k(x)} \mathrm{Spec}\, \bar{k}$ instead of $\mathrm{Spec}\, \mathcal{O}_x$ for the proof of formula (6.8) and the scheme $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, (\bar{k})$ instead of X for the proof of formula (6.9). So, we assume that $k = \bar{k}$ is an algebraically closed field. Besides, we can omit the norm maps in formulas (6.7) and (6.9).

We can assume that X is connected.

By Corollary 5.14 of Theorem 5.9 we know that $(\cdot,\cdot,\cdot) = C_3$. Therefore we will prove the reciprocity laws (6.8) and (6.9) for the map C_3. Our strategy to prove these reciprocity laws is to repeat the proof of Theorem 5.3 from [OsZh], but to change the scheme X to the scheme $X_R = X \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ and consider local rings and adelic complexes on the scheme X_R which has the same topological space as the scheme X.

(1) We will give a sketch of the proof of formula (6.8). We fix a point $x \in X$. Similarly to [OsZh, § 5B] we look at the scheme $U_{x,R} = (\mathrm{Spec}\, \hat{\mathcal{O}}_x \otimes_k R) \setminus x$. For any $f \in (\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$ we consider the coherent subsheaf $f \cdot (\hat{\mathcal{O}}_x \otimes_k R)$ of the constant sheaf $\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R$ on the scheme $\mathrm{Spec}\, \hat{\mathcal{O}}_x \otimes_k R$. The adelic complex $\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))$ of the restriction of the sheaf $f \cdot (\hat{\mathcal{O}}_x \otimes_k R)$ to the scheme $U_{x,R}$ looks as follows[16]:

$$\mathbb{A}_{X,x,0,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \oplus \mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \longrightarrow \mathbb{A}_{X,x,01,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))$$

and this complex is isomorphic to the subcomplex $f \cdot (\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x) \otimes_k R)$ of the complex $\mathcal{A}_{X,x}(\mathrm{Frac}\hat{\mathcal{O}}_x) \otimes_k R$, where $\mathcal{A}_{X,x}(\cdot)$ is the corresponding adelic complex when $R = k$. Therefore we have

$$\begin{aligned} \mathbb{A}_{X,x,0,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) &= \mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R \\ \mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) &= \prod_{\mathbf{C} \ni x} f \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) \\ \mathbb{A}_{X,x,01,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) &= \prod_{\mathbf{C} \ni x}{}' K_{x,\mathbf{C}} \otimes_k R, \end{aligned}$$

where $\mathbf{C}$ runs over all one-dimensional integral subschemes in $\mathrm{Spec}\, \hat{\mathcal{O}}_x$, a ring $\hat{\mathcal{O}}_{x,\mathbf{C}}$ is the discrete valuation ring in the two-dimensional local field $K_{x,\mathbf{C}}$, an expression $f \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R)$ is considered inside of the ring $K_{x,\mathbf{C}} \otimes_k R$, and $\prod'$ is the restricted product with respect to the rings $\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R$. Besides,

$$H^0(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))) = f \cdot (\hat{\mathcal{O}}_x \otimes_k R)$$

[16] For various f the corresponding adelic complexes $\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))$ are isomorphic because the sheaves are isomorphic. We are interested in the position of this adelic complex inside of the adelic complex of the restriction of the constant sheaf $\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R$ to the scheme $U_{x,R}$

is a compact Tate R-module (i.e. it is dual to the free discrete R-module), and $H^1(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)))$ is isomorphic to $H^1(\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x)) \otimes_k R$ which is a discrete Tate R-module.

Let f and g are from $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$. For any $\mathbf{C}$ we have that f and g belong to $(K_{x,\mathbf{C}} \otimes_k R)^*$. Therefore for any $\mathbf{C}$ there is $n_{\mathbf{C}} \in \mathbf{N}$ such that inside the ring $K_{x,C} \otimes_k R$ we have

$$t_{\mathbf{C}}^{n_{\mathbf{C}}} \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R \subset f \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) \qquad \text{and} \qquad t_{\mathbf{C}}^{n_{\mathbf{C}}} \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R \subset g \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R),$$

where $t_{\mathbf{C}} \in \hat{\mathcal{O}}_x$ gives an equation $t_{\mathbf{C}} = 0$ of $\mathbf{C}$ in $\mathrm{Spec}\, \hat{\mathcal{O}}_x$. Since we can take $n_{\mathbf{C}} = 0$ for almost all $\mathbf{C}$, an element $h = \prod\limits_{\mathbf{C} \ni x} t_{\mathbf{C}}^{n_{\mathbf{C}}}$ from $\hat{\mathcal{O}}_x$ is well-defined. Thus, we have constructed the element $h \in (\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$ such that we have the following embeddings of adelic complexes

$$\mathcal{A}_{X,x,R}(h \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \subset \mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))$$
$$\mathcal{A}_{X,x,R}(h \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \subset \mathcal{A}_{X,x,R}(g \cdot (\hat{\mathcal{O}}_x \otimes_k R)).$$

We note that among such constructed elements h there is a "minimal" element when all the integers $n_{\mathbf{C}}$ are minimal. We have that

$$\mathbb{A}_{X,x,1,R}(f_1 \cdot (\hat{\mathcal{O}}_x \otimes_k R)) / \mathbb{A}_{X,x,1,R}(f_2 \cdot (\hat{\mathcal{O}}_x \otimes_k R)) = \bigoplus_{C \ni x} (f_1 \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R)) / (f_2 \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R)),$$

where f_1 and f_2 are from $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$ such that $f_1 \cdot (\hat{\mathcal{O}}_x \otimes_k R) \supset f_2 \cdot (\hat{\mathcal{O}}_x \otimes_k R)$ is a Tate R-module, because for almost all $\mathbf{C} \ni x$ we have

$$f_1 \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) = f_2 \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) = \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R.$$

We have that

$$\begin{aligned}\mathcal{D}et(\mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \mid \mathbb{A}_{X,x,1,R}(g \cdot (\hat{\mathcal{O}}_x \otimes_k R))) = \\ = \mathcal{D}et(\mathbb{A}_{X,x,1,R}(g \cdot (\hat{\mathcal{O}}_x \otimes_k R)) / \mathbb{A}_{X,x,1,R}(h \cdot (\hat{\mathcal{O}}_x \otimes_k R))) - \\ - \mathcal{D}et(\mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) / \mathbb{A}_{X,x,1,R}(h \cdot (\hat{\mathcal{O}}_x \otimes_k R)))\end{aligned}$$

is a well-defined $\mathcal{P}ic_R^{\mathbb{Z}}$-torsor. Using the explicit description of the cohomology groups of the adelic complex given above, we have that for any $f \in (\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$

$$\begin{aligned}\mathcal{D}et(H^*(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)))) = \mathcal{D}et(H^0(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)))) - \\ - \mathcal{D}et(H^1(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R))))\end{aligned}$$

is a well-defined $\mathcal{P}ic_R^{\mathbb{Z}}$-torsor. Using Remark 6.1 and decompose long exact cohomological sequences into the split-exact short sequences of Tate R-modules (when $f \cdot (\hat{\mathcal{O}}_x \otimes_k R) \subset g \cdot (\hat{\mathcal{O}}_x \otimes_k R)$) we obtain that

$$\begin{aligned}\mathcal{D}et(\mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)) \mid \mathbb{A}_{X,x,1,R}(g \cdot (\hat{\mathcal{O}}_x \otimes_k R))) = \\ = \mathcal{D}et(H^*(\mathcal{A}_{X,x,R}(g \cdot (\hat{\mathcal{O}}_x \otimes_k R)))) - \mathcal{D}et(H^*(\mathcal{A}_{X,x,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)))).\end{aligned}$$

Hence we have a trivialization (given by multiplications on the elements of the group $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$) of a categorical central extension

$$f \mapsto \mathcal{D}et(\mathbb{A}_{X,x,1,R}(\hat{\mathcal{O}}_x \otimes_k R) \mid \mathbb{A}_{X,x,1,R}(f \cdot (\hat{\mathcal{O}}_x \otimes_k R)))$$

over the group $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$. From this fact, as in the proof of Theorem 5.3 from [OsZh] we obtain the reciprocity law around the point x for any elements f, g and h from the group $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$ when we take in formula (6.8) the product over all one-dimensional integral subschemes $\mathbf{C}$ of the scheme $\mathrm{Spec}\, \hat{\mathcal{O}}_x$. This product contain only finitely many terms distinct from 1, because for almost all $\mathbf{C}$ we have that the elements f, g and h preserve the lattice $\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R$ in $K_{x,\mathbf{C}} \otimes_k R$. Therefore the coresponding $C_3(f,g,h)$ is equal to 1 for such $\mathbf{C}$. To obtain formula (6.8) itself, i.e. when we take elements f, g and h from the ring $K_x \otimes_k R$ and when the product in this formula is taken over all irreducible curves $C \subset X$ such that $C \ni x$, we note that

$$f \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) = g \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) = h \cdot (\hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R) = \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_k R$$

when $\mathbf{C}$ is not a formal branch of some irreducible curve $C \subset X$. Therefore $C_3(f,g,h) = 1$ for such $\mathbf{C}$. Thus formula (6.8) follows from the previous product formula for elements from the group $(\mathrm{Frac}(\hat{\mathcal{O}}_x) \otimes_k R)^*$ and all $\mathbf{C}$.

(2) We will give a sketch of the proof of formula (6.9). We fix an irreducible projective curve C on X. At first, we prove formula (6.9) when elements f, g and h are from the group $(k(X) \otimes_k R)^*$. On the scheme X_R we consider the following set E of invertible subsheaves of the constant sheaf $k(X) \otimes_k R$ on the scheme X_R:

$$\mathcal{F} \in E \qquad \text{iff} \qquad \mathcal{F} = g \cdot \mathcal{O}_{X_R}(D)$$

for some $g \in (k(X) \otimes_k R)^*$ and a divisor D on X. It is clear that for any subsheaves $\mathcal{F}$ and $\mathcal{G}$ from the set E there is a sheaf $\mathcal{O}_{X_R}(D)$ for some divisor D on X such that inside the sheaf $k(X) \otimes_k R$ we have

$$\mathcal{F} \subset \mathcal{O}_{X_R}(D) \qquad , \qquad \mathcal{G} \subset \mathcal{O}_{X_R}(D).$$

Moreover, we can find a "minimal" sheaf with the above property. Using it, we can find also a divisor G on X such that

$$\mathcal{F} \supset \mathcal{O}_{X_R}(G) \qquad , \qquad \mathcal{G} \supset \mathcal{O}_{X_R}(G). \tag{6.10}$$

Let J_C be the ideal sheaf of the curve C on X. Let $J_{C,R} = J_C \otimes_k R$. For any sheaf $\mathcal{F} \in E$ we consider a complex

$$\mathcal{A}_{X,C,R}(\mathcal{F}) = \varinjlim_{n} \varprojlim_{m>n} \mathcal{A}_{(C_R, \mathcal{O}_{x,R}/J_{C,R}^{m-n})}(\mathcal{F} \otimes_{\mathcal{O}_{X,R}} J^n_{C,R}/J^m_{C,R}),$$

where $(C_R, \mathcal{O}_{x,R}/J_{C,R}^{m-n})$ is the scheme with the topological space as the topological space of the scheme $C_R = C \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ and the structure sheaf as the sheaf $\mathcal{O}_{x,R}/J_{C,R}^{m-n}$, and $\mathcal{A}_{(C_R, \mathcal{O}_{x,R}/J_{C,R}^{m-n})}(\cdot)$ is the functor of the adelic complex on this scheme applied to coherent sheaves on it. We have that the cohomology groups $H^*(\mathcal{A}_{X,C,R}(\mathcal{F}))$ are isomorphic to $\varinjlim_{n} \varprojlim_{m>n} H^*(X, \mathcal{O}(D) \otimes_{\mathcal{O}_X} J^n_C/J^m_C) \otimes_k R$ for some divisor D on X. Using the case when $R = k$ (see [OsZh, § 5B] and the proof of Proposition 12 from [O2]), we have[17] that $H^0(\mathcal{A}_{X,C,R}(\mathcal{F}))$ is a Tate R-module, and $\tilde{H}^1(\mathcal{A}_{X,C,R}(\mathcal{F}))$ which is the quotient space of $H^1(\mathcal{A}_{X,C,R}(\mathcal{F}))$ by the closure of 0 is a Tate R-module. We introduce a $\mathcal{P}ic^{\mathbb{Z}}_R$-module

$$\mathcal{D}et(H^*(\mathcal{A}_{X,C,R}(\mathcal{F}))) = \mathcal{D}et(H^0(\mathcal{A}_{X,C,R}(\mathcal{F}))) - \mathcal{D}et(\tilde{H}^1(\mathcal{A}_{X,C,R}(\mathcal{F}))).$$

[17] It is important that C is a projective curve. Therefore for any $m > n$ we have $\dim_k H^i(X, \mathcal{O}(D) \otimes_{\mathcal{O}_X} J^n_C/J^m_C) < \infty$ where i is equal to 1 or to 2.

The adelic complex $\mathcal{A}_{X,C,R}(\mathcal{F})$ looks as follows

$$\mathbb{A}_{X,C,0,R}(\mathcal{F}) \oplus \mathbb{A}_{X,C,1,R}(\mathcal{F}) \longrightarrow \mathbb{A}_{X,C,01,R}(\mathcal{F}),$$

where

$$\mathbb{A}_{X,C,0,R}(\mathcal{F}) = K_C \otimes_k R \qquad , \qquad \mathbb{A}_{X,C,01,R}(\mathcal{F}) = \mathbb{A}_C((t_C)) \otimes_k R$$

and t_C is a local parameter of the curve C on some open affine subset of X, $\mathbb{A}_C$ is the ring of adeles on the curve C. Besides,

$$\mathbb{A}_{X,C,1,R}(\mathcal{F}) = (\prod_{x \in C} ((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{F})) \cap (\mathbb{A}_C((t_C)) \otimes_k R),$$

where[18] the intersection is taken inside of the ring $\prod\limits_{x \in C} K_{x,C} \otimes_k R$, and B_x is the subring of the ring $K_{x,C}$ given as $\varinjlim\limits_{n>0} s_C^{-n} \hat{\mathcal{O}}_x$ for $s_C \in \mathcal{O}_x$ which defines the curve C on some local affine set on X containing the point x (clearly, the ring B_x does not depend on the choice of s_C).

We fix two sheaves $\mathcal{F} \supset \mathcal{H}$ from the set E. For almost all points $x \in C$ we have

$$((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{F})/((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{H}) = 0.$$

Therefore

$$\mathbb{A}_{X,C,1,R}(\mathcal{F})/\mathbb{A}_{X,C,1,R}(\mathcal{H}) = \bigoplus_{x \in C} ((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{F})/((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{H})$$

is a Tate R-module, because for any point $x \in C$ we have that

$$((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{F})/((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{H})$$

is a Tate R-module as it follows from a lemma.

Lemma 6.2. *Let $g \in (\mathrm{Frac}\hat{\mathcal{O}}_x \otimes_k R)^*$ such that $g \cdot (\hat{\mathcal{O}}_x \otimes_k R) \subset \hat{\mathcal{O}}_x \otimes_k R$. Then we have that $(B_x \otimes_k R)/(g \cdot (B_x \otimes_k R))$ is a Tate R-module.*

Proof. The beginning of the proof is similar to the proof of Proposition 5.4. Let $K = K_{x,C} \otimes_k R$ and $B_R = B_x \otimes_k R$. Then the map $K \to K/B_R$ is splittable (i.e. admits a continuous splitting), because the map $K_{x,C} \to K_{x,C}/B_x$ is splittable. The multiplication by g is a continuous automorphism of K. Therefore the map $K \to K/gB_R$ is splittable. The restriction of the last splitting gives the splitting of the map $B_R \to B_R/gB_R$. Thus we have a topological decomposition:

$$B_R = (B_R/gB_R) \oplus gB_R.$$

Since R is a local finite k-algebra, there is an element $f \in (\mathrm{Frac}\hat{\mathcal{O}}_x)^*$ such that $g \cdot (\hat{\mathcal{O}}_x \otimes_k R) \supset f \cdot (\hat{\mathcal{O}}_x \otimes_k R)$ (it is easy to see it directly by localizing the regular ring $\hat{\mathcal{O}}_x$ at all prime ideals of height 1, or one can look at the adelic complex $\mathcal{A}_{X,x,R}(\cdot)$ considered above and its zero cohomology group). Therefore $gB_R \supset fB_R$. As before, we prove that there is a topological decomposition: $gB_R = (gB_R/fB_R) \oplus fB_R$. Therefore we have

$$B_R = (B_R/gB_R) \oplus (gB_R/fB_R) \oplus fB_R.$$

[18]Here and later we use the following notation. The ring $B_x \otimes_k R$ is an $\mathcal{O}_{x,R} = \mathcal{O}_x \otimes_k R$-module. We mean $(B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} \mathcal{F} = (B_x \otimes_k R) \otimes_{\mathcal{O}_{x,R}} \mathcal{F}_x$, where $\mathcal{F}_x$ is a stalk of the sheaf $\mathcal{F}$ at the point $x \in X_R$, i.e. $\mathcal{F}_x$ is an $\mathcal{O}_{x,R}$-module.

Hence we have that the R-module B_R/gB_R is a topological direct summand of the R-module B_R/fB_R. Since $f \in (\mathrm{Frac}\hat{\mathcal{O}}_x)^*$, we obtain $B_R/fB_R = (B_x/fB_x) \otimes_k R$. Now B_x/fB_x is a Tate k-vector space. Therefore B_R/fB_R is a Tate R-module. Hence, B_R/gB_R is a Tate R-module. □

Now, using Remark 6.1 (for long cohomological sequences of Tate R-modules) we have for any sheaves $\mathcal{F}$ and $\mathcal{G}$ from E

$$\mathcal{D}et(H^*(\mathcal{A}_{X,C,R}(\mathcal{F}))) - \mathcal{D}et(H^*(\mathcal{A}_{X,C,R}(\mathcal{G}))) = \\ = \mathcal{D}et(\mathbb{A}_{X,C,1,R}(\mathcal{G}) \mid \mathbb{A}_{X,C,1,R}(\mathcal{F})), \tag{6.11}$$

where

$$\mathcal{D}et(\mathbb{A}_{X,C,1,R}(\mathcal{G}) \mid \mathbb{A}_{X,C,1,R}(\mathcal{F})) = \\ = \mathcal{D}et(\mathbb{A}_{X,C,1,R}(\mathcal{F})/\mathbb{A}_{X,C,1,R}(\mathcal{O}_{X_R}(G))) - \mathcal{D}et(\mathbb{A}_{X,C,1,R}(\mathcal{G})/\mathbb{A}_{X,C,1,R}(\mathcal{O}_{X_R}(G)))$$

and the divisor G on X is chosen by formula (6.10) with the "maximality" condition for the sheaf $\mathcal{O}_{X_R}(G)$.

From formula (6.11) we have that the categorical central extension

$$g \longmapsto \mathcal{D}et(\mathbb{A}_{X,C,1,R}(\mathcal{O}_{X_R}) \mid \mathbb{A}_{X,C,1,R}(g\mathcal{O}_{X_R})), \quad g \in (k(X) \otimes_k R)^*$$

is isomorphic to the trivial categorical central extension over the group $(k(X)\otimes_k R)^*$. It gives the reciprocity law for any elements f, g and h from the group $(k(X) \otimes_k R)^*$ along the curve C on X, but for any point $x \in C$ the local generalized commutator (depending on 3 commuting elements) has to be constructed from the following categorical cental extension

$$d \longmapsto \mathcal{D}et\Big((B_x \otimes_k R) \mid ((B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} (d \cdot \mathcal{O}_{X_R}))\Big), \quad d \in (k(X) \otimes_k R)^*. \tag{6.12}$$

In § 5.4 we proved that the two-dimensional Contou-Carrère symbol coincides with the generalized commutator C_3 when this commutator is calculated from another categorical central extension[19]:

$$d \longmapsto \mathcal{D}et\Big((\hat{\mathcal{O}}_{x,C} \otimes_k R) \mid (g \cdot (\hat{\mathcal{O}}_{x,C} \otimes_k R))\Big), \quad d \in (k(X) \otimes_k R)^*, \tag{6.13}$$

where $\hat{\mathcal{O}}_{x,C} \otimes_k R = \prod\limits_{i=1}^{s} (\hat{\mathcal{O}}_{x,\mathbf{C}_i} \otimes_k R)$ (see formula (6.3)) is a subring in the ring $K_{x,C} \otimes_k R$.

We will show that categorical central extension (6.12) is inverse to categorical central extension (6.13). From this fact we have that generalized commutator constructed by central extension (6.12) is equal to the minus one power of the generalized commutator constructed by central extension (6.13). Thus we will prove formula (6.9) for any elements f, g and h from the group $(k(X) \otimes_k R)^*$.

For any point $x \in C$ and any element $d \in (\mathrm{Frac}\hat{\mathcal{O}}_x \otimes_k R)^*$ we consider a complex $\mathcal{A}_{X,C,x,R}(d \cdot (\hat{\mathcal{O}}_x \otimes_k R))$:

$$(B_x \otimes_{\hat{\mathcal{O}}_x} (d \cdot (\hat{\mathcal{O}}_x \otimes_k R))) \oplus (d \cdot (\hat{\mathcal{O}}_{x,C} \otimes_k R)) \longrightarrow K_{x,C} \otimes_k R.$$

[19] More exactly, in § 5.4 we have considered the ring $R((u))((t))$, and the ring $K_{x,C} \otimes_k R$ is a finite direct product of these rings. Therefore the generalized commutator will be the finite product of two-dimensional Contou-Carrère symbols calculated for every formal branch of the curve C at the point x.

We have that $H^i(\mathcal{A}_{X,C,x,R}(d \cdot (\hat{\mathcal{O}}_x \otimes_k R))) = H^i(U_{x,R},\, d \cdot (\hat{\mathcal{O}}_x \otimes_k R) \mid_{U_{x,R}}) \otimes_k R$ (for i equal to 1 or to 2) is a Tate R-module, where (we recall) the scheme $U_{x,R} = \operatorname{Spec}(\hat{\mathcal{O}}_x \otimes_k R) \setminus x$. Inside of the ring $\mathrm{Frac}\hat{\mathcal{O}}_x \otimes_k R$ we have an equality for any $(d \in k(X) \otimes_k R)^*$:

$$B_x \otimes_{\hat{\mathcal{O}}_x} (d \cdot (\hat{\mathcal{O}}_x \otimes_k R)) = (B_x \otimes_k R) \otimes_{\mathcal{O}_{X_R}} (d \cdot \mathcal{O}_{X_R}).$$

Therefore from formula

$$\mathcal{D}et\Big((B_x \otimes_k R) \mid (B_x \otimes_{\hat{\mathcal{O}}_x} (d \cdot (\hat{\mathcal{O}}_x \otimes_k R)))\Big) + \mathcal{D}et\Big((\hat{\mathcal{O}}_{x,C} \otimes_k R) \mid (d \cdot (\hat{\mathcal{O}}_{x,C} \otimes_k R))\Big)$$
$$= \mathcal{D}et(H^*(\mathcal{A}_{X,C,x,R}(d \cdot (\hat{\mathcal{O}}_x \otimes_k R)))) - \mathcal{D}et(H^*(\mathcal{A}_{X,C,x,R}(\hat{\mathcal{O}}_x \otimes_k R))$$

we obtain that categorical central extension (6.12) is inverse to categorical central extension (6.13).

Now we will obtain formula (6.9) for any elements f, g and h from the group $(K_C \otimes_k R)^*$. We note that $K_C = k(C)((t_C))$, where $k(C)$ is the field of rational functions on the curve C, and $t_C = 0$ is an equation of the curve C on some open affine subset of X. Using that R is an Artinian ring and $k(X) \otimes_k R$ is dense in $K(C)((t_C)) \otimes_k R$ (when we consider the discrete topology on the field $k(C)$) it is easy to construct for any $n \in \mathbf{N}$ and $d \in (K_C \otimes_k R)^*$ elements $d_1 \in (k(X) \otimes_k R)^*$ and $d_2 \in 1 + t_C^n \cdot (k(C) \otimes_k R)[[t_C]]$ such that $d = d_1 d_2$. The two-dimensional Contou-Carrère symbol is a tri-multiplicative map. Besides, for any elements f and g from the group $(K_C \otimes_k R)^*$ there is[20] $n \in \mathbf{N}$ such that $(f, g, e) = 1$ for any element $e \in 1 + t_C^n \cdot (k(C) \otimes_k R)[[t_C]]$ and any point $x \in C$. Hence we obtain formula (6.9) in general case from the case when elements f, g and h are from the group $(K_C \otimes_k R)^*$. $\square$

7. Contou-Carrère symbols via algebraic K-theory

This section briefly discuss the K-theoretical approach to the usual and the higher-dimensional Contou-Carrère symbols. This approach develops some ideas suggested to us by one of the editors.

In Remark 3.4, we indicated that there exists the n-dimensional Contou-Carrère symbol given by an explicit formula (when $\mathbf{Q} \subset R$). It could also be obtained via algebraic K-theory for any (commutative) ring R in the following way.

For a (commutative) ring A and an integer $i \geq 0$, let $K_i(A)$ denote its ith algebraic K-group, as defined by D. Quillen. Recall that there is a canonical decomposition $K_1(A) = A^* \times SK_1(A)$, and there is the product structure in algebraic K-theory (see, e.g., [S, § 2]):

$$K_1(A) \times \cdots \times K_1(A) \longrightarrow K_{n+1}(A).$$

In addition, for any integer $m \geq 1$ there is the following canonical homomorphism

$$\partial_m : K_m(A((t))) \longrightarrow K_{m-1}(A) \tag{7.1}$$

which was constructed by K. Kato in [Ka1, § 2.1]. We briefly review the construction. Let H be the exact category of $A[[t]]$-modules that are annihilated by some power of t and that admit a resolution of length 1 by finitely generated projective $A[[t]]$-modules. Then the "localization theorem for projective modules" (or localization

[20]This kind of continuous property for all points $x \in C$ is obvious when $\mathbf{Q} \subset R$ from formula (3.15). The general case follows by means of the lift to the case $\mathbf{Q} \subset R$, see the proof of Lemma 3.15.

theory of algebraic K-theory for singular varieties, see [Gr], [S, § 9]) produces a canonical homomorphism

$$\tilde{\partial}_m \ : \ K_m(A((t))) \longrightarrow K_{m-1}(H), \tag{7.2}$$

We claim that any $A[[t]]$-module from the category H is a finitely generated projective A-module (see Propositon 7.1 below). Thus we have an exact functor from the category H to the category of finitely generated projective A-modules, and therefore a homomorphism $K_{m-1}(H) \to K_{m-1}(A)$. The composition of the homomorphism $\tilde{\partial}_m$ with the last homomorphism gives the homomorphism ∂_m.

Proposition 7.1 (Compare with [G, § 3.3])**.** *Let $\alpha : M \hookrightarrow N$ be an embedding of finitely generated projective $A[[t]]$-modules such that α becomes an isomorphism after inverting of the element t. Then N/M is a finitely generated projective A-module.*

Proof. By adding a finitely generated projective $A[[t]]$-module T to N and to M we can assume that $N = A[[t]]^s$ for some $s \in \mathbf{N}$. For some $l \in \mathbf{N}$ we have

$$t^l N \subset M \subset N.$$

Let $P = N/M$. Then we have an embedding $P \hookrightarrow t^{-l}M/M$. We claim that this embedding splits as a map of A-modules. From this claim we obtain that P is a finitely generated projective A-module, since $t^{-l}M/M$ is a finitely generated projective A-module (because we can add to M an $A[[t]]$-module Q such that $M \oplus Q = A[[t]]^r$ for some $r \in \mathbf{N}$). To prove the required splitting, it is enough to show that an embedding $N \hookrightarrow t^{-l}M$ splits as a map of A-modules. The last splitting will follow if we will show that the composed embedding

$$N \hookrightarrow t^{-l}M \hookrightarrow t^{-l}N$$

splits, again as a map of A-modules, but this is clear. □

Combining the above facts, one can produce for any (commutative) ring R a map

$$\begin{aligned}(7.3)\quad (R((t_n))\cdots((t_1))^*)^{n+1} \longrightarrow (K_1(R((t_n))\cdots((t_1))))^{n+1} \longrightarrow \\ \longrightarrow K_{n+1}(R((t_n))\cdots((t_1))) \xrightarrow{\partial_2\cdots\partial_{n+1}} K_1(R) \to R^*.\end{aligned}$$

By construction and the properties of the product structure in algebraic K-theory, this map is an n-multiplicative, anti-symmetric and functorial with respect to R map. Moreover, map (7.3) satisfies the Steinberg relation, since this relation is satisfied for the product structure in K-theory (see [S, §1 -§2]).

We believe that the map defined in (7.3) coincides with the one given in Remark (3.4) when $\mathbf{Q} \subset R$. We outline the proof of this fact in the case $n = 1$ and $n = 2$. From the structures of group ind-schemes $L\mathbb{G}_m$ and $L^2\mathbb{G}_m$ described in Section 2 and Sections 3.1 and 3.2, the map (7.3) for $n = 1$ and 2 will coincide with the usual and the two-dimensional Contou-Carrère symbols respectively. (Note that the case $n = 1$ answers a question of [KV, Remark 4.3.7].)

Theorem 7.2. *Let R be any ring. For $n = 1$ (resp. $n = 2$), the map constructed via K-theory by expression* (7.3) *coincides with the Contou-Carrère symbol defined via Lemma-Definition 2.2 (resp. via the map C_3 by formula* (5.4)*).*

Proof. Since the K-theoretical definition by formula (7.3) is functorial with respect to R, we have by Lemma-Definition 2.2 and by Lemma 3.10 that it is enough to prove the proposition when R is a $\mathbf{Q}$-algebra. To proceed, we note the following obvious lemma that ∂_m is invariant under some change of local parameter t (compare also with [Ka1, § 2.1, Lemma 2]).

Lemma 7.3. *Let $f_* : K_m(A((t))) \to K_m(A((t)))$ be the map induced by the automorphism $f : A((t)) \to A((t))$ given either by $f(t) = \sum_{i \geq 1} a_i t^i$, where $a_1 \in A^*$, $a_i \in A$ $(i > 1)$, or by $f(t) = a_0 + t$, where $a_0 \in \mathcal{N}A$. Then $\partial_m f_* = \partial_m : K_m(A((t))) \to K_m(A)$.*

Now, we prove that when $n = 1$, the map given by formula (7.3) coincides with the usual Contou-Carrère symbol. If $R = k$ is a field, then both definitions equal to the usual tame symbol, because the tame symbol is the boundary map in Milnor K-theory and this boundary map coincides with the map ∂_2 (see, e.g., [Ka1, § 2.4, Cor. 1]). Formula (7.3) defines the bimultiplicative and anti-symmetric morphism from $L\mathbb{G}_m \times L\mathbb{G}_m$ to $\mathbb{G}_m$. We denote this morphism as $(\cdot,\cdot)_{K-\mathrm{th}}$. Using the fact that $L\mathbb{G}_m = \widehat{\mathbb{W}} \times \mathbb{Z} \times \mathbb{G}_m \times \mathbb{W}$, and by similar (but easier) arguments as in the proof of Theorem 5.9, it remains to calculate the following pairing

$$(\widehat{\mathbb{W}} \times \mathbb{W}) \times (\widehat{\mathbb{W}} \times \mathbb{W}) \longrightarrow \mathbb{G}_m$$

induced by the morphism $(\cdot,\cdot)_{K-\mathrm{th}}$. In the sequel, we change $R((t_1))$ to $R((t))$ for simplicity. By an easy analog of part 1 of Proposition 3.6 (applied to the affine group scheme given by a functor $R \mapsto tR[[t]]$) and by an easy analog of Lemma 5.10 (see also formula (5.23)), we can assume that $R = \mathbf{Q}[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. But by Lemma 7.3, by Lemma 5.7 and similar (but easier) arguments as in the proof of Lemma 5.11, it remains to show that

$$(1 - \varepsilon_1 t, 1 - \varepsilon_2 t^{-1})_{K-\mathrm{th}} = 1 - \varepsilon_1 \varepsilon_2. \tag{7.4}$$

To check expression (7.4) we need the following observation. We recall a fact from [Ka1, § 2.4, Prop. 5]. For $a \in K_*(A((t)))$ and $t \in K_1(A((t)))$, let $\{a,t\} \in K_{*+1}(A((t)))$ denote the product.

Lemma 7.4. *The composite*

$$\alpha \ : \ K_*(A[[t]]) \longrightarrow K_*(A((t))) \xrightarrow{a \mapsto \{a,t\}} K_{*+1}(A((t))) \xrightarrow{\bigoplus_m \tilde{\partial}_m} K_*(H)$$

coincides with the homomorphism $\beta : K_*(A[[t]]) \to K_*(H)$ induced by the exact functor from the exact category of finitely generated projective $A[[t]]$-modules to the exact category H given as $M \mapsto M/tM$.

Therefore the composition of α with the homomorphism $K_*(H) \to K_*(A)$ is the homomorphism $K_*(A[[t]]) \to K_*(A)$ induced by the functor $M \mapsto M/tM$. As a consequence we obtain that for any $m \geq 1$, for any elements $a_1, \ldots, a_m \in A[[t]]^*$:

$$\partial_{m+1}\{a_1, \ldots, a_m, t\} = \{\overline{a_1}, \ldots, \overline{a_m}\}, \tag{7.5}$$

where for any $b \in A[[t]]^*$ we put $\overline{b} \in A^*$ under the homomorphism $A[[t]] \to A$.

Since $t \mapsto at$, where $a \in A[[t]]^*$, is a well-defined change of local parameter in $A((t))$, from Lemma 7.3, Formula (7.5) and using n-multiplicativity we obtain also for any elements $a_1, \ldots, a_{m+1}$ from the group $A[[t]]^*$:

$$\partial_{m+1}\{a_1, \ldots, a_{m+1}\} = 1. \tag{7.6}$$

(We note that formula (7.6) follows also from the localizing exact sequence (for singular varieties), since the composition $K_{m+1}(A[[t]]) \to K_{m+1}(A((t))) \xrightarrow{\tilde{\partial}_{m+1}} K_m(H)$ is the zero map.)

From formula (7.5) we have

$$(1-\varepsilon_1 t, 1-\varepsilon_2 t^{-1})_{K-\mathrm{th}} = (1-\varepsilon_1 t, t)^{-1}_{K-\mathrm{th}}(1-\varepsilon_1 t, t-\varepsilon_2)_{K-\mathrm{th}} = (1-\varepsilon_1 t, t-\varepsilon_2)_{K-\mathrm{th}}.$$

Using an automorphism of $R[[t]]$ given as $t \mapsto t+\varepsilon_2$ (which is the change of the local parameter) and Lemma 7.3 we have

$$(1-\varepsilon_1 t, t-\varepsilon_2)_{K-\mathrm{th}} = (1-\varepsilon_1\varepsilon_2 - \varepsilon_1 t, t)_{K-\mathrm{th}} = 1-\varepsilon_1\varepsilon_2.$$

Thus we have checked expression (7.4). This finishes the proof of the case $n=1$.

Now we consider the case $n=2$. Similarly and from Lemma 7.3, the homomorphism

$$\partial_{m-1}\partial_m \ : \ K_m(A((u))((t))) \longrightarrow K_{m-2}(A) \tag{7.7}$$

is invariant under the change of local parameters (at least) of the following type:

$$t \longmapsto \sum_{i\geq 1} a_i t^i, \quad a_1 \in A((u))^*, \quad a_i \in A((u)) \ (i>1);$$

$$u \longmapsto \sum_{i\geq 1} b_i u^i + gt, \quad b_1 \in A^*, \quad b_i \in A \ (i>1), \quad g \in A((u))[[t]].$$

We note that from Lemma 7.3 we have that homomorphism (7.7) is invariant under the following change of the local parameter t: $t \mapsto t+a$, where $a \in \mathcal{N}(A((u)))$. (We will use the last change of local parameter later in an explict calculation.)

The map given by formula (7.3) defines for $n=2$ the morphism from the ind-scheme $(L^2\mathbb{G}_m)^3$, where

$$L^2\mathbb{G}_m = \mathbb{P} \times \mathbb{Z}^2 \times \mathbb{G}_m \times \mathbb{M}$$

to the scheme $\mathbb{G}_m$. We denote this tri-multiplicative and anti-symmetric morphism as $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$, and we will use the notation $R((u))((t))$ instead of $R((t_2))((t_1))$ in formula (7.3). If $R=k$ is a field, then the map $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$ coincides with the two-dimensional tame symbol, because the two-dimensional tame symbol is the composition of two boundary maps in Milnor K-theory (see, e.g, [OsZh, § 4A]) and ∂_m restricted to Milnor K-theory coincides with the boundary map there (see formula (7.5) or [Ka1, § 2.4, Cor. 1]).

From Lemma 3.10 and Theorem 5.9 we see that to check the case $n=2$ it is enough to prove that the map $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$ satisfies properties (3.15)-(3.18) for a $\mathbf{Q}$-algebra R. Now, using the invariance of the map $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$ under the change of local parameters u and t and formulas (7.5) and (7.6) we can repeat the proof of Theorem 5.9 for the map $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$. Thus we reduce the proof to calculate some particular cases.

As in the proof of Lemma 5.11 we have to calculate for the ring $R=\mathbf{Q}[\varepsilon_1,\varepsilon_2,\varepsilon_3]/(\varepsilon_1^2,\varepsilon_2^2,\varepsilon_3^2)$ the following expression $(1+u\varepsilon_1, 1+t\varepsilon_2, 1+u^{-1}t^{-1}\varepsilon_3)_{K-\mathrm{th}}$. We have

$$(1+u\varepsilon_1, 1+t\varepsilon_2, 1+u^{-1}t^{-1}\varepsilon_3)_{K-\mathrm{th}} =$$
$$= (1+u\varepsilon_1, 1+t\varepsilon_2, t)^{-1}_{K-\mathrm{th}}(1+u\varepsilon_1, 1+t\varepsilon_2, t+u^{-1}\varepsilon_3)_{K-\mathrm{th}}$$

From formula (7.5) we have $(1+u\varepsilon_1, 1+t\varepsilon_2, t)_{K-\mathrm{th}} = 1$. Using an automorphism of the ring $R((u))((t))$ obtained by the change of the local parameter t: $t \mapsto t-u^{-1}\varepsilon_3$, and using again formula (7.5) we obtain

$$(1+u\varepsilon_1, 1+t\varepsilon_2, t+u^{-1}\varepsilon_3)_{K-\mathrm{th}} = (1+u\varepsilon_1, 1-u^{-1}\varepsilon_3\varepsilon_2+t\varepsilon_2, t)_{K-\mathrm{th}} = \\ = (1+u\varepsilon_1, 1-u^{-1}\varepsilon_3\varepsilon_2)_{K-\mathrm{th}}.$$

Now using the case $n=1$ of this theorem and an explicit formula (2.7) we have that

$$(1+u\varepsilon_1, 1-u^{-1}\varepsilon_3\varepsilon_2)_{K-\mathrm{th}} = 1+\varepsilon_1\varepsilon_2\varepsilon_3.$$

Thus, we have checked that $(1+u\varepsilon_1, 1+t\varepsilon_2, 1+u^{-1}t^{-1}\varepsilon_3)_{K-\mathrm{th}} = 1+\varepsilon_1\varepsilon_2\varepsilon_3$.

As in the proof of Lemma 5.12 we have the following calculation for the ring $R=\mathbf{Q}[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_2^2)$:

$$(t+\varepsilon_1, 1+t\varepsilon_2, u)_{K-\mathrm{th}} = (1+t\varepsilon_2, u, t+\varepsilon_1)_{K-\mathrm{th}} = (1+t\varepsilon_2, u, t+\varepsilon_1)_{K-\mathrm{th}} = \\ = (1-\varepsilon_1\varepsilon_2+t\varepsilon_2, u, t)_{K-\mathrm{th}} = (1-\varepsilon_1\varepsilon_2, u)_{K-\mathrm{th}} = 1-\varepsilon_1\varepsilon_2,$$

where we used the change of the local parameter $t \mapsto t-\varepsilon_1$ and formula (7.5).

As in the proof of Lemma 5.13 we have the following calculation for the ring $R=\mathbf{Q}[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_2^2)$:

$$(1+u^{-1}\varepsilon_1, 1+u\varepsilon_2, t)_{K-\mathrm{th}} = (1+u^{-1}\varepsilon_1, 1+u\varepsilon_2)_{K-\mathrm{th}} = 1+\varepsilon_1\varepsilon_2.$$

The above explicit calculations are the only calculations needed to repeat the proof of Theorem 5.9 for the case of the map $(\cdot,\cdot,\cdot)_{K-\mathrm{th}}$. Thus we have checked the case $n=2$ of Theorem 7.2. This finishes the proof of this theorem. □

The following corollary follows from the corresponding property of the product structure in algebraic K-theory.

Corollary 7.5. *For any ring R the two-dimensional Contou-Carrère symbol defined via the map C_3 by formula* (5.4) *satisfies the Steinberg relations.*

Remark 7.1. Let C be a smooth projective curve over a perfect field k, R is a commutative k-algebra and $m \ge 1$ is an integer. There is the following reciprocity law. For a closed point p of C with residue field $k(p)$, let t_p be a local coordinate around p. Using the ring homomorphism $k(C)\otimes_k R \to (k(p)\otimes_k R)((t_p))$ we obtain a homomorphism
(7.8)

$$s_p : K_m(k(C)\otimes_k R) \longrightarrow K_m(k(p)\otimes_k R)((t_p)) \xrightarrow{\partial_{p,m}} K_{m-1}(k(p)\otimes_k R) \longrightarrow K_{m-1}(R),$$

where the map $\partial_{p,m}$ is homomorphism (7.1) applied to $K_m((k(p)\otimes_k R)((t_p)))$, and the last arrow in (4.3) denotes the tranfer (or pushforward) map. Then the reciprocity law is: for any $x \in K_m(k(C)\otimes_k R)$ we have that $s_p(x)$ is nonzero for only finitely many points p, and $\sum_{p\in C} s_p(x) = 0$.

We briefly explain its proof. Since $K_m(\cdot)$ commutes with filtered direct limits of rings (see, e.g., [S, Lemma 5.9]) we can assume that x comes from $K_m(U\otimes_k R')$, where U is an affine open subset of C and $R' \subset R$ is a *Noetherian* subring. In the sequel, we will write R instead of R' for simplicity (so we can suppose that R is a Noetherian ring). We consider $C\setminus U = \bigcup_{i=1}^q p_i$, where p_i is a point on C. Let $j : U\otimes_k R \hookrightarrow C\otimes_k R$ be the corresponding open embedding. We write the localizing exact sequence for singular varieties (see [S, Th. 9.1]):

$$\dots \longrightarrow K_m(U\otimes_k R) \xrightarrow{\partial} K_{m-1}(\mathcal{H}) \xrightarrow{\alpha} K_{m-1}(C\otimes_k R) \longrightarrow \dots, \tag{7.9}$$

where $\mathcal{H}$ is an exact category of coherent $\mathcal{O}_{C\otimes_k R}$-modules $\mathcal{F}$ such that $j^*\mathcal{F}=0$, and $\mathcal{F}$ has a resolution of length 1 by locally free $\mathcal{O}_{C\otimes_k R}$-modules of finite rank.

Let $f : C\otimes_k R \to \mathrm{Spec} R$ be the projection. Since f is a proper flat morphism, there is a well-defined map (see [S, Prop. 5.12])

$$f_* \ : \ K_{m-1}(C\otimes_k R) \to K_{m-1}(R).$$

We consider a commutative diagram
(7.10)

$$\begin{array}{ccccc} K_m(U\otimes_k R) & \xrightarrow{\partial} & K_{m-1}(\mathcal{H}) & \xrightarrow{\alpha} & K_{m-1}(C\otimes_k R) \\ \downarrow & & \downarrow & & \downarrow f_* \\ \bigoplus\limits_{i=1}^{q} K_m((k(p_i)\otimes_k R)((t_{p_i}))) & \xrightarrow{\bigoplus\limits_{i=1}^{q}\tilde{\partial}_{p_i,m}} & \bigoplus\limits_{i=1}^{q} K_{m-1}(H_{p_i}) & \xrightarrow{\beta} & K_{m-1}(R), \end{array}$$

where for any $1\le i\le q$, H_{p_i} is the exact category attached to $(k(p_i)\otimes_k R)((t_{p_i}))$ and constructed as H at the beginning of Section 7, and the map $\tilde{\partial}_{p_i,m}$ is homomorphism (7.2) applied to $K_m((k(p_i)\otimes_k R)((t_{p_i})))$. The map β in this diagram is given as sum over i of composition of maps $K_{m-1}(H_{p_i})\to K_{m-1}(k(p_i)\otimes_k R)$ and $K_{m-1}(k(p_i)\otimes_k R)\to K_{m-1}(R)$.

Using $\alpha\circ\partial=0$ in sequence (7.9) and commutative diagram (7.10) we obtain the reciprocity law for elements from the group $K_m(U\otimes_k R)$.

When $m=2$, using the product structure in algebraic K-theory and Theorem 7.2, from this reciprocity law we derive the reciprocity law on the curve C over a perfect field k for the usual Contou-Carrère symbol over a k-algebra R (compare with Theorem 2.4).

Remark 7.2. Let R be a finite k-algebra. We now give a short explanation how from the above reciprocity law on a projective curve (see Remark 7.1) and Theorem 7.2 it is possible to obtain the reciprocity law for the two-dimensional Contou-Carrère symbol along a projective curve C on a smooth algebraic surface X over a perfect field k (see formula (6.9)). Previously, we proved this reciprocity law in Theorem (6.1) by means of categorical central extensions and semilocal adelic complexes on X.

We recall that the field $K_C=k(C)((t_C))$. Therefore $K_C\otimes_k R=(k(C)\otimes_k R)((t_C))$. We suppose for simplicity that C is a smooth curve. (If C is not smooth then one has to work with the normalization of C.) Then for any point $x\in C$ we have that $K_{x,C}=k(x)((u))((t_C))$ is a two-dimensional local field, where $k(x)((u))$ is the completion of the field $k(C)$ with respect to the discrete valuation given by the point x. Besides, $K_{x,C}\otimes_k R=(k(x)((u))\otimes_k R)((t_C))$ We have a commutative diagram

$$\begin{array}{ccc} K_3((k(C)\otimes_k R)((t_C))) & \xrightarrow{\partial_{C,3}} & K_2(k(C)\otimes_k R) \\ \downarrow & & \downarrow \\ K_3((k(x)((u))\otimes_k R)((t_C))) & \xrightarrow{\partial_{x,C,3}} & K_2(k(x)((u))\otimes_k R). \end{array}$$

This diagram and Theorem 7.2 allow us to reduce the reciprocity law along C on X for the two-dimensional Contou-Carrère symbol for any elements f, g and h from $(K_C\otimes_k R)^*$ to the reciprocity law on C (which we considered in Remark 7.1) for the element $\partial_{C,3}\{f,g,h\}$ in $K_2(k(C)\otimes_k R)$. It is useful also to note that the homomorphism $(k(x) \otimes_k R)^* \to R^*$ obtained from the transfer homomorphism $K_1(k(x) \otimes_k R) \to K_1(R)$ coincides with the norm map.

We note that it is not clear for us how it would be possible to apply the localizing exact sequence for singular varieties to obtain via algebraic K-theory the reciprocity law for the two-dimensional Contou-Carrère symbol around a point on a smooth algebraic surface (see formula (6.8)), which is another reciprocity law obtained in Theorem 6.1!

8. The two-dimensional Contou-Carrère symbol and two-dimensional class field theory

Two-dimensional class field theory was developed by A.N. Parshin, K. Kato and others, see [Pa1, Pa2, Pa3] and [Ka1, KS, Ka2].

By Proposition 4.3, the two-dimensional Contou-Carrère symbol coincides with the two-dimensional tame symbol when the ground ring R is equal to a field k. The two-dimensional tame symbol was used in the local two-dimensional class field theory for the field $\mathbf{F}_q((u))((t))$ to describe the generalization of the Kummer duality and, consequently, Kummer extensions of the field $\mathbf{F}_q((u))((t))$ where $\mathbf{F}_q$ is a finite field and $q = p^n$ for some prime p, see [Pa3, § 3.1].

We will construct some one-parametric deformation of the two-dimensional tame symbol. It will be given as the two-dimensional Contou-Carrère symbol over some Artinian ring. From this deformation we will obtain the local symbol[21] which was used by Parshin in [Pa3, § 3.1-3.2] to obtain the generalization of the Artin-Schreier-Witt duality for the two-dimensional local field $\mathbf{F}_q((u))((t))$. The generalization of the Artin-Schreier-Witt duality describes abelian extensions of exponent p^m of the field $\mathbf{F}_q((u))((t))$.

Let R be any commutative ring. Let S be the set of positive integers which is closed under passage to divisors. We denote by $W_S(R)$ the ring of (big) Witt vectors[22], i.e. $W_S(R) = \{(x_i)_{i \in S}\}$ where $x_i \in R$. We will need the ghost (or auxiliary) coordinates which are defined as $x(i) = \sum\limits_{d|i} d x_d^{i/d}$ where $i \in S$. The addition and multiplication in ghost coordinates of Witt vectors are coordinate-wise, but in usual coordinates (i.e. in coordinates x_i) the addition and multiplication are given by some universal polynomials with integer coefficients in the variables x_i. We have an equality in the ring $\mathbf{Q}[[s]]$:

$$-\log \prod_{i=1}^{\infty} (1 - x_i s^i) = \sum_{l=1}^{\infty} x(l) s^l / l. \tag{8.1}$$

For any positive integer n we denote by $W_n(R)$ the truncated Witt vectors, i.e. in our previous notation $W_n(R) = W_{\{1,\ldots,n\}}(R)$. The additive group of the ring $W_n(R)$ is isomorphic to the group of invertible elements of the following kind $\{1 + r_1 s + \ldots + r_i s^i + \ldots + r_n s^n\}$ in the ring $R[s]/s^{n+1}$ by means of the map

$$(x_i)_{1 \le i \le n} \longmapsto \prod_{i=1}^{n} (1 - x_i s^i) \mod s^{n+1}. \tag{8.2}$$

[21] The relation between one-dimensional Contou-Carrère symbol and the Witt symbol for a usual one-dimensional local field $k((t))$ was noticed in [AP, § 4.3].

[22] This is a ring scheme.

Remark 8.1. An obvious generalization of the map (8.2) gives an isomorphism between the additive group of the ring $\varprojlim_{n\geq 1} W_n(R)$ and the group $\mathbb{W}(R)$ introduced in § 2.

We denote $W_n^p(R) = W_{\{1,p,\ldots,p^{n-1}\}}(R)$. The ring scheme W_n^p is a natural quotient of the ring scheme $W_{p^{n-1}}$. We denote[23] $W^p(R) = \varprojlim_{n\geq 1} W_n^p(R)$.

We fix a positive integer n and a perfect field k. Let $R = k[s]/s^{n+1}$. Let $(\cdot,\cdot,\cdot)$ be the two-dimensional Contou-Carrère symbol: $(R((u))((t))^*)^3 \to R^*$. We define a tri-multiplicative map:

$$k((u))((t))^* \times k((u))((t))^* \times W_n(k((u))((t))) \longrightarrow W_n(k), \tag{8.3}$$

where on $W_n(k((u))((t)))$ and on $W_n(k)$ we consider only the additive structure (so "multiplicativity" means "with respect to additive structure of these rings"). Let map (8.3) be denoted by

$$(g_1, g_2 \mid y_1, \ldots, y_n] \; \in \; W_n(k),$$

where $g_i \in k((u))((t))^*$ and $(y_1, \ldots, y_n) \in W_n(k((u))((t)))$. Then this map is defined as follows:

$$\prod_{i=1}^{n}(1-(g_1,g_2 \mid y_1,\ldots,y_n]_i \cdot s^i) \mod s^{n+1} = (\prod_{i=1}^{n}(1-y_i s^i)\,, g_1, g_2). \tag{8.4}$$

If char$k = p$, and we consider only non-zero coordinates such as $(y_1, y_p, \ldots, y_{p^{m-1}})$, then we will show that the image of the expression $(g_1, g_2 \mid y_1, \ldots, y_{p^{m-1}}]$ in the group $W_m^p(k)$ is equal to the generalization of the Witt symbol given by Parshin in [Pa3, §3][24]. Indeed, we consider the field Frac$W^p(k)$. We have that $\mathbf{Q} \subset \mathrm{Frac}W^p(k)$ and $W^p(k)/pW^p(k) = k$. We choose some lifts $\widetilde{y_i}, \widetilde{g_j} \in W^p(k)((u))((t))$ of elements $y_i, g_j \in k((u))((t))$. Using formulas (3.15) and (8.1) we have

$$-\log(\prod_{i=1}^{p^{m-1}}(1-\widetilde{y_i}s^i), \widetilde{g_1}, \widetilde{g_2}) = -\mathrm{Res}(\log\prod_{i=1}^{p^{m-1}}(1-\widetilde{y_i}s^i)\frac{d\widetilde{g_1}}{\widetilde{g_1}} \wedge \frac{d\widetilde{g_2}}{\widetilde{g_2}}) =$$
$$= \mathrm{Res}(\sum_{i=1}^{p^{m-1}} \frac{\widetilde{y}(i)s^i}{i}\frac{d\widetilde{g_1}}{\widetilde{g_1}} \wedge \frac{d\widetilde{g_2}}{\widetilde{g_2}}) = \sum_{i=1}^{p^{m-1}} \mathrm{Res}(\widetilde{y}(i)\frac{d\widetilde{g_1}}{\widetilde{g_1}} \wedge \frac{d\widetilde{g_2}}{\widetilde{g_2}})\frac{s^i}{i}.$$

Using formulas (8.1) and (8.4) we obtain

$$(\widetilde{g_1}, \widetilde{g_2} \mid \widetilde{y_1, \ldots, y_{p^{m-1}}}](i) = \mathrm{Res}(\widetilde{y}(i)\frac{d\widetilde{g_1}}{\widetilde{g_1}} \wedge \frac{d\widetilde{g_2}}{\widetilde{g_2}}). \tag{8.5}$$

Besides, we have

$$(g_1, g_2 \mid y_1, \ldots, y_{p^{m-1}}]_i = (\widetilde{g_1}, \widetilde{g_2} \mid \widetilde{y_1, \ldots, y_{p^{m-1}}}]_i \mod p. \tag{8.6}$$

[23] We note that usually when p is fixed, the rings $W_n^p(R)$ and $W^p(R)$ are denoted as $W_n(R)$ and $W(R)$.

[24] A. N. Parshin considered only the case $k = \mathbf{F}_q$. Besides, we have to compose the above expression with the trace map from $W_m^p(\mathbf{F}_q)$ to $W_m^p(\mathbf{F}_p) = \mathbf{Z}/p^m\mathbf{Z}$ to obtain the symbol for the generalization of the Witt duality, see [Pa3, § 3, Prop. 7]

Now if $i = 1, p, \ldots, p^{m-1}$ and $k = \mathbf{F}_q$, then formulas (8.5)-(8.6) coincide with Parshin's Definition 5 from [Pa3, § 3.3].

We note that we have just constructed the following map when $\mathrm{char} k = p$:

$$k((u))((t))^* \times k((u))((t))^* \times W^p_m(k((u))((t))) \longrightarrow W^p_m(k). \tag{8.7}$$

From formula (8.5) we have that map (8.7) is additive with respect to the groups $W^p_m(k((u))((t)))$ and $W^p_m(k)$ (and, consequently, is tri-multiplicative with respect to all arguments like formula (8.3)), since if $i = 1, p, \ldots, p^{m-1}$, then passage to the ghost coordinates and to the usual coordinates depends only on these integers.

Remark 8.2. Clearly, the reciprocity laws which were proved for the two-dimensional Contou-Carrère symbol in Theorem 6.1 imply analogous reciprocity laws for maps (8.3) and (8.7) such that the norm map $\mathrm{Nm}_{k''/k'} : (k'' \otimes_k R)^* \to (k' \otimes_k R)^*$ is converted to the trace map $\mathrm{Tr}_{k''/k'} : W_n(k'') \to W_n(k')$ for any finite extensions of fields $k'' \supset k' \supset k$.

We can interpret the reciprocity laws for the two-dimensional tame symbol and for map (8.7) as follows. Let X be a smooth projective algebraic surface over a finite field $\mathbf{F}_q$. Then, according to [Pa2], some K_2-adelic object $K_{2,\mathbb{A}_X}$ and a reciprocity map $\theta : K_{2,\mathbb{A}_X} \to \mathrm{Gal}(\mathbf{F}_q(X)^{\mathrm{ab}}/\mathbf{F}_q(X))$ should exist such that

$$K_{2,\mathbb{A}_X} = \prod_{x\in\mathbf{C}}{}' K_2(K_{x,\mathbf{C}}) \subset \prod_{x\in\mathbf{C}} K_2(K_{x,\mathbf{C}})$$

is a complicated "restricted" product and

$$\theta = \sum_{x\in\mathbf{C}} \theta_{x,\mathbf{C}},$$

where $\theta_{x,\mathbf{C}} : K_2(K_{x,\mathbf{C}}) \to \mathrm{Gal}(K^{\mathrm{ab}}_{x,\mathbf{C}}/K_{x,\mathbf{C}})$ is the local reciprocity map from two-dimensional local class field theory (for any field L by L^{ab} we denote its maximal abelian extension and $x \in \mathbf{C}$ runs over all points $x \in X$ and all formal branches $\mathbf{C}$ of all irreducible curves on X which contain a point x).

For any point $x \in X$ the ring $K^M_2(K_x)$ is diagonally mapped to the ring $K_{2,\mathbb{A}_X}$ (via maps $K^M_2(K_x) \to K_2(K_{x,\mathbf{C}})$ for all $\mathbf{C} \ni x$ and we put $1 \in K_2(K_{y,\mathbf{F}})$ for all pairs $y \in \mathbf{F}$ such that $y \neq x$). For any irreducible curve C on X the ring $K_2(K_C)$ is also diagonally mapped to the ring $K_{2,\mathbb{A}_X}$ (via maps $K_2(K_C) \to K_2(K_{x,\mathbf{C}})$ for all $x \in \mathbf{C}$ and we put $1 \in K_2(K_{y,\mathbf{F}})$ for all pairs $y \in \mathbf{F}$ such that $\mathbf{F}$ is not a formal branch of C). Let an extension $N \supset \mathbf{F}_q(X)$ be either maximal Kummer extension (i.e. the union of all finite Galois extensions of exponent $q-1$ contained in $\mathbf{F}_q(X)^{\mathrm{ab}}$) or the maximal abelian p-extension (i.e the union of all finite Galois p-extensions contained in $\mathbf{F}_q(X)^{\mathrm{ab}}$). Let $\gamma : \mathrm{Gal}(\mathbf{F}_q(X)^{\mathrm{ab}}/\mathbf{F}_q(X)) \to \mathrm{Gal}(N/\mathbf{F}_q(X))$ be the natural quotient map. Then, similar to the case of algebraic curves over finite fields, we obtain from reciprocity laws that

$$\gamma \circ \theta\,(K^M_2(K_x)) = 0 \qquad \text{and} \qquad \gamma \circ \theta\,(K_2(K_C)) = 0$$

for any $x \in X$ and $C \subset X$.

Remark 8.3. We obtained reciprocity laws on an algebraic surface for the Parshin generalization of the Witt symbol as the consequence of our reciprocity laws for the two-dimensional Contou-Carrère symbol. We note that earlier K. Kato and S. Saito obtained by another methods similar reciprocity laws for the generalization of the

Witt symbol, see Lemma 4 and Lemma 5 from [KS, Ch. III]. We explain the relation between two generalizations of the Witt symbol. Let k be a perfect field, $\mathrm{char} k = p$. Let G be a commutative smooth connected algebraic group over k. The authors constructed in [KS] some local symbol map:

$$k((u))((t))^* \times k((u))((t))^* \times G(k((u))((t))) \longrightarrow G(k) \tag{8.8}$$

and proved the two-dimensional reciprocity laws for this symbol map. When $G = W_m^p$, then map (8.8) is the generalization of the Witt symbol. We note that the construction for this symbol map given in [KS] was very indirect. An explicit easy formula for map (8.8) (when $G = W_m^p$) can be found in [KR, § 7.1]. From this formula one can easily see that map (8.8) (when $G = W_m^p$) coincides with map (8.7), which is described by formulas (8.5)-(8.6).

References

[AP] Anderson G., Pablos Romo F. *Simple proofs of classical explicit reciprocity laws on curves using determinant groupoids over an artinian local ring*, Commun. Algebra 32, no. 1, 79-102 (2004).

[BBE] Beilinson A., Bloch S., Esnault H. *ε-factors for Gauss-Manin determinants.*, Mosc. Math. J. 2, No. 3, 477-532 (2002).

[Br1] Breen L. *Bitorseurs et cohomologie non-abélienne*, in The Grothendieck Festschrift I, P. Cartier et al. eds., Progress in Mathematics 86, pp. 401-476 (1990).

[Br2] Breen L. *On the classification of 2-gerbes and 2-stacks*, Astérisque. 225. Paris: Société mathématique de France (1994).

[BM] Brylinski J.-L., McLaughlin D.A. *The geometry of two-dimensional symbols*, K-theory 10, no. 3, 215-237 (1996).

[CC1] Contou-Carrère C. *Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré*, C. R. Acad. Sci. Paris, t. 318, S'erie I, pp. 743-746, 1994.

[CC2] Contou-Carrère C. *Jacobienne locale d'une courbe formelle relative*, Rend. Sem. Mat. Univ. Padova, vol. 130, 2013, pp. 1-106.

[Del] Deligne P. *Le symbole modéré*, Publ. Math. IHES 73(1991), 147-181.

[Dr] Drinfeld V. *Infinite-dimensional vector bundles in algebraic geometry: an introduction*, The unity of mathematics, In honor of the ninetieth birthday of I. M. Gelfand, Papers from the conference held in Cambridge, MA, USA, August 31 – September 4, 2003, Progr. Math. 244 (2006), 263-304.

[FZ] Frenkel E., Zhu X. *Gerbal representation of double loop groups*, Int. Math. Res. Not. 2012, no. 17, 3929-4013.

[G] Gaitsgory D. *Affine grassmannian and the loop group*, Seminar Notes written by D. Gaitsgory and N. Rozenblyum, 2009, 12 pp., available at http://www.math.harvard.edu/∼gaitsgde/grad_2009/SeminarNotes/Oct13(AffGr).pdf

[Gr] Grayson D. *Higher algebraic K -theory. II (after Daniel Quillen).* Algebraic K -theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976), pp. 217-240. Lecture Notes in Math., Vol. 551, Springer, Berlin, 1976.

[HL] Horozov I., Luo Zh. *On the Contou-Carrere Symbol for Surfaces*, preprint arXiv:1310.7065 [math.AG].

[Kap] Kapranov M. *Semiinfinte symmetric powers*, preprint arXiv:math/0107089 [math.QA].

[KV] Kapranov M., Vasserot E. *Formal loops II. A local Riemann-Roch theorem for determinantal gerbes*, Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 1, 113-133.

[Ka1] Kato K. *A generalization of local class field theory by using K -groups. II.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 3, 603-683.

[KS] Kato K., Saito S. *Two Dimensional Class Field Theory* , Advanced Studies in Pure Mathematics 2, 1983, Galois Groups and their Represenations, pp. 103-152.

[Ka2] Kato K. *Generalized Class Field Theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 419-428, Math. Soc. Japan, Tokyo, 1991.

[KR] Kato K., Russell H. *Modulus of a rational map into a commutative algebraic group*, Kyoto J. Math. 50 (2010), no. 3, 607-622.

[Mor] Morava J. *An algebraic analog of the Virasoro group*, Czechoslovak Journal of Physics, Vol. 51 (2001), No. 12, pp. 1395-1400.

[O1] Osipov D. V. *Adele constructions of direct images of differentials and symbols*, Mat. Sb., 188:5 (1997), 59–84; English transl. in Sb. Math. 188:5 (1997), 697-723.

[O2] Osipov D. V. *Central extensions and reciprocity laws on algebraic surfaces*, (Russian) Mat. Sb. 196:10 (2005), 111-136; translation in Sb. Math. 196:10 (2005), 1503-1527.

[O3] Osipov D. *Adeles on n-dimensional schemes and categories C_n*, International Journal of Mathematics, vol. 18, no. 3 (2007), 269-279.

[O4] Osipov D. V. *n-dimensional local fields and adeles on n-dimensional schemes*, Surveys in Contemporary Mathematics, Edited by N. Young, Y. Choi; London Mathematical Society Lecture Note Series, No. 347 (2007), pp. 131-164, Cambridge University Press.

[OsZh] Osipov D., Zhu X., *A categorical proof of the Parshin reciprocity laws on algebraic surfaces*, Algebra & Number Theory, vol. 5, 2011, No. 3, 289-337.

[Pal] Pál A. *On the kernel and the image of the rigid analytic regulator in positive characteristic.*, Publ. Res. Inst. Math. Sci. 46 (2010), no. 2, 255288.

[Pa1] Parshin A. N. *Class fields and algebraic K-theory*. Uspekhi Mat. Nauk 30 (1975), 253-254 (in russian), English translation in Russian Math. Surveys.

[Pa2] Parshin A. N. *Abelian coverings of arithmetic schemes*, Dokl. Akad. Nauk SSSR, Tom 243 (1978), No. 4 (in russian); English translation: Soviet Math. Dokl. Vol. 19 (1978), No. 6, 1438-1442.

[Pa3] Parshin A. N. *Local class field theory*, Trudy Matem. Inst. Steklov. 165 (1984) (in russian), English translation: Proceedings of the Steklov Institute of Mathematics 1985, Issue 3, 157-185.

[S] Srinivas V., *Algebraic K-theory*. Second edition. Progress in Mathematics, 90. Birkhauser Boston, Inc., Boston, MA, 1996.

Denis Osipov,
Steklov Mathematical Institute, Gubkina str. 8, Moscow, 119991, Russia
E-mail address: *d_osipov@mi.ras.ru*

Xinwen Zhu,
Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA
E-mail address: *xinwenz@math.northwestern.edu*