GEOMETRIC LANGLANDS IN PRIME CHARACTERISTIC

TSAO-HSIEN CHEN, XINWEN ZHU

Abstract. Let G be a semisimple algebraic group over an algebraically closed field k, whose characteristic is positive and does not divide the order of the Weyl group of G, and let $\breve{G}$ be its Langlands dual group over k. Let C be a smooth projective curve over k. Denote by Bun_G the moduli stack of G-bundles on C and $\mathrm{LocSys}_{\breve{G}}$ the moduli stack of $\breve{G}$-local systems on C. Let D_{Bun_G} be the sheaf of crystalline differential operators on Bun_G. In this paper we construct an equivalence between the bounded derived category $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}}))$ of quasi-coherent sheaves on some open subset $\mathrm{LocSys}^0_{\breve{G}}\subset \mathrm{LocSys}_{\breve{G}}$ and bounded derived category $D^b(D^0_{\mathrm{Bun}_G}\text{-mod})$ of modules over some localization $D^0_{\mathrm{Bun}_G}$ of D_{Bun_G}. This generalizes the work of Bezrukavnikov-Braverman in the GL_n case.

Contents

1. Introduction

1.1. **Geometric Langlands conjecture in prime characteristic.** Let G be a reductive algebraic algebraic group over $\mathbb{C}$ and let $\breve{G}$ be its Langlands dual group. Let C be a smooth projective curve over $\mathbb{C}$. Let Bun_G be the stack of G-bundles on C and $\mathrm{LocSys}_{\breve{G}}$ be the stack of de Rham $\breve{G}$-local systems on C. The geometric Langlands conjecture, as proposed by Beilinson and Drinfeld, is a conjectural equivalence between certain appropriate defined category of quasi-coherent sheaves on $\mathrm{LocSys}_{\breve{G}}$ and certain appropriate defined category of $\mathcal{D}$-module on Bun_G. A precise formulation of this conjecture (over $\mathbb{C}$) can be found in the recent work of Dima Arinkin and Dennis Gaitsgory [AG, Ga].

The geometric Langlands duality has a quasi-classical limits which amounts to the duality of Hitchin fibrations. The classical duality is established "generically" by Dogani and Pentev in [DP] over $\mathbb{C}$ and is extended in this paper to any algebraically closed field whose characteristic does not divide the order of the Weyl group of G (see §3 for details).

In this paper, we establish a "generic" characteristic p version of the geometric Langlands conjecture. Namely, we assume that G is a semi-simple algebraic group over an algebraically closed field k of characteristic p, where p does not divide the order of the Weyl gorup of G. Let $\breve{G}$ be its Langlands dual group over k. Let C be a curve over k. We establish an equivalence of bounded derived category

$$D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{Qcoh}(\mathrm{LocSys}_{\breve{G}})^0), \tag{1.1.1}$$

where $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ (resp. $\mathrm{Qcoh}(\mathrm{LocSys}_{\breve{G}})^0$) is certain localization of the category of $\mathcal{D}$-modules on Bun_G (resp. localizations of the category of quasi-coherent sheaves on $\mathrm{LocSys}_{\breve{G}}$). We call this a "generic" version of the GLC.

One remark is in order. Recall that over a field of positive characteristic, there are different objects that can be called $\mathcal{D}$-modules. In this paper, we use the notion of crystalline $\mathcal{D}$-modules, i.e. $\mathcal{D}$-modules are quasi-coherent sheaves with flat connections. Likewise, the stack $\mathrm{LocSys}_{\breve{G}}$ are the stack of $\breve{G}$-bundles on C with flat connections.

1.2. **Summary of the construction.** The case $G = \mathrm{GL}_n$ has been considered by R. Bezrukavnikov and A. Braverman in [BB] (see [Groe, Trav] for various extensions). The main observation is that, the geometric Langlands duality in characteristic p formulated in the above form can be thought as a twisted version of its classical limit. Since the classical duality holds "generically", they proved a "generic" version of this conjecture in the case when $G = \mathrm{GL}_n$.

Our generalization to any semisimple group G is based on the same observation, but some new ingredients are needed in this general situation.

One of the main difficulties for general G is that the classical duality is more complicated. For $G = \mathrm{GL}_n$, the generic fibers of the Hitchin fibration are the Picard stack of the line bundles on the corresponding spectral curves and the duality of Hitchin fibration in this case essentially amounts to the self-duality of the Jacobian of an algebraic curve. However, for general G, the fibers of the Hitchin fibration involve more general Picard stacks, such as the Prym varieties, etc., and the duality of Hitchin fibrations for G and $\breve{G}$ over $\mathbb{C}$ are the main theme of [DP] (see [HT] for the case $G = \mathrm{SL}_n$). As commented by the authors, the argument in [DP] uses transcendental methods in an essential way and therefore cannot be applied to our situation directly.

Our first step is to extend the classical duality to any reductive group G over any algebraically closed field whose characteristic does not divide the order of the Weyl group of G. Let us first give the statement of the classical duality, for details see §3. For a reductive group G and a curve C, let $\mathrm{Higgs}_G \to B$ be the corresponding Hitchin fibration, on which the Picard stack $\mathscr{P} \to B$ acts (see §2 for a review of the Hitchin fibrations). There is an open subset $B^0 \subset B$ such that $\mathscr{P}|_{B^0}$ is a Beilinson 1-motive (a Picard stack that is essentially an abelian variety, see Appendix B). Fix a nondegenerate bilinear form on the Lie algebra $\mathfrak{g}$ of G, one can identify the Hitchin base B and the corresponding open subset B^0 for G and $\breve{G}$. The classical duality is the following assertion.

Theorem 1.2.1. *There is a canonical isomorphism of Picard stacks*

$$\mathfrak{D}_{cl} : (\mathscr{P}|_{B^0})^\vee \simeq \breve{\mathscr{P}}|_{B^0},$$

where $(\mathscr{P}|_{B^0})^\vee$ is the dual Picard stack of $\mathscr{P}|_{B^0}$ (as defined in Appendix B).

Now assume that the characteristic is positive. The second step is to construct a twisted version of the classical duality in this situation. To explain this, let us first introduce a notation: If X is a stack over k, we denote by X' its Frobenius twist, i.e., the pullback of X along the absolute Frobenius endomorphism of k. Let $F_X : X \to X'$ be the relative Frobenius morphism. We will replace both sides of (1.2.1) by certain torsors under $\mathscr{P}'^{\vee}$ and $\breve{\mathscr{P}}'$.

We begin to explain the $\breve{\mathscr{P}}'$-torsor $\breve{\mathscr{H}}$, which was introduced in [CZ]. Recall that there is a smooth commutative group scheme $\breve{J}'$ on $C' \times B'$ and $\breve{\mathscr{P}}'$ classifies $\breve{J}'$-torsors. Let us denote by $\breve{J}^p$ the pullback of $\breve{J}'$ along the relative Frobenius $F_{C' \times B'/B'} : C \times B' \to C' \times B'$. This is a group scheme with a canonical connection along C, and therefore it makes sense to talk about $\breve{J}^p$-local systems on $C \times B'$ and their p-curvatures (see [CZ, Appendix] for generalities). Let $\breve{\mathscr{H}}$ be the stack of $\breve{J}^p$-local systems with some specific p-curvature $\breve{\tau}'$. This is a $\breve{\mathscr{P}}'$-torsor.

Next we explain the $\mathscr{P}'^{\vee}$-torsor $\mathscr{T}_{\mathscr{D}(\theta_m)}$. According to general nonsense (Appendix B), such a torsor gives a multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}$ on $\mathscr{P}'$ and vice versa, so that it is enough to explain this multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta_m)$ on $\mathscr{P}'$. First recall that the sheaf of crystalline differential operators on $\mathscr{P}$ can be regarded as a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{P}}$ on the cotangent bundle $T^*\mathscr{P}'$. We will construct a one-form θ_m on $\mathscr{P}'$, which is multiplicative (in the sense of §C.2). Now, $\mathscr{D} = \mathscr{D}(\theta_m)$ is the gerbe on $\mathscr{P}'$ obtained via pullback of $\mathscr{D}_{\mathscr{P}}$ on $T^*\mathscr{P}'$ along $\theta_m : \mathscr{P}' \to T^*\mathscr{P}'$.

The twisted version of the classical duality is the following assertion

Theorem 1.2.2. *Over B'^0, there is a canonical isomorphism of $\mathscr{P}'^{\vee} \simeq \breve{\mathscr{P}}'$-torsors*

$$\mathfrak{D} : \mathscr{T}_{\mathscr{D}(\theta_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}.$$

The final step towards to establish (1.1.1) is two abelianisation theorems. Another difference between the geometric Langlands correspondence for GL_n and for general group G is that in the latter case, there is no canonical equivalence in general. As is widely known to experts (e.g. see [FW]), the geometric Langlands correspondence for general G should depend on a choice of theta characteristic of the curve C.

Let us fix a square root κ of ω_C. Then the Kostant section for $\mathrm{Higgs}'_G \to B'$ induces a map $\epsilon_{\kappa'} : \mathscr{P}' \to \mathrm{Higgs}'_G$. The first abelianisation theorem asserts a canonical isomorphism

$$\epsilon_{\kappa'}^* \mathscr{D}_{\mathrm{Bun}_G} \simeq \mathscr{D}(\theta_m),$$

where $\mathscr{D}_{\mathrm{Bun}_G}$ is the $\mathbb{G}_m$-gerbe on $\mathrm{Higgs}'_G = T^* \mathrm{Bun}'_G$ of crystalline differential operators on Bun_G and $\mathscr{D}(\theta_m)$ is the $\mathbb{G}_m$-gerbe on $\mathscr{P}'$ mentioned above.

On the dual side, we constructed a canonical morphism in [CZ]

$$\mathfrak{C} : \breve{\mathscr{H}} \times^{\breve{\mathscr{P}}'} \mathrm{Higgs}'_{\breve{G}} \to \mathrm{LocSys}_{\breve{G}},$$

and the Kostant section for $\mathrm{Higgs}'_{\breve{G}} \to B'$ induces an isomorphism

$$\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}^{reg}_{\breve{G}},$$

where $\mathrm{LocSys}^{reg}_{\breve{G}}$ is certain open substack of $\mathrm{LocSys}_{\breve{G}}$.

Combining the above three steps and a general version of the Fourier-Mukai transform (Appendix B) gives the desired equivalence (1.1.1).

Let us mention that the morphism $\mathfrak{C}$ was obtained in [CZ] as a version of Simpson correspondence for smooth projective curves in positive characteristic.

Finally in §5.5 and §5.6, we discuss how the equivalence constructed above depends on the choice of the theta characteristic. This can be regarded as a verification of the predictions of [FW, §10] in our settings.

1.3. **The Langlands transform.** To claim that the above equivalence is the conjectural geometric Langlands transform, one needs to verify several properties the above equivalence is supposed to satisfy. We will only briefly discuss these properties (see [Ga] for more details), but leave the verifications to our next work.

The first property is that the equivalence should intertwine the action of the Hecke operators on the automorphic side and the action of the Wilson operators on the spectral side. Recall that in the case $k = \mathbb{C}$, both categories $D(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D(\mathrm{Qcoh}(\mathrm{LocSys}_{\check{G}}))$ admit actions of a family of commuting operators, labeled by points x on the curve and representations V of the group $\check{G}$. Namely, for $x \in C$ and $V \in \mathrm{Rep}(\check{G})$, there is a so-called Wilson operator $W_{V,x}$ acting on $\mathrm{Qcoh}(\mathrm{LocSys}_{\check{G}})$ by tensoring with the locally free sheaf $V_{E_{univ}}|_{\mathrm{LocSys}_{\check{G}} \times \{x\}}$. On the other side, there is the Hecke operator $H_{V,x}$ acting on $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)$ via certain integral transform (e.g. see [BD, §5]). The second property is that the equivalence should satisfy the Whittaker normalization. Namely, the Whittaker $\mathcal{D}$-module $\mathcal{F}_\Psi$ on Bun_G are supposed to transformed to the structure sheaf $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$.

However, in the positive characteristic, it is yet not clear how to define the Hecke operators (except those corresponding to the minuscule coweights) due to the lack of notion of intersection cohomology $\mathcal{D}$-modules. Our observation is that by the geometric Casselman-Shalika formula ([FGV]), the two properties together will imply that the Whittaker coefficients of $\mathcal{D}$-modules on Bun_G can be calculated by applying the Wilson operators on their Langlands transforms and then taking the global sections. This is a well formulated statement in characteristic p and we will verify in the future work that this is satisfied by the equivalence constructed here.

The third property that the equivalence should satisfy is that it is compatible with Beilinson-Drinfeld's construction of automorphic $\mathcal{D}$-modules via opers ([BD]). We will also verify this property in the future work.

1.4. **Structure of the article.** Let us now describe the contents of this paper in more detail.

In §2 we collect some facts about Hitchin fibration that are used in this paper. Main reference is [N1, N2].

In §3 we prove the classical duality, i.e., duality of Hitchin fibration. This extends the work of [DP] (over $\mathbb{C}$) to any algebraically closed field whose characteristic does not divide the order of the Weyl group of G. In §3.8, we explain the classical duality constructed here is compatible with certain twist by $Z(\check{G})$-torsors, which is used in the discussion of the dependence of the equivalence (1.1.1) on the choice of the theta characteristic.

In §4 we constructed a canonical multiplicative one form θ_m on $\mathscr{P}'$.

In §5 we deduce our main Theorem 5.0.4 from twisted duality (see §5.2) and abelianisation Theorems (see §5.3).

There are three appendices at the end of the paper.

In §A we collect some basic fact about Beilinson's 1-motive and Duality on Beilinson's 1-motive. In particular, we prove a general version of Fourier-Mukai transforms for Beilinson 1-motives.

In §B we recall the basic theory of $\mathcal{D}$-modules over varieties and stacks in positive characteristic, following [BMR, BB, OV, Trav].

In §C we proved the abelian duality for good Beilinson 1-motives. It asserts that the derived category of $\mathcal{D}$-modules on a "good" Beilinson 1-motive $\mathscr{A}$ is equivalent to the derived category of quasi-coherent sheaves on the universal extension $\mathscr{A}^\natural$ by vector groups of its dual $\mathscr{A}^\vee$.

1.5. **Notations.**

1.5.1. *Notations related to algebraic stacks.* Our terminology of algebraic stacks follow the book [LB]. Let k be an algebraically closed field and let p be the characteristic component of k. Let S be a Noetherian scheme over k. In this paper, an algebraic stack $\mathscr{X}$ over S is a stack such that the diagonal morphism

$$\Delta_S : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

is representable and quasi-compact and there exists a smooth presentation, i.e., a smooth, surjective morphism $X \to \mathscr{X}$ from a scheme X.

An algebraic stack $\mathscr{X}$ is called smooth if for every S-scheme U maps smoothly to $\mathscr{X}$, the structure morphism $U \to S$ is smooth.

For any algebraic stack $\mathscr{X}$, we denote by $\mathscr{X}_{Et}$ the big étale site of $\mathscr{X}$. We denote by $\mathscr{X}_{sm}$ the smooth site on $\mathscr{X}$, i.e., it is the site for which the underling category has objects consisting of S-scheme U together with a smooth morphism $U \to \mathscr{X}$ and for which morphisms are smooth 2-morphisms and for which covering maps are smooth surjective maps of schemes. If $\mathscr{X}$ is a Deligne-Mumford stack, we denote by $\mathscr{X}_{et}$ the small étale sit of $\mathscr{X}$.

Assuming that $\mathscr{X}/S$ is smooth and proper. Let $\mathscr{Y} \to \mathscr{X}$ quasi-projective morphism of algebraic stack. We denote by $\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})$ to be the stack of "sections" of $\mathscr{Y}$ over $\mathscr{X}$, i.e., for any $u : U \to S$ we have

$$\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})(U) = \mathrm{Hom}_{\mathscr{X}}(\mathscr{X} \times_S U, \mathscr{Y}).$$

If the base scheme $S = \mathrm{Spec}(k)$, we write $\mathrm{Sect}(\mathscr{X}, \mathscr{Y}) = \mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})$.

Let $\mathscr{X}$ be a smooth algebraic stack over S. We define the relative tangent stack $T(\mathscr{X}/S)$ to be the following stack: for any Spec $\to S$ we have

$$T(\mathscr{X}/S) := \mathscr{X}(R[\epsilon]/\epsilon^2).$$

This stack is algebraic and the natural inclusion $R \to R[\epsilon]/\epsilon^2$ induces a morphism

$$\tau_{\mathscr{X}} : T(\mathscr{X}/S) \to \mathscr{X}.$$

One can show that $T(\mathscr{X}/S)$ is a relative Picard stack over $\mathscr{X}$, therefore, we can associate to it a complex in $D^{[-1,0]}(\mathscr{X}, \mathbb{Z})$ called the relative tangent complex:

$$T^\bullet_{\mathscr{X}/S} = \{T_{\mathscr{X}/S} \to T_{\mathscr{X}}\}.$$

The relative cotangent stack is then defined as

$$T^*(\mathscr{X}/S) := \mathrm{Spec}_{\mathscr{X}}(\mathrm{Sym}_{\mathcal{O}_{\mathscr{X}}} H^0(T^\bullet_{\mathscr{X}/S})).$$

Let $f : \mathscr{X} \to \mathscr{Y}$ be a (representable)morphism between two algebraic stacks over S. We denote the cotangent morphism by

$$\begin{array}{ccc} T^*(\mathscr{Y}/S) \times_{\mathscr{Y}} \mathscr{X} & \xrightarrow{f_d} & T^*(\mathscr{X}/S) \\ \downarrow{\scriptstyle f_p} & & \\ T^*(\mathscr{Y}/S) & & \end{array} \quad (1.5.1)$$

1.5.2. *Notations related Frobenius morphism.* Let S be a Noetherian scheme and $\mathscr{X} \to S$ be an algebraic stack over S. If $p\mathcal{O}_S = 0$, we denote by $Fr_S : S \to S$ be the absolute Frobenius map of S. We have the following commutative diagram

$$\begin{array}{ccccc} \mathscr{X} & \xrightarrow{F_{\mathscr{X}/S}} & \mathscr{X}^{(S)} & \xrightarrow{\pi_{\mathscr{X}/S}} & \mathscr{X} \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{Fr_S} & S \end{array}$$

where the square is Cartesian. We call $\mathscr{X}^{(S)}$ the Frobenius twist of $\mathscr{X}$ along S, and $F_{\mathscr{X}/S} : \mathscr{X} \to \mathscr{X}^{(S)}$ the relative Frobenius morphism. If the base scheme S is clear, $\mathscr{X}^{(S)}$ is also denoted by $\mathscr{X}'$ for simplicity.

1.5.3. *Notation related to torsors.* Let $\mathcal{G}$ be a smooth affine group scheme over X. Let E be a $\mathcal{G}$-torsor on X, we denote by $\mathrm{Aut}(E) = E \times^{\mathcal{G}} \mathcal{G}$ for the adjoint torsor and $\mathrm{ad}(E)$ or $\mathfrak{g}_E = E \times^{\mathcal{G}} \mathrm{Lie}\mathcal{G}$ for the adjoint bundle.

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2. The Hitchin fibration

In this section, we review some basic geometric facts of Hitchin fibrations, following [N1, N2]. Only §2.7 is probably new.

2.1. **Notation related to reductive groups.** Let G be a reductive algebraic group over k of rank l. We denote by $\breve{G}$ its Langlands dual group over k. We denote by $\mathfrak{g}$ and $\breve{\mathfrak{g}}$ Lie algebras of G and $\breve{G}$. We fixed a Borel subgroups $B = TN$ of G with the unipotent radical N and a maximal torus T. The counterpart on the Langlands dual side are denoted by $\breve{T}, \breve{B}$. We denote the corresponding Lie algebras by $\mathfrak{b}$, $\breve{\mathfrak{b}}$, $\mathfrak{t}$, $\breve{\mathfrak{t}}$. We denote by W the canonical Weyl groups of (G, B). We denote by $\mathbb{X}^\bullet$ and $\mathbb{X}_\bullet$ the character and co-character group of T. We denote by $\mathbb{X}_\bullet^+$ the set of dominant coweight.

From now on, we assume that the $char(k) = p$ is zero or $p \nmid |W|$ and we fix a W-invariant non-degenerate bilinear form $(\ ,\ ) : \mathfrak{t} \times \mathfrak{t} \to k$ and identify $\mathfrak{t}$ with $\breve{\mathfrak{t}}$ using $(\ ,\ )$. This invariant form also determines a unique G-invariant non-degenerate bilinear form $\mathfrak{g} \times \mathfrak{g} \to k$, still denoted by $(\ ,\ )$. Let $\mathfrak{g} \to \mathfrak{g}^*$ be the resulting G-equivariant isomorphism.

2.2. **Hitchin map.** Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebra of polynomial function on $\mathfrak{g}$ and $\mathfrak{t}$. By Chevalley's theorem, we have an isomorphism $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^W$. Moreover, $k[\mathfrak{t}]^{\mathrm{W}}$ is isomorphic to a polynomial ring of l variables $u_1, \ldots, u_l$ and each u_i is homogeneous in degree e_i. Let $\mathfrak{c} = \mathrm{Spec}(k[\mathfrak{t}]^W)$. The natural $\mathbb{G}_m$ action on $\mathfrak{g}$ induces a $\mathbb{G}_m$-action on $\mathfrak{c}$ and under the isomorphism $\mathfrak{c} \simeq \mathrm{Spec}(k[u_1, \ldots, u_l]) \simeq \mathbb{A}^l$ the action is given by

$$h \cdot (a_1, \ldots, a_l) = (h^{e_1} a_1, \ldots, h^{e_l} a_l).$$

Let $\chi : \mathfrak{g} \to \mathfrak{c}$ be the map induced by $k[\mathfrak{c}] \simeq k[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$. It is $G \times \mathbb{G}_m$-equivariant map where G acts trivially on $\mathfrak{c}$. Let $\mathcal{L}$ be an invertible sheave on C and we denote by $\mathcal{L}^\times$ the corresponding $\mathbb{G}_m$-torsor. We denote by $\mathfrak{g}_{\mathcal{L}} = \mathfrak{g} \times^{\mathbb{G}_m} \mathcal{L}^\times$ and $\mathfrak{c}_{\mathcal{L}} = \mathfrak{c} \times^{\mathbb{G}_m} \mathcal{L}^\times$ the $\mathbb{G}_m$-twist of $\mathfrak{g}$ and $\mathfrak{c}$ with respect to the natural $\mathbb{G}_m$-action.

Let $\mathrm{Higgs}_{G,\mathcal{L}} = \mathrm{Sect}(C, [\mathfrak{g}_{\mathcal{L}}/G])$ be the stack of section of $[\mathfrak{g}_{\mathcal{L}}/G]$ over C, i.e., for each k-scheme S the groupoid $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consists of maps:

$$h_{E,\phi} : C \times S \to [\mathfrak{g}_{\mathcal{L}}/G].$$

Equivalently, $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consists of a pair (E, ϕ) (called the a Higgs field), where E is an G-torsor over $C \times S$ and ϕ is an element in $\Gamma(C \times S, \mathrm{ad}(E) \otimes \mathcal{L})$. If the group G is clear from the content, we simply write $\mathrm{Higgs}_{\mathcal{L}}$ for $\mathrm{Higgs}_{G,\mathcal{L}}$.

Let $B_{\mathcal{L}} = \mathrm{Sect}_{\mathrm{Spec}\, k}(C, \mathfrak{c}_{\mathcal{L}})$ be the scheme of section of $\mathfrak{c}_{\mathcal{L}}$ over C, i.e., for each k-scheme S, $B_{\mathcal{L}}(S)$ is the set of section

$$b : C \times S \to \mathfrak{c}_{\mathcal{L}}.$$

This is called the Hitchin base of G.

The natural G-invariant projection $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a map

$$[\chi_{\mathcal{L}}] : [\mathfrak{g}_{\mathcal{L}}/G] \to \mathfrak{c}_{\mathcal{L}}.$$

The map $[\chi_{\mathcal{L}}]$ induces a natural map

$$h_{\mathcal{L}} : \mathrm{Higgs}_{\mathcal{L}} = \mathrm{Sect}(C, [\mathfrak{g}_{\mathcal{L}}/G]) \to \mathrm{Sect}(C, \mathfrak{c}_{\mathcal{L}}) = B_{\mathcal{L}}.$$

Definition 2.2.1. We call $h_{\mathcal{L}} : \mathrm{Higgs}_{\mathcal{L}} \to B_{\mathcal{L}}$ the Hitchin map associated to $\mathcal{L}$.

For any $b \in B_{\mathcal{L}}(S)$ we denote by $\mathrm{Higgs}_{\mathcal{L},b}$ the fiber product $S \times_{B_{\mathcal{L}}} \mathrm{Higgs}_{\mathcal{L}}$.

Observe that the invariant bilinear form $\mathfrak{t} \times \mathfrak{t} \to k$ induces a canonical isomorphism $\mathfrak{t} \simeq \mathfrak{t}^* =: \breve{\mathfrak{t}}$, compatible with the W-action. Therefore, there is a canonical isomorphism $\mathfrak{c} \simeq \breve{\mathfrak{c}}$ and $B_{\mathcal{L}} \simeq \breve{B}_{\mathcal{L}}$. In what follows, we will denote the Hitchin base for G and $\breve{G}$ by B.

Let $\omega = \omega_C$ be the canonical line bundle of C. We are mostly interested in the case $\mathcal{L} = \omega$. For simplicity, from now on we denote $B = B_\omega$, $\mathrm{Higgs} = \mathrm{Higgs}_\omega$, $h = h_\omega : \mathrm{Higgs} \to B$, and $\mathrm{Higgs}_b = \mathrm{Higgs}_{\omega_C,b}$. We sometimes also write Higgs_G for Higgs to emphasize the group G. Observe that the bilinear form as in 2.1 induces an isomorphism $\mathrm{Higgs} \simeq T^* \mathrm{Bun}_G$.

2.3. **Kostant section.** In this section, we recall the construction of the Kostant section of Hitchin map $h_{\mathcal{L}}$. For each simple root α_i we choose a nonzero vector $f_i \in \mathfrak{g}_{-\alpha_i}$. Let $f = \oplus_{i=1}^l f_i \in \mathfrak{g}$. We complete f into a sl_2 triple $\{f, h, e\}$ and we denote by $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$. A theorem of Kostant says that $f + \mathfrak{g}^e$ consists of regular element in $\mathfrak{g}$ and the restriction of $\chi : \mathfrak{g} \to \mathfrak{c}$ to $f + \mathfrak{g}^e$ is an isomorphism onto $\mathfrak{c}$. We denote by $kos : \mathfrak{c} \simeq f + \mathfrak{g}^e$ to be the inverse of $\chi|_{f+\mathfrak{g}^e}$. Let $\rho(\mathbb{G}_m)$ be the $\mathbb{G}_m$ action on $\mathfrak{g}$ described below. It acts trivially on $\mathfrak{t}$ and on $\mathfrak{g}_\alpha$ the action is given by $\rho(t)x = t^{\mathrm{ht}(\alpha)} x$ where $\mathrm{ht}(\alpha) = \sum n_i$ if $\alpha = \sum n_i \alpha_i$. We have $\rho(t) f = t^{-1} f$ and

$\rho(t)e = te$, in particular $\mathfrak{g}^e$ is invariant under $\rho(\mathbb{G}_m)$. We define a new $\mathbb{G}_m$-action on $\mathfrak{g}$ by $\rho^+(t) = t\rho(t)$. We have $\rho^+(t)f = f$ and $\rho^+(\mathbb{G}_m)$ preverse $f + \mathfrak{g}^e$. The isomorphism $kos : \mathfrak{c} \simeq f + \mathfrak{g}^e$ is $\mathbb{G}_m$-equivariant where $f + \mathfrak{g}^e$ is acted by $\rho^+(\mathbb{G}_m)$.

For any k scheme S, the groupoid $\mathrm{Higgs}_{\mathcal{L}}(S)$ consists of maps

$$h_{E,\phi} : S \times C \to [\mathfrak{g}/G \times \mathbb{G}_m]$$

such that the composition of $h_{E,\phi}$ with the projection $[\mathfrak{g}/G \times \mathbb{G}_m] \to B\mathbb{G}_m$ is given by the $\mathbb{G}_m$-torsor ρ_L. Similarly, $B_L(S)$ can be regarded as maps

$$b : S \times C \to [\mathfrak{c}/\mathbb{G}_m]$$

such that the composition of b with the projection $[\mathfrak{c}/\mathbb{G}_m] \to B\mathbb{G}_m$ is given by $\mathcal{L}^\times$. Clearly, the Hithch map $h_{\mathcal{L}}$ is induced by the natural map $[\chi/G\times\mathbb{G}_m] : [\mathfrak{g}/G\times\mathbb{G}_m] \to [\mathfrak{c}/\mathbb{G}_m]$.

The diagonal map $\mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ induced a map

$$[\mathfrak{g}/\rho^+(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)].$$

By precomposing with the map $[\mathfrak{c}/\mathbb{G}_m] \overset{kos}{\simeq} [f + \mathfrak{g}^e/\rho^+(\mathbb{G}_m)] \to [\mathfrak{g}/\rho^+(\mathbb{G}_m)]$ we obtain

$$[\mathfrak{c}/\mathbb{G}_m] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)].$$

If the acton of $\rho(\mathbb{G}_m)$ on $\mathfrak{g}$ factors through the adjoint action of G, for example when G is adjoint, then there is a map $[\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times G]$ and it defines a section

$$[\mathfrak{c}/\mathbb{G}_m] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times G]$$

of $[\chi/G\times\mathbb{G}_m]$, in particular, we get a section of $h_{\mathcal{L}}$. In general case, the action $\rho(\mathbb{G}_m)$ does not factor through G, but its square dose and it is given by the co-character $2\rho : \mathbb{G}_m \to G$ where 2ρ is the sum of positive coroots. So if we consider the square map $\mathbb{G}_m^{[2]} \to \mathbb{G}_m$, we get a map

$$\eta^{1/2} : [\mathfrak{c}/\mathbb{G}_m^{[2]}] \to [\mathfrak{g}/\mathbb{G}_m^{[2]} \times \rho(\mathbb{G}_m^{[2]})] \to [\mathfrak{g}/\mathbb{G}_m^{[2]} \times G].$$

Let $\mathcal{L}^{1/2}$ be a square root of $\mathcal{L}$. For any $b : S \times C \to [\mathfrak{c}/\mathbb{G}_m]$ in $B_{\mathcal{L}}(S)$, it factors through a unique map $b^{1/2} : S \times C \to [\mathfrak{c}/\mathbb{G}_m^{[2]}]$. Therefore, by composing with $\eta^{1/2}$, we get a lift of b:

$$\eta^{1/2}(b) : S \times C \xrightarrow{b^{1/2}} [\mathfrak{c}/\mathbb{G}_m^{[2]}] \xrightarrow{\eta^{1/2}} [\mathfrak{g}/\mathbb{G}_m^{[2]} \times G] \to [\mathfrak{g}/\mathbb{G}_m \times G].$$

The assignment $b \to \eta^{1/2}(b)$ defines a section

$$\eta_{\mathcal{L}^{1/2}} : B_{\mathcal{L}} \to \mathrm{Higgs}_{\mathcal{L}}$$

of Hitchin map $h_{\mathcal{L}}$.

We fix a square root $\kappa = \omega^{1/2}$ (called a theta characteristic) of ω and write $\kappa = \eta_\kappa : B \to \mathrm{Higgs}$.

2.4. **Cameral curve.** Let $C \times B \to \mathfrak{c}_{\mathcal{L}}$ be the natural map and let $\widetilde{C}_{\mathcal{L}} = (C \times B) \times_{\mathfrak{c}_{\mathcal{L}}} \mathfrak{t}_{\mathcal{L}}$ universal cameral curve. We have a natural projection $p_B : \widetilde{C}_{\mathcal{L}} \to B_{\mathcal{L}}$ and for any $b : S \to B_{\mathcal{L}}$ we denote by $\widetilde{C}_b$ the fiber product $S \times_{B_{\mathcal{L}}} \widetilde{C}_{\mathcal{L}}$.

2.5. **The universal centralizer group schemes.** Consider the group scheme I over $\mathfrak{g}$ consisting of pair

$$I = \{(g, x) \in G \times \mathfrak{g} \mid \mathrm{Ad}_g(x) = x\}.$$

We define $J = kos^*I$, where $kos : \mathfrak{c} \to \mathfrak{g}$ is the Kostant section. This is called the universal centralizer group scheme of $\mathfrak{g}$ (see Proposition 2.5.1). To study it, it is convenient to introduce two auxiliary group schemes. Let $\pi : \mathfrak{t} \to \mathfrak{c}$ be the projection. We define $J^1 = \mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(T)^{\mathrm{W}}$ and let J^0 to be the neutral component of J^1. All the group schemes J, J^0 and J^1 are smooth commutative group schemes over $\mathfrak{c}$. The following proposition is proved in [N1] (see also [DG]).

Proposition 2.5.1.

(1) There is a canonical isomorphism of group schemes $\chi^*J|_{\mathfrak{g}^{reg}} \simeq I|_{\mathfrak{g}^{reg}}$, which extends to a morphism of group schemes $a : \chi^*J \to I \subset G \times \mathfrak{g}$.

(2) There are natural inclusions $J^0 \subset J \subset J^1$.

All the above constructions can be twisted. Namely, there are $\mathbb{G}_m$-actions on I, J, J^1 and J^0. Moreover, the $\mathbb{G}_m$-action on I can be extended to a $G \times \mathbb{G}_m$-action given by $(h, t) \cdot (x, g) = (tx, hgh^{-1})$. The natural morphisms $J \to \mathfrak{c}$ and $I \to \mathfrak{g}$ are $\mathbb{G}_m$-equvariant, therefore we can twist everything by the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$ and get $J_\mathcal{L} \to \mathfrak{c}_\mathcal{L}$, $I_\mathcal{L} \to \mathfrak{g}_\mathcal{L}$ where $J_\mathcal{L} = J \times^{\mathbb{G}_m} \mathcal{L}^\times$ and $I_\mathcal{L} = I \times^{\mathbb{G}_m} \mathcal{L}^\times$. Similarly, we have $J^0_\mathcal{L} \to \mathfrak{c}_\mathcal{L}$ and $J^1_\mathcal{L} \to \mathfrak{c}_\mathcal{L}$. The group scheme $I_\mathcal{L}$ over $\mathfrak{g}_\mathcal{L}$ is equivariant under the G-action, hence it descends to a group scheme $[I_\mathcal{L}]$ over $[\mathfrak{g}_\mathcal{L}/G]$. For simplicity, we will write $J^0 = J^0_\omega$, $J^1 = J^1_\omega$, $J = J_\omega$ and $I = I_\omega$ if no confusion will arise.

2.6. **Symmetries of Hitchin fibration.** Let $b : S \to B_\mathcal{L}$ be S-point of $B_\mathcal{L}$. It corresponds to a map $b : C \times S \to \mathfrak{c}_\mathcal{L}$. Pulling back $J \to \mathfrak{c}_\mathcal{L}$ using b we get a smooth groups scheme $J_b = b^*J$ over $C \times S$.

Let $\mathscr{P}_b$ be the Picard category of J_b-torsors over $C \times S$. The assignment $b \to \mathscr{P}_b$ defines a Picard stack over B, denoted by $\mathscr{P}_\mathcal{L}$. Let $b \in B_\mathcal{L}(S)$. Let $(E, \phi) \in \mathrm{Higgs}_{\mathcal{L},b}$ and let $h_{E,\phi} : C \times S \to [\mathfrak{g}_\mathcal{L}/G]$ be the corresponding map. Observe that the morphism $\chi^*J \to I$ in Proposition 2.5.1 induces $[\chi_\mathcal{L}]^*J \to [I]$ of group schemes over $[\mathfrak{g}_\mathcal{L}/G]$. Pulling back to $C \times S$ using $h_{E,\phi}$, we get a map

$$a_{E,\phi} : J_b \to h^*_{E,\phi}[I] = \mathrm{Aut}(E, \phi) \subset \mathrm{Aut}(E). \tag{2.6.1}$$

Therefore, using the map $a_{E,\phi}$ we can twist $(E, \phi) \in \mathrm{Higgs}_{\mathcal{L},b}$ by a J_b-torsor. This defines an action of $\mathscr{P}_\mathcal{L}$ on $\mathrm{Higgs}_\mathcal{L}$ over $B_\mathcal{L}$.

Let $\mathrm{Higgs}^{reg}_\mathcal{L}$ be the open stack of $\mathrm{Higgs}_\mathcal{L}$ consists of $(E, \phi) : C \to [\mathfrak{g}_\mathcal{L}/G]$ that factors through $C \to [(\mathfrak{g}^{reg})_\mathcal{L}/G]$. If $(E, \phi) \in \mathrm{Higgs}^{reg}_\mathcal{L}$, then $a_{E,\phi}$ above is an isomorphism. The Kostant section $\eta_{\mathcal{L}^{1/2}} : B_\mathcal{L} \to \mathrm{Higgs}_\mathcal{L}$ factors through $\eta_{\mathcal{L}^{1/2}} : B_\mathcal{L} \to \mathrm{Higgs}^{reg}_\mathcal{L}$. Following [N1, §4], we define by $B^0_\mathcal{L}$ the open sub-scheme of $B_\mathcal{L}$ consisting of $b \in B^0(k)_\mathcal{L}$ such that the image of $b : C \to \mathfrak{c}_\mathcal{L}$ in $\mathfrak{c}_\mathcal{L}$ intersects the discriminant divisor transversally. The following proposition can be extracted from [DG, DP, N1]:

Proposition 2.6.1. *(1) The stack $\mathrm{Higgs}^{reg}_\mathcal{L}$ is a $\mathscr{P}_\mathcal{L}$-torsor, which is trivialized by a choice of Kostant section $\eta_{\mathcal{L}^{1/2}}$.*

(2) One has $\mathrm{Higgs}^{reg}_\mathcal{L} \times_{B_\mathcal{L}} B^0_\mathcal{L} = \mathrm{Higgs}_\mathcal{L} \times_{B_\mathcal{L}} B^0_\mathcal{L}$.

(3) The restriction of the cameral curve $\widetilde{C}|_{B^0} \to B^0_\mathcal{L}$ is smooth. The restriction $\mathscr{P}_\mathcal{L}|_{B^0_\mathcal{L}}$ is a Beilinson 1-motive .

2.7. **The tautological section $\tau : \mathfrak{c} \to \mathrm{Lie}J$.** Recall that by Proposition 2.5.1, there is a canonical isomorphism $\chi^* J|_{\mathfrak{g}^{reg}} \simeq I|_{\mathfrak{g}^{reg}}$. The sheaf of Lie algebras $\mathrm{Lie}(I|_{\mathfrak{g}^{reg}}) \subset \mathfrak{g}^{reg} \times \mathfrak{g}$ admits a tautological section given by $x \mapsto x \in I_x$ for $x \in \mathfrak{g}^{reg}$. Clearly, this section descends to give a tautological section $\tau : \mathfrak{c} \to \mathrm{Lie}J$. Recall the following property of τ [CZ, Lemma 2.2]

Lemma 2.7.1. *Let $x \in \mathfrak{g}$, and $a_x : J_{\chi(x)} \to I_x \subset G$ be the homomorphism as in Proposition 2.5.1 (1). Then $da_x(\tau(x)) = x$.*

If we regard $\mathrm{Lie}J$ as a scheme over $\mathfrak{c}$, besides the section τ, there is a canonical map $c : \mathrm{Lie}J \to \mathfrak{c}$ such that $c\tau = \mathrm{id}$. Namely, if we regard $\mathrm{Lie}(I|_{\mathfrak{g}^{reg}})$ as a scheme, then there is a natural map $\mathrm{Lie}(I|_{\mathfrak{g}^{reg}}) \to \mathfrak{c}$ given by

$$\mathrm{Lie}(I|_{\mathfrak{g}^{reg}}) \subset \mathfrak{g}^{reg} \times \mathfrak{g} \to \mathfrak{g}^{reg} \times \mathfrak{c} \to \mathfrak{c},$$

which also descends to a morphism $c : \mathrm{Lie}J \to \mathfrak{c}$.

The morphisms τ and c have global counterparts (see also [CZ, §2.3]). Observe that $\mathbb{G}_m$ acts on $\mathfrak{g}^{reg} \times \mathfrak{g}$ via natural homothethies on both factors, and therefore on $\chi^*\mathrm{Lie}J|_{\mathfrak{g}^{reg}} \simeq \mathrm{Lie}(I|_{\mathfrak{g}^{reg}}) \subset \mathfrak{g}^{reg} \times \mathfrak{g}$. This $\mathbb{G}_m$-action on $\chi^*\mathrm{Lie}J|_{\mathfrak{g}^{reg}}$ descends to a $\mathbb{G}_m$-action on $\mathrm{Lie}J$ and for any line bundle $\mathcal{L}$ on C, the $\mathcal{L}^\times$-twist $(\mathrm{Lie}J) \times^{\mathbb{G}_m} \mathcal{L}^\times$ under this $\mathbb{G}_m$ action is $\mathrm{Lie}(J_{\mathcal{L}}) \otimes \mathcal{L}$, where $J_{\mathcal{L}}$ is introduced in §2.5. In addition, both maps τ and c are $\mathbb{G}_m$-equivariant with respect to this $\mathbb{G}_m$ action on $\mathrm{Lie}J$ and the natural $\mathbb{G}_m$ action on $\mathfrak{c}$. Therefore, if we define a vector bundle $B_{J,\mathcal{L}}$ over $B_{\mathcal{L}}$, whose fiber over $b \in B_{\mathcal{L}}$ is $\Gamma(C, \mathrm{Lie}J_b \otimes \mathcal{L})$, then by twisting τ and c by $\mathcal{L}$, we obtain

$$\tau_{\mathcal{L}} : B_{\mathcal{L}} \to B_{J,\mathcal{L}}. \tag{2.7.1}$$

which is a canonical section of the projection $\mathrm{pr} : B_{J,\mathcal{L}} \to B_{\mathcal{L}}$, and a canonical map

$$c_{\mathcal{L}} : B_{J,\mathcal{L}} \to B_{\mathcal{L}} \tag{2.7.2}$$

such that $c_{\mathcal{L}}\tau_{\mathcal{L}} = \mathrm{id}$. As before, we omit the subscript $_{\mathcal{L}}$ if $\mathcal{L} = \omega$ for brevity.

Likewise, we introduce the vector bundle $B^*_{J,\mathcal{L}}$ over $B_{\mathcal{L}}$ whose fiber over b is $\Gamma(C, (\mathrm{Lie}J_b)^* \otimes \mathcal{L})$. Observe that $B^*_{J,\mathcal{L}}$ is not the dual of $B_{J,\mathcal{L}}$. Rather, it is the pullback $e^*T^*(\mathscr{P}_{\mathcal{L}}/B_{\mathcal{L}})$ of the cotangent bundle of $\mathscr{P}_{\mathcal{L}} \to B_{\mathcal{L}}$ along the unit section $e : B_{\mathcal{L}} \to \mathscr{P}_{\mathcal{L}}$ and will also be denoted by $\mathbb{T}^*_e(\mathscr{P}_{\mathcal{L}})$ interchangeably later on. We construct a section

$$\tau^*_{\mathcal{L}} : B_{\mathcal{L}} \to B^*_{J,\mathcal{L}} \tag{2.7.3}$$

as follows. The non-degenerate bilinear form $(\ ,\ )$ we fixed in 2.1 induces $\mathfrak{g} \sim \mathfrak{g}^*$, which restricts to a map $\mathrm{Lie}I_x \to (\mathrm{Lie}I_x)^*$ for every $x \in \mathfrak{g}^{reg}$. This map descends to give

$$\iota : \mathrm{Lie}J \to (\mathrm{Lie}J)^*, \tag{2.7.4}$$

which is $\mathbb{G}_m$-equivariant. We define $\tau^*_{\mathcal{L}}$ as the twist of $\mathfrak{c} \to \mathrm{Lie}J \to (\mathrm{Lie}J)^*$. As before, we omit the subscript $_{\mathcal{L}}$ if $\mathcal{L} = \omega$. We give another interpretation of this map.

Observe that the Kostant section κ induces the map

$$v_\kappa : \mathscr{P} \to \mathrm{Higgs}_G \to \mathrm{Bun}_G \times B$$

over B, and therefore,

$$\begin{array}{ccc} T^*(\mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathscr{P} & \xrightarrow{(v_\kappa)_d} & T^*(\mathscr{P}/B) \; . \\ \big\downarrow{\scriptstyle (v_\kappa)_p} & & \\ T^* \mathrm{Bun}_G & & \end{array}$$

Lemma 2.7.2. *The map*

$$\mathscr{P} \overset{\kappa \times \mathrm{id}}{\to} T^*(\mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathscr{P} \overset{(v_\kappa)_d}{\to} T^*(\mathscr{P}/B) \simeq \mathbb{T}_e^*\mathscr{P} \times_B \mathscr{P},$$

can be identified with

$$\mathscr{P} \overset{\mathrm{pr} \times \mathrm{id}}{\to} B \times \mathscr{P} \overset{\tau^* \times \mathrm{id}}{\longrightarrow} \mathbb{T}_e^*\mathscr{P} \times \mathscr{P}.$$

Proof. For $b \in B$, we write the restriction of v_κ over b by $v_{\kappa,b} : \mathscr{P}_b \to \mathrm{Bun}_G$. We need to show that for $x \in \mathscr{P}_b$, the image of the point

$$\kappa(x) \in T^*_{v_{\kappa,b}(x)} \mathrm{Bun}_G \to T^*_x \mathscr{P}_b \simeq (\mathbb{T}_e^*\mathscr{P})_b$$

coincides with $\tau^*(b)$. Let E denote the G-bundle $v_{k,b}(x)$.

Observe that there is a universal G-torsor E_{univ} over $[\mathfrak{g}/G]$ given by $\mathfrak{g} \to [\mathfrak{g}/G]$, and that $\mathrm{ad}(E_{univ}) \to [\mathfrak{g}/G]$ is canonically isomorphic to $[\mathfrak{g}/G] \times_{BG} [\mathfrak{g}/G] \overset{\mathrm{pr}_1}{\to} [\mathfrak{g}/G]$. The cotangent map

$$(v_{\kappa,b})_d : T^*_{v_{\kappa,b}(x)} \mathrm{Bun}_G \to T^*_x \mathscr{P}_b$$

is induced by twisting

$$kos^*(\mathrm{ad}(E_{univ}))^* \to (\mathrm{Lie}J)^*$$

by the $(G \times \mathbb{G}_m)$-torsor $(E \times \omega^\times)$. Therefore, it is enough to show that

$$\kappa(x) \in T^*_{v_{\kappa,b}(x)} \mathrm{Bun}_G = \Gamma(C, \mathfrak{g}_E \otimes \omega)$$

can be identified with the image of b under

$$\tau(b) \in \Gamma(C, \mathrm{Lie}J_b \otimes \omega) \to \Gamma(C, \mathfrak{g}_E \otimes \omega).$$

Let us consider the universal situation. Therefore, we need to show that

$$\mathfrak{c} \overset{\tau}{\to} \mathrm{Lie}J \to kos^*\mathrm{ad}(E_{univ}) \simeq \mathfrak{c} \times_{BG} [\mathfrak{g}/G]$$

is the same as

$$\mathfrak{c} \overset{\mathrm{id} \times kos}{\to} \mathfrak{c} \times_{BG} [\mathfrak{g}/G].$$

However, the composition

$$[\mathfrak{g}/G] \overset{[\chi]^*(\tau)}{\to} [\chi]^*\mathrm{Lie}J \to \mathrm{ad}(E_{univ}) \simeq [\mathfrak{g}/G] \times_{BG} [\mathfrak{g}/G]$$

restricts to a map $[\mathfrak{g}^{reg}/G] \to [\mathfrak{g}^{reg}/G] \times_{BG} [\mathfrak{g}/G]$, which is easily checked to be the diagonal map using the definition of τ. By pulling back this identification along $kos : \mathfrak{c} \to [\mathfrak{g}^{reg}/G]$, we obtain the claim. □

3. Classical duality

In this section, we show that the $\check{\mathscr{P}} \simeq \mathscr{P}^\vee$ as Picard stacks over B^0. Note that this duality for $k = \mathbb{C}$ is the main theorem of [DP] (for $G = \mathrm{SL}_n$, see [HT]). However, as mentioned by the authors, transcendental arguments are used in *loc. cit.* in an essential way, and therefore cannot be applied directly to our situation. Our argument works for any algebraically closed field k of characteristic zero or p with $p \nmid |\mathrm{W}|$.

In fact, it is not hard to construct a canonical isogeny $\mathfrak{D}_{cl}$ between $\check{\mathscr{P}}$ and $\mathscr{P}^\vee$. If the adjoint group of G does not contain a simple factor of type B or C, then to show that $\mathfrak{D}_{cl}$ is an isomorphism is relatively easy. It is to show that $\mathfrak{D}_{cl}$ is an isomorphism in the remaining cases that some complicated calculations are needed.

Observe in this section, we do not need to assume that $\mathcal{L} = \omega_C$. However, to simplify the notations, we still omit the subscript $_{\mathcal{L}}$.

3.1. **Galois description of $\mathscr{P}$.** We first introduce several auxiliary Picard stacks.

Let $\widetilde{C} \to B$ be he universal cameral curve. There is a natural action of W on $\widetilde{C}$. For a T-torsor E_T on $\widetilde{C}$, and an element $w \in \mathrm{W}$, there are two ways to produce a new T-torsor. Namely, the first is via the pullback $w^*E_T = \widetilde{C} \times_{w,\widetilde{C}} E_T$, and the second is via the induction $E_T \times^{T,w} T$. We denote

$$w(E_T) = ((w^{-1})^*E_T) \times^{T,w} T.$$

Clearly, the assignment $E_T \mapsto w(E_T)$ defines an action of W on $\mathrm{Bun}_T(\widetilde{C}/B)$, i.e. for every $w, w' \in \mathrm{W}$, there is a canonical isomorphism $w(w'(E_T)) \simeq (ww')(E_T)$ satisfying the usual cocycle conditions.

Example 3.1.1. Let us decribe $w(E_T)$ more explicitly in the case $G = \mathrm{SL}_2$. Let s be the unique nontrivial element in the Weyl group, acting on the spectral curve $s : \widetilde{C}_b \to \widetilde{C}_b$. If we identify $T = \mathbb{G}_m$-torsors with invertible sheaves $\mathcal{L}$, then

$$s(\mathcal{L}) = s^*\mathcal{L}^{-1}.$$

Let $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ (or $\mathrm{Bun}_T^{\mathrm{W}}$ for simplicity) be the Picard stack of strongly W-equivariant T-torsors on $\widetilde{C}/B$. By definition, for a B-scheme S, $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is the groupoid of $(E_T, \gamma_w, w \in W)$, where E_T is a T-torsor on $\widetilde{C}_S$, and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism, satisfying the natural compatibility conditions. Another way to formulate these compatibility conditions is provided in [DG]. Namely, for a T-torsor E_T, let $\mathrm{Aut}_{\mathrm{W}}(E_T)$ be the group consists of (w, γ_w), where $w \in \mathrm{W}$ and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism. Then there is a natural projection $\mathrm{Aut}_{\mathrm{W}}(E_T) \to \mathrm{W}$. Then an object of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is a pair (E_T, γ), where $\gamma : \mathrm{W} \to \mathrm{Aut}_{\mathrm{W}}(E_T)$ is a splitting of the projection.

For later purpose, it is worthwhile to give another description of $\mathrm{Bun}_T^{\mathrm{W}}$. Namely, there is a non-constant group scheme $\mathcal{T} = \widetilde{C} \times^{\mathrm{W}} T$ on the stack $[\widetilde{C}/\mathrm{W}]$. Then the pullback functor induces an isomorphism from the stack $\mathrm{Bun}_{\mathcal{T}}$ of $\mathcal{T}$-torsors on $[\widetilde{C}/W]$ to $\mathrm{Bun}_T^{\mathrm{W}}$.

In [DG], a Galois description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}$ is given. We here refine their description.

Let $\mathscr{P}^1$ be the Picard stack over B classifying J^1-torsors on $C \times B$. First, we claim that there is a canonical morphism

$$j^1 : \mathscr{P}^1 \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B). \tag{3.1.1}$$

To construct j^1, recall that $J^1 = (\pi_*(T \times \widetilde{C}))^W$, where $\pi : \widetilde{C} \to C \times B$ is the projection, and therefore, for any J^1-torsor E_{J^1} on $C \times S$ (where $b : S \to B$ is a test scheme), one can form a T-torsor on $\widetilde{C}_S$ by

$$E_T := \pi^* E_{J^1} \times^{\pi^* J^1} T. \tag{3.1.2}$$

Clearly, E_T carries on a strongly W-equivariant structure γ, and $j^1(E_{J^1}) = (E_T, \gamma)$ defines the morphism j^1.

The morphism j^1, in general, is not an isomorphism. Let us describe the image. Let $\alpha \in \Phi$ be a root and let $i_\alpha : \widetilde{C}_\alpha \to \widetilde{C}$ be the inclusion of the fixed point subscheme of the reflection s_α. Let $T_\alpha = T/(s_\alpha - 1)$ be the torus of coinvariants of the reflection s_α. Then $s_\alpha(E_T)|_{\widetilde{C}_\alpha} \times^T T_\alpha$ is canonically isomorphic to $E_T|_{\widetilde{C}_\alpha} \times^T T_\alpha$ and therefore $\gamma_{s_\alpha}|_{\widetilde{C}_\alpha}$ induces an automorphism of the T_α-torsor $E_T \times^T T_\alpha$. In other words, there is a natural map

$$r = \prod_{\alpha \in \Phi} r_\alpha : \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B) \to (\prod_{\alpha \in \Phi} \mathrm{Res}_{\widetilde{C}_\alpha/B}(T_\alpha \times \widetilde{C}_\alpha))^W.$$

It is easy to see that rj_1 is trivial, and one can show that

Lemma 3.1.2. $\mathscr{P}^1 \simeq \ker r$. *In other words, $\mathscr{P}^1(S)$ consists of those strongly* W-*equivariant T-torsors (E_T, γ) such that the induced automorphism of $E_T \times^T T_\alpha|_{\widetilde{C}_\alpha}$ is trivial for every $\alpha \in \Phi$.*

Proof. One needs to show that every strongly W-equivaraint T-torsor (E_T, γ) such that $r(E_T, \gamma) = 1$ is étale locally on $\widetilde{C}$ isomorphic to the trivial one, i.e., the trivial T-torsor together with the canonical W-equivariance structure. If this is the case, then the inverse map from $\ker r \to \mathscr{P}^1$ is given as follows. For every strongly W-equivariant T-torsor (E_T, γ), $\pi_* E_T$ carries on an action of W. Namely, let $x : S \to C$ be a point and $m : S \times_C \widetilde{C}_b \to E_T$ be a point of $\pi_* E_T$ over x. Then $w(m)$ is the point of $\pi_* E_T$ over x given by

$$S \times_C \widetilde{C}_b \stackrel{1 \times w^{-1}}{\to} S \times_C \widetilde{C}_b \stackrel{w^{-1}(m)}{\to} (w^{-1})^* E_T \to w(E_T) \stackrel{s(w)}{\to} E_T.$$

This W-action on $\pi_* E_T$ is compatible with the action of $\pi_*(T \times \widetilde{C})$ in the sense that $w(mt) = w(m)w(t)$. Now let $E_{J^1} = (\pi_* E_T)^W$, then as (E, γ) is locally isomorphic to the trivial one, E_{J^1} is locally isomorphic to J^1, and therefore is a J^1-torsor on C.

To prove the local triviality, we follow the argument as in [DG, Proposition 16.4]. One reduces to prove the statement for a neighborhood around a point $x \in \cap_\alpha \widetilde{C}_\alpha$. By replacing $\widetilde{C}$ by the local ring around x, one can assume that E_T is trivial. Pick up a trivialization, then the W-equivariance structure on E_T amounts to a 1-cocycle $W \to T(\widetilde{C})$. By evaluating $T(\widetilde{C})$ at the unique closed point x, there is a short exact sequence $1 \to K \to T(\widetilde{C}) \to T(k) \to 1$. The condition $r(E_T, \gamma) = 1$ would mean that the cocycle takes value in K. As K is an $\mathbb{F}_p$-vector space and $p \nmid |\mathrm{W}|$, this cocycle is trivial. $\square$

3.2. Recall that in [DG, N1], an open embedding $J \to J^1$ is constructed. To describe the cokernel, we need some notations. Let $\breve{\alpha} \in \breve{\Phi}$ be a coroot. Let

$$\mu_{\breve{\alpha}} := \ker(\breve{\alpha} : \mathbb{G}_m \to T).$$

This is either trivial, or μ_2, depending on whether $\breve{\alpha}$ is primitive or not. Let $\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha$ be the constant group scheme over $\widetilde{C}_\alpha$, regarded as a sheaf of groups over $\widetilde{C}_\alpha$, and let $(i_\alpha)_*(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha)$ be its push forward to $\widetilde{C}$. Now, the result of [DG] can be reformulated as: there is a natural exact sequence of sheaves of groups on C.

$$1 \to J \to J^1 \to \pi_*(\bigoplus_{\alpha\in\Phi}(i_\alpha)_*(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha))^{\mathrm{W}} \to 1. \tag{3.2.1}$$

As a result, we obtain a short exact sequence (by taking the cohomology $R\Gamma(C, -)$)

$$1 \to (\prod_{\alpha\in\Phi} \mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha))^{\mathrm{W}} \to \mathscr{P} \to \mathscr{P}^1 \to 1. \tag{3.2.2}$$

To simplify the notation, we will denote $\mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha)$ by $\mu_{\breve{\alpha}}(\widetilde{C}_\alpha)$ in what follows.

Consider the composition

$$j : \mathscr{P} \to \mathscr{P}^1 \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B).$$

By combining Lemma 3.1.2 and (3.2.2), we recover a description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ as given in [DG]. Namely, given a strongly W-equivariant T-torsor (E_T, γ), one obtains a canonical trivialization

$$E_T^{\breve{\alpha}\circ\alpha} := (E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \times^{\mathbb{G}_m,\breve{\alpha}} T \simeq E_T^0|_{\widetilde{C}_\alpha}, \tag{3.2.3}$$

as $(E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \times^{\mathbb{G}_m,\breve{\alpha}} T \simeq E_T|_{\widetilde{C}_\alpha} \otimes s_\alpha(E_T^{-1})|_{\widetilde{C}_\alpha}$. The condition that $r_\alpha(E, \gamma) = 1$ is equivalent to the condition that (3.2.3) comes from a trivialization

$$c_\alpha : E_T^\alpha := (E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha. \tag{3.2.4}$$

In addition, the set of all such c_α form a $\mu_{\breve{\alpha}}$-torsor. Therefore, we can describe $\mathscr{P}(S)$ as the Picard groupoid of triples

$$\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}_S)^+ := (E_T, \gamma, c_\alpha, \alpha \in \Phi) \tag{3.2.5}$$

where (E_T, γ) is a strongly W-equivariant T-torsor on $\widetilde{C}_S$, and $c_\alpha : (E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha$ is a trivialization, which induces (3.2.3), and is compatible with the W-equivariant structure. We called those trivializations $\{c_\alpha\}_{\alpha\in\Phi}$ a $+$-structure on (E_T, γ).

Here is an application of the above discussion. Observe there is the norm map

$$\mathrm{Nm} : \mathrm{Bun}_T(\widetilde{C}/B) \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B), \quad E_T \mapsto (\bigotimes_{w\in W} w(E_T), \gamma_{\mathrm{can}}).$$

We claim that Nm admits a canonical lifting

$$\mathrm{Nm}^{\mathscr{P}} : \mathrm{Bun}_T(\widetilde{C}/B) \to \mathscr{P}. \tag{3.2.6}$$

To show this, we need to exhibit a canonical trivialization

$$c_\alpha : \bigotimes_{w\in W} w(E_T)|_{\widetilde{C}_\alpha} \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha.$$

compatible with the strongly W-equivariant structure. However, for any T-torsor E_T, there is a canonical isomorphism $(E_T|_{\widetilde{C}_\alpha} \otimes s_\alpha(E_T)|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha$, and therefore, we obtain c_α by noting

$$\bigotimes_{w\in W} w(E_T)|_{\widetilde{C}_\alpha} \times^{T,\alpha} \mathbb{G}_m \simeq \bigotimes_{w\in s_\alpha\backslash W} (w(E_T)|_{\widetilde{C}_\alpha} \otimes s_\alpha w(E_T)|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m.$$

The compatibility of the collection $\{c_\alpha\}$ with the W-equivariant structure is clear.

3.3. **Galois description of $\mathscr{P}$-torsors.** The above description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ can be generalized as follows. Let $\mathscr{D}$ be a J-gerbe on $C \times B$. Similar to (3.1.2), we define

$$\mathscr{D}_T := (\pi^*\mathscr{D})^{j^1}$$

to be the T-gerbe on $\widetilde{C}$ induced from $\mathscr{D}$ using maps $\pi : \widetilde{C} \to C \times B$ and $j^1 : \pi^*J \to T \times \widetilde{C}$ (see A.4 and A.5 for the notion of grebes and functors between them). Since the map j^1 is W-equivariant the grebe $\mathscr{D}_T$ is strongly W-equivariant. Equivalently, this means that $\mathscr{D}_T$ descends to a $\mathfrak{T}$-gerbe on $[\widetilde{C}/\mathrm{W}]$.

Let $\mathscr{T}_{\mathscr{D}}$ be the stack of splittings of $\mathscr{D}$ over B. By definition, for every $S \to B$, $\mathscr{T}_{\mathscr{D}}(S)$ is the groupoid of the splittings of the gerbe $\mathscr{D}|_{C\times S}$. Clearly, this is a (pseudo) $\mathscr{P}$-torsor. On the other hand, let $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}$ denote the stack of strongly W-equivariant splittings of $\mathscr{D}_T$, i.e., $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}(S)$ is the groupoid of the splitting of $\mathscr{D}_T|_{[\widetilde{C}/\mathrm{W}]\times_B S}$. Our goal is to give a description of $\mathscr{T}_{\mathscr{D}}$ in terms of $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}$.

Let $\alpha \in \Phi$. Similar to E_T^α and $E_T^{\check{\alpha}\circ\alpha}$ as defined in (3.2.3) and (3.2.4), let $\mathscr{D}_T^\alpha$, $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ be the restriction to $\widetilde{C}_\alpha$ of the $\mathbb{G}_m$- and T-gerbes on $\widetilde{C}$ induced from $\mathscr{D}_T$ using the maps $\alpha : T \to \mathbb{G}_m$ and $\check{\alpha} \circ \alpha : T \to T$. The strongly W-equivariant structure on $\mathscr{D}_T$ implies the T-gerbe $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ has a canonical splitting F_α^0. Moreover, by a similar argument in §3.2, one can show that: (i) there is a canonical splitting E_α^0 of the $\mathbb{G}_m$-gerbe $\mathscr{D}_T^\alpha$, which induces F_α^0 via the canonical map $\mathscr{D}_T^\alpha \to \mathscr{D}_T^{\check{\alpha}\circ\alpha}$ and (ii) for any strongly W-equivariant splitting (E,γ) of $\mathscr{D}_T$ there is a canonical isomorphism of splittings

$$E^{\check{\alpha}\circ\alpha}|_{\widetilde{C}_\alpha} \simeq F_\alpha^0, \tag{3.3.1}$$

here $E^{\check{\alpha}\circ\alpha}$ is the splitting of $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ induces by E via the canonical map $\mathscr{D}_T^\alpha \to \mathscr{D}_T^{\check{\alpha}\circ\alpha}$. We define $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$ to be the stack over B whose S-points consist of

$$\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}(S) := (E, \gamma, t_\alpha, \alpha \in \Phi)$$

where (E,γ) is a strongly W-equivariant splittings of $\mathscr{D}_T$ and

$$t_\alpha : E^\alpha|_{\widetilde{C}_\alpha} \simeq E_\alpha^0$$

is an isomorphism of splittings of $\mathscr{D}_T^\alpha$, which induces (3.3.1), and is compatible with the W-equivariant structure. It is clear that $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$ is a $\mathscr{P} = \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+$-torsor.

Lemma 3.3.1. *There is a canonial isomorphism of $\mathscr{P}$-torsor $\mathscr{T}_{\mathscr{D}} \simeq \mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$.*

Proof. Let $E \in \mathscr{T}_{\mathscr{D}}$ be a splitting of $\mathscr{D}$. Then $E_T := (\pi^*(E))^{j^1}$ defines a splittings of $\mathscr{D}_T$. Since both maps j^1 and π are W-equivariant the splitting E_T has a canonial W-equivariant structure, which we denote by γ. Moreover, by the same reasoning as in §3.2, there is a canonical isomorphism of splittings $t_\alpha : E_T^\alpha|_{\widetilde{C}_\alpha} \simeq E_\alpha^0$ such that the induced isomorphism $E_T^{\check{\alpha}\circ\alpha}|_{\widetilde{C}_\alpha} \simeq (E^0_\alpha)^\alpha \simeq F^0_\alpha$ is equal to the one coming form the W-equivariant structure γ. The assignment $E \to (E_T, \gamma, t_\alpha, \alpha \in \Phi)$ defines a morphism $\mathscr{T}_{\mathscr{D}} \to \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}_T}$ and one can check that it is compatible with their $\mathscr{P}$-torsor structures, hence an isomorphism. □

3.4. **The Abel-Jacobi map.** From now on till the end of this section, we restrict to the open subset B^0 of the Hitchin base. To simplify the notation, sometimes we use B to denote B^0 unless specified. Recall from Proposition 2.6.1 that the cameral curve $\widetilde{C}$ is smooth over B^0.

Let

$$\mathrm{AJ} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathrm{Bun}_T(\widetilde{C}/B)$$

be the Abel-Jacobi map given by $(x, \check{\lambda}) \mapsto \mathcal{O}(\check{\lambda}x) := \mathcal{O}(x) \times^{\mathbb{G}_m, \check{\lambda}} T$.

Therefore we obtain

$$\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathscr{P}.$$

by composing AJ with $\mathrm{Nm}^{\mathscr{P}}$. This morphism is W-equivariant, where W acts on $\widetilde{C} \times \mathbb{X}_\bullet(T)$ diagonally and on $\mathscr{P}$ trivially, and is commutative and multiplicative with respect to the group structures on $\mathbb{X}_\bullet(T)$ and on $\mathscr{P}$. Observe that for any $x \in \widetilde{C}_\alpha$, $\mathrm{AJ}^{\mathscr{P}}(x, \check{\alpha})$ is the unit in $\mathscr{P}$. This follows from

$$\bigotimes_{w \in W} w\mathcal{O}(\check{\alpha}x) \simeq \bigotimes_{w \in W/s_\alpha} w\mathcal{O}(\check{\alpha}x + s_\alpha(\check{\alpha})x)$$

is canonically trivialized, and the trivialization is compatible with the W-equivariant structure.

By pulling back the line bundles, we thus obtain

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Pic}^m(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}},$$

where $\mathrm{Pic}^m(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}$ denotes the Picard stack over B of W-equivariant line bundles on $\widetilde{C} \times \mathbb{X}_\bullet(T)$ whic are multiplicative with respect to $\mathbb{X}_\bullet(T)$. Observe that there is the canonical isomorphism $\mathrm{Bun}^{\mathrm{W}}_{\check{T}}(\widetilde{C}/B) \to \mathrm{Pic}^m(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}$ given by $(E_{\check{T}}, \gamma) \mapsto \mathscr{L}$, where $\mathscr{L}|_{(x,\check{\lambda})} = E^{\lambda}_{\check{T}}|_x$. Therefore, we regard $(\mathrm{AJ}^{\mathscr{P}})^\vee$ as a morphism

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Bun}^{\mathrm{W}}_{\check{T}}(\widetilde{C}/B).$$

We claim that $(\mathrm{AJ}^{\mathscr{P}})^\vee$ canonically lifts to a morphism

$$\mathfrak{D}_{cl} : \mathscr{P}^\vee \to \check{\mathscr{P}}.$$

Let $\mathcal{L}$ be a multiplicative line bundle on $\mathscr{P}$. We thus need to show that

$$(\mathrm{AJ}^{\mathscr{P}})^*\mathcal{L}|_{(\widetilde{C}_\alpha, \check{\alpha})}$$

admits a canonical trivialization, which is compatible with the W-equivariance structure. However, this follows from $\mathrm{AJ}^{\mathscr{P}}((x, \check{\alpha}))$ is the unit of $\mathscr{P}$ and a multiplicative line bundle on $\mathscr{P}$ is canonically trivialized over the unit. To summarize, we have constructed the following commutative diagram

(3.4.1)
$$\begin{array}{ccc} \mathscr{P}^\vee & \xrightarrow{\mathfrak{D}_{cl}} & \check{\mathscr{P}} \\ & {\scriptstyle (\mathrm{AJ}^{\mathscr{P}})^\vee} \searrow \quad \swarrow {\scriptstyle \check{j}} & \\ & \mathrm{Bun}^{\mathrm{W}}_{\check{T}} & \end{array}$$

Now, the classical duality theorem reads as

Theorem 3.4.1. *$\mathfrak{D}_{cl}$ is an isomorphism.*

The rest of the section is devoted to the proof of this theorem.

3.5. **First reductions.** We first show that $\mathfrak{D}_{cl}$ induces an isomorphism

$$\pi_0(\mathfrak{D}_{cl}) : \pi_0(\mathscr{P}^\vee) \to \pi_0(\breve{\mathscr{P}}).$$

For any S-point $b \in B^0$, $\mathscr{P}_b$ is a Beilinson 1-motive (Appendix B). We have

$$\underline{\mathrm{Aut}}(e) \simeq \mathrm{H}^0(C, J_b), \quad \pi_0(\mathscr{P}_b) = \mathscr{P}_b / W_1 \mathscr{P}_b.$$

Observe that

$$\mathrm{H}^0(C, J_b) \simeq \ker(T^{\mathrm{W}} \to (\prod_{\alpha \in \Phi} \mathrm{Res}_{\widetilde{C}_\alpha / b}(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha))^{\mathrm{W}}) = Z(G).$$

By Corollary A.3.3

$$\pi_0(\mathscr{P}^\vee) \simeq (\mathrm{Aut}_{\mathscr{P}}(e))^*.$$

Let us also recall the description of $\pi_0(\mathscr{P})$ as given in [N2, §4.10, §5.5]. As we restrict $\mathscr{P}$ to B^0, the answer is very simple. Namely, the Abel-Jacobi map

$$\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathscr{P}$$

induces a surjective map

$$\pi_0(\widetilde{C} \times \mathbb{X}_\bullet(T)) \simeq \mathbb{X}_\bullet(T) \twoheadrightarrow \pi_0(\mathscr{P}).$$

which induces

$$\pi_0(\mathscr{P})^* \simeq Z(\breve{G}) \subset \breve{T}^{\mathrm{W}}.$$

Therefore, as abstract groups $\pi_0(\mathscr{P}^\vee) \simeq \pi_0(\breve{\mathscr{P}})$.

Since $\pi_0(\mathscr{P}^\vee) \simeq \pi_0(\breve{\mathscr{P}})$ are finitely generated abelian groups and are isomorphic abstractly, to show that $\pi_0(\mathfrak{D}_{cl})$ is an isomorphism, it is enough to show that

Lemma 3.5.1. *The induced map $\pi_0(\mathfrak{D}_{cl})$ is surjective.*

Proof. According to the above description, it is enough to construct a morphism $\widetilde{C} \times \mathbb{X}^\bullet(T) \to \mathscr{P}^\vee$ making the following diagram is commutative.

$$\begin{array}{ccccc} & & \widetilde{C} \times \mathbb{X}^\bullet(T) & & \\ & \swarrow & & \searrow^{\mathrm{AJ}^{\breve{\mathscr{P}}}} & \\ \mathscr{P}^\vee & & \xrightarrow{\quad \mathfrak{D}_{cl} \quad} & & \breve{\mathscr{P}}. \end{array}$$

To this goal, observe that there is the universal line bundle $\mathscr{L}_{\mathrm{univ}}$ on $(\widetilde{C} \times \mathbb{X}^\bullet(T)) \times \mathrm{Bun}_T$. Then the pullback of this line bundle to $(\widetilde{C} \times \mathbb{X}^\bullet(T)) \times \mathscr{P}$ gives rise to the desired map. The commutativity of this diagram is an easy exercise. □

Next, we see that

$$W_0(\mathfrak{D}_{cl}) : W_0 \mathscr{P}^\vee \to W_0 \breve{\mathscr{P}}$$

is an isomorphism. Indeed, we can construct $\mathrm{AJ}^{\breve{\mathscr{P}}} : \widetilde{C} \times \mathbb{X}^\bullet(T) \to \breve{\mathscr{P}}$, and therefore $\breve{\mathfrak{D}}_{cl} : \breve{\mathscr{P}}^\vee \to \mathscr{P}$. By the same argument, it induces an isomorphism $\pi_0(\breve{\mathfrak{D}}_{cl}) : \pi_0(\breve{\mathscr{P}}^\vee) \to \pi_0(\mathscr{P})$. It is easy to check that $\breve{\mathfrak{D}}_{cl} = \mathfrak{D}_{cl}^\vee$, and therefore $W_0(\mathfrak{D}_{cl})$ is also an isomorphism.

Therefore, it is enough to show that $D_{cl}: P^\vee \to \check{P}$ is an isomorphism, where P (resp. $\check{P}$) is the neutral connected component of the coarse space of $\mathscr{P}$ (resp. $\check{\mathscr{P}}$), and D_{cl} is the map induced by $\mathfrak{D}_{cl}$. We can prove this fiberwise, and therefore we fix $b \in B(k)$. However, to simplify the notation, in the following discussion we write $\widetilde{C}, \mathscr{P}$ instead of $\widetilde{C}_b, \mathscr{P}_b$, etc.

3.6. **The calculation of the coarse moduli.** We introduce a few more notations. Let $\mathscr{P}^0$ be the Picard stack of J^0-torsors on C, and let P^0 (resp. P^1) be the neutral connected components of the coarse space of $\mathscr{P}^0$ (resp. $\mathscr{P}^1$).

We first understand P^1. Let Jac denote the Jacobi variety of $\widetilde{C}$. Then $\mathrm{Jac}\otimes\mathbb{X}_\bullet$ is the neutral connected component of the coarse space of Bun_T.

Lemma 3.6.1. *The map $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is an embedding, and P^1 can be identified with $(\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W},0}$, the neutral connected component of the* W*-fixed point subscheme of* $\mathrm{Jac}\otimes\mathbb{X}_\bullet$.

Proof. We first show that $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is injective. Indeed, up to isomorphism, the strongly W-equivariant structures on a trivializable T-torsor on $\widetilde{C}$ are classified by $\mathrm{H}^1(\mathrm{W}, T(k))$. By Lemma 3.1.2, the kernel of $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ can be identified with the kernel of

$$\mathrm{H}^1(\mathrm{W}, T(k)) \to \bigoplus_{\widetilde{C}_\alpha} T_\alpha(\widetilde{C}_\alpha).$$

Therefore, it is then enough to show that this latter map is injective. Over B^0, $\widetilde{C}_\alpha$ is nonempty for every root α. Then the injectivity of this map follows from [HMS, Proposistion 2.6, (iii)].

To complete the proof, observe that the restriction of the norm

$$\mathrm{Nm}: \mathrm{Jac}\otimes\mathbb{X}_\bullet \to P \to P^1 \to (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}}$$

to $\mathrm{Nm}: (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}} \to (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}}$ is the multiplication by $|\mathrm{W}|$. Therefore, the image of $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is $(\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W},0}$. $\square$

As a result, for any $\ell \neq p$ a prime,

$$T_\ell P^1 \simeq (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet)^{\mathrm{W}}.$$

In addition, observe that from the definition of D_{cl}, the map $P^1 \subset \mathrm{Jac}\otimes\mathbb{X}_\bullet \stackrel{\mathrm{Nm}}{\to} P^1$ factors as

$$P^1 \subset \mathrm{Jac}\otimes\mathbb{X}_\bullet \simeq (\mathrm{Jac}\otimes\mathbb{X}^\bullet)^\vee \to (\check{P}^1)^\vee \to \check{P}^\vee \stackrel{\check{D}_{cl}}{\to} P \to P^1.$$

Therefore D_{cl} is a prime-to-p isogeny. In addition, the map

$$T_\ell\mathrm{Nm}: T_\ell(\mathrm{Jac}\otimes\mathbb{X}_\bullet) \twoheadrightarrow T_\ell(\check{P}^1)^\vee \hookrightarrow T_\ell P^1$$

can be identified with

$$\mathrm{Nm}: \mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1))\otimes\mathbb{X}_\bullet \twoheadrightarrow (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1))\otimes\mathbb{X}_\bullet)_{\mathrm{W}}/(\mathrm{torsion}) \hookrightarrow (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1))\otimes\mathbb{X}_\bullet)^{\mathrm{W}}.$$

We need the following key result.

Lemma 3.6.2. *The quasi-isogeny $(\check{P}^1)^\vee \to P^1 \leftarrow P^0$ is an isomorphism.*

Proof. Consider the norm map $\mathrm{Nm} : \pi_*T \to J^1 = (\pi_*T)^{\mathrm{W}}$. As J^0 is connected, Nm factors as $\pi_*T \to J^0 \to J^1$. Therefore, $\mathrm{Nm} : \mathrm{Jac} \otimes \mathbb{X}_\bullet \to P^1$ also factors as $\mathrm{Nm} : \mathrm{Jac} \otimes \mathbb{X}_\bullet \to P^0 \to P^1$. Clearly, $P^0 \to P^1$ is a prime-to-p isogeny. Therefore, the lemma then is equivalent to saying that the induced map of Tate modules $\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet \to T_\ell P^0$ is surjective for every $\ell \neq p$.

The Kummer sequence for J^0 gives rise to

$$T_\ell P^0 \simeq \varprojlim \mathrm{H}^1(C, J^0[\ell^n]).$$

Observe that as J^0 is connected, the system $\{J^0[\ell^n]\}$ form a $\mathbb{Z}_\ell$-sheaf M on C. This means that $J^0[\ell^n]$ is a $\mathbb{Z}/\ell^n$-module and that the multiplication by ℓ map $J^0[\ell^n] \to J^0[\ell^{n-1}]$ induces an isomorphism $J^0[\ell^n] \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^{n-1} \simeq J^0[\ell^{n-1}]$ (observe that the system $\{J[\ell^n]\}$ does not satisfy these conditions if J is not connected). Therefore, using the standard notation, $T_\ell P^0 \simeq \mathrm{H}^1(C, M)$.

Let $j : U \subset C$ be the complement of the ramification loci of $\pi : \widetilde{C} \to C$ and let $\widetilde{U}$ be its preimage in $\widetilde{C}$. Then our assumption implies that

$$\widetilde{C} = \coprod_{\alpha \in \Phi} \widetilde{C}_\alpha \sqcup \widetilde{U},$$

Let $L = \pi_*(\mathbb{X}_\bullet \otimes \mathbb{Z}_\ell|_{\widetilde{U}})$. This is a local system on U with an action of W. Clearly, $j^*M \simeq L^{\mathrm{W}}(1)$ and there is a short exact sequence

$$0 \to j_! L^{\mathrm{W}}(1) \to M \to N \to 0,$$

where N is finite and supports on the ramification locus. The norm maps $\mathrm{Nm} : L \to L^{\mathrm{W}}(1)$ induces the following commutative diagram with arrows in each row surjective

$$\begin{array}{ccc} \mathrm{H}^1_c(\widetilde{U}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet & \longrightarrow & \mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet \\ \mathrm{Nm}\big\downarrow & & \mathrm{Nm}\big\downarrow \\ (\mathrm{H}^1_c(\widetilde{U}, L^{\mathrm{W}}(1)) & \longrightarrow & \mathrm{H}^1(\widetilde{C}, M). \end{array}$$

Therefore, it remains to show that

$$\mathrm{Nm} : \mathrm{H}^1_c(\widetilde{U}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet = \mathrm{H}^1_c(U, L(1)) \to \mathrm{H}^1_c(U, L^{\mathrm{W}}(1))$$

is surjective.

Pick up $x \in U$. Then the stalk of L at x

$$L_x \simeq \mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}_\bullet,$$

and the monodromy representation $\rho : \pi_1(U, x) \to \mathrm{GL}(L_x)$ is given by $\rho(\gamma)(a \otimes b) = \rho(\gamma)a \otimes b$ (to make these identifications canonical, one needs to pick up $\bar{x} \in \widetilde{U}$ lying over x). There is another action of W on L_x given by $w(a \otimes b) = aw^{-1} \otimes wb$, which gives rise to the W-action on L. Let $L^\vee$ be the dual local system of L.

As L is a tame local system on U, it is well-known that

$$\mathrm{H}^1_c(U, L(1)) \simeq \mathrm{H}^1(U, L^\vee)^\vee \simeq \mathrm{H}^1(\pi_1^{\mathrm{tame}}(U, x), L_x^\vee)^\vee.$$

We recall the following theorem of Grothendieck (SGA 1, XIII, cor. 2.12, p. 392).

Theorem 3.6.3. *The tame fundamental group $\pi_1^{\mathrm{tame}}(U, x)$ is the profinite completion $\hat{\Gamma}_{g,n}$ of a free group of $2g + \sharp(C \setminus U)(k) - 1$ generators $\{\gamma_i\}$. In addition, this surjective map $\hat{\Gamma}_{g,n} \to \pi_1^{\mathrm{tame}}(U, x)$ induces an isomorphism of the maximal prime-to-p quotients.*

As $p \nmid |\mathrm{W}|$, we have

$$\mathrm{H}^1_c(U, L(1)) \simeq \mathrm{H}^1(\pi_1^{\mathrm{tame}}(U,x), L_x^\vee) \simeq \bigoplus_i \frac{L_x^\vee}{(1-\rho(\gamma_i))L_x^\vee}.$$

Similarly,

$$\mathrm{H}^1_c(U, L^{\mathrm{W}}(1)) \simeq \bigoplus_i \frac{(L_x^\vee)_{\mathrm{W}}}{(1-\rho(\gamma_i))(L_x^\vee)_{\mathrm{W}}}.$$

The map $\mathbb{X}^\bullet \to \mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet$ given by $\lambda \mapsto 1 \otimes \lambda$ induces an isomorphism $\mathbb{X}^\bullet \simeq (L_x^\vee)_{\mathrm{W}}$. Therefore, it is enough to show that

$$\mathrm{Nm} : \mathrm{Hom}\Big(\frac{\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet}{(1-\gamma)\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet}, \mathbb{Z}_\ell\Big) \to \mathrm{Hom}\Big(\frac{\mathbb{X}^\bullet}{(1-\gamma)\mathbb{X}^\bullet}, \mathbb{Z}_\ell\Big) \tag{3.6.1}$$

is surjective for any $\gamma \in \mathrm{W}$. Observe that

$$\mathrm{Hom}\Big(\frac{\mathbb{X}^\bullet}{(1-\gamma)\mathbb{X}^\bullet}, \mathbb{Z}_\ell\Big) \simeq \mathrm{Hom}\Big(\frac{\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet}{(1-\gamma)\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet}, \mathbb{Z}_\ell\Big)^{\mathrm{W}}$$

and under this identification, the norm map (3.6.1) then is given by the formula $\mathrm{Nm}(\varphi)(a \otimes b) = \varphi(\sum_{w \in \mathrm{W}} aw^{-1} \otimes wb)$.

Choose a splitting $\frac{\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet}{(1-\gamma)\mathbb{Z}_\ell[\mathrm{W}] \otimes \mathbb{X}^\bullet} \simeq M \oplus (\mathbb{X}^\bullet/(1-\gamma)\mathbb{X}^\bullet)/(\text{torsion})$, and let $\varphi|_M = 0$ and $\varphi = \psi$ on the second factor, then $\mathrm{Nm}\varphi = \psi$. □

Now, let $A' = \ker(P^0 \to P)$, and $A = \ker(P \to P^1)$. Then by the lemma, we have $\ker D_{cl} = A'/(\breve{A})^*$. As both groups are finite étale groups, it is enough to show that $|A'| = |\breve{A}|$, where for a finite group Γ, $|\Gamma|$ denotes the number of its elements. Indeed, it is enough to show that $|\breve{A}| \geq |A'|$. This is the subject of the next subsection.

3.7. **Calculation of finite groups.** Let us understand A. In fact, it is better to pick up $\infty \in C$ which does not lie in the ramification loci. Let $\mathcal{O}_\infty$ denote the completed local ring of C at ∞. Let J_∞ be the dilatation of J along the unit of the fiber of J at ∞, i.e. there is a natural map $J_\infty \to J$ such that $J_\infty(\mathcal{O}_\infty)$ is identified with the first congruent subgroup of $J(\mathcal{O}_\infty)$. Let $\mathscr{P}_\infty$ be the Picard stacke of J_∞-torsors on C. One can also interpret $\mathscr{P}_\infty$ as the Picard stack of J-torsors on C together with a trivialization at ∞. Observe that $\mathscr{P}_\infty$ is in fact a scheme. Let P_∞ denote the neutral connected component of $\mathscr{P}_\infty$. Similarly, one can define $J^0_\infty, J^1_\infty, P^0_\infty, P^1_\infty$ etc. Let $A_\infty = \ker(P_\infty \to P^1_\infty)$ and $A'_\infty = \ker(P^0_\infty \to P_\infty)$.

Lemma 3.7.1. *There are the following two exact sequences*

$$1 \to A_\infty \to \Gamma(C, J^1/J) \to \pi_0(\mathscr{P}) \to \pi_0(\mathscr{P}^1) \to 1$$

and

$$1 \to \mathrm{Aut}_{\mathscr{P}}(e) \to \mathrm{Aut}_{\mathscr{P}^1}(e) \to A_\infty \to A \to 1.$$

Similarly,

$$1 \to A'_\infty \to \Gamma(C, J/J^0) \to \pi_0(\mathscr{P}^0) \to \pi_0(\mathscr{P}) \to 1$$

and

$$1 \to \mathrm{Aut}_{\mathscr{P}^0}(e) \to \mathrm{Aut}_{\mathscr{P}}(e) \to A'_\infty \to A' \to 1.$$

Proof. Consider $1 \to J \to J^1 \to J^1/J \to 1$. Taking $R\Gamma(C, -)$ and noting that $J^1_\infty/J_\infty = J^1/J$ and $\pi_0(\mathscr{P}_\infty) = \pi_0(\mathscr{P}), \pi_0(\mathscr{P}^1_\infty) = \pi_0(\mathscr{P}^1)$, we obtain the first two sequences. The proof of the other two is similar. □

As a corollary, we can write

$$|A| = \frac{|\Gamma(C, J^1/J)|}{|\operatorname{coker}(\operatorname{Aut}_{\mathscr{P}}(e) \to \operatorname{Aut}_{\mathscr{P}^1}(e))||\ker(\pi_0(\mathscr{P}) \to \pi_0(\mathscr{P}^1)|}.$$

and

$$|A'| = \frac{|(\Gamma(C, J/J^0)|}{|\operatorname{coker}(\operatorname{Aut}_{\mathscr{P}^0}(e) \to \operatorname{Aut}_{\mathscr{P}}(e))||\ker(\pi_0(\mathscr{P}^0) \to \pi_0(\mathscr{P}))|}.$$

Therefore to show that $|\breve{A}| \geq |A'|$, it is enough to show that

(1) $|\Gamma(C, \breve{J}^1/\breve{J})| = |\Gamma(C, J/J^0)|$;
(2) $|\operatorname{coker}(\operatorname{Aut}_{\breve{\mathscr{P}}}(e) \to \operatorname{Aut}_{\breve{\mathscr{P}}^1}(e))| = |\operatorname{coker}(\pi_0(\mathscr{P})^* \to \pi_0(\mathscr{P}^0)^*)|$;
(3) $|\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1))| \leq |\ker(\operatorname{Aut}_{\mathscr{P}}(e)^* \to \operatorname{Aut}_{\mathscr{P}^0}(e)^*)|$.

We first prove (1). By (3.2.1), $\Gamma(C, \breve{J}^1/\breve{J}) = (\bigoplus_\alpha \mu_\alpha(\widetilde{C}_\alpha))^{\mathrm{W}}$. Observe that $\mu_\alpha \neq 0$ if and only if α is not a primitive root, i.e. $\alpha/2 \in \mathbb{X}^\bullet$. On the other hand, it is easy to see that the character group of $\Gamma(C, J/J_0)$ is $(\bigoplus_{x \in \sqcup \widetilde{C}_\alpha} \frac{\mathbb{Q}\alpha \cap \mathbb{X}^\bullet}{\mathbb{Z}\alpha})^{\mathrm{W}}$. Then (1) follows.

(2) has been essentially treated in §3.5. Namely, both maps can be identified with the natural inclusion $Z(\breve{G}) \to \breve{T}^{\mathrm{W}}$. Finally, we show (3). Note that we have

$$\Gamma(C, J/J^0)^* \to \operatorname{Aut}_{\mathscr{P}}(e)^* \to \operatorname{Aut}_{\mathscr{P}^0}(e)^* \to 1,$$

and from the description of $\Gamma(C, J/J^0)$ above , it is easy to see that

$$\ker(\operatorname{Aut}_{\mathscr{P}}(e)^* \to \operatorname{Aut}_{\mathscr{P}^0}(e)^*) = \frac{\mathbb{Q}\Phi \cap \mathbb{X}^\bullet}{\mathbb{Z}\Phi}.$$

On the other hand, recall that

$$\widetilde{C} \times \mathbb{X}^\bullet \overset{\mathrm{AJ}^{\breve{\mathscr{P}}}}{\to} \breve{\mathscr{P}} \to \breve{\mathscr{P}}^1 \to \mathrm{Bun}_{\breve{T}}^{\mathrm{W}} \to \mathrm{Bun}_{\breve{T}}$$

induce the maps between π_0

$$\mathrm{Nm} : \mathbb{X}^\bullet \twoheadrightarrow \pi_0(\breve{\mathscr{P}}) \twoheadrightarrow \pi_0(\breve{\mathscr{P}}^1) \to \mathbb{X}^\bullet = \pi_0(\mathrm{Bun}_{\breve{T}}).$$

It is clear that the kernel of the norm map $\mathrm{Nm} : \mathbb{X}^\bullet \to \mathbb{X}^\bullet$ is $\mathbb{Q}\Phi \cap \mathbb{X}^\bullet$, and therefore the kernel of $\pi_0(\breve{\mathscr{P}}) \to \mathbb{X}^\bullet$ is $\frac{\mathbb{Q}\Phi \cap \mathbb{X}^\bullet}{\mathbb{Z}\Phi}$, which contains $\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1))$ as a subgroup. The claim follows and therefore the proof of the theorem is complete.

Remark 3.7.2. Observe that since $|\breve{A}| = |A'|$, we must have

$$\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1)) = \frac{\mathbb{Q}\Phi \cap \mathbb{X}^\bullet}{\mathbb{Z}\Phi}.$$

Therefore,

$$\pi_0(\breve{\mathscr{P}}^1) = \frac{\mathbb{X}^\bullet}{\mathbb{Q}\Phi \cap \mathbb{X}^\bullet}.$$

It seems that this expression of $\pi_0(\breve{\mathscr{P}}^1)$ did not appear in literature before.

3.8. **A property of $\mathfrak{D}_{cl}$.** In this subsection, we show that the classical duality $\mathfrak{D}_{cl}$ intertwines certain homomorphisms of Picard stacks

$$\mathfrak{l}_J : Z(\breve{G})\text{-tors}(C) \times B \to \mathscr{P}^\vee, \quad \breve{\mathfrak{l}}_J : Z(\breve{G})\text{-tors}(C) \times B \to \breve{\mathscr{P}}.$$

We start with the construction of $\mathfrak{l}_J$ and $\breve{\mathfrak{l}}_J$.

The definition of $\breve{\mathfrak{l}}_J$ is easy. It is induced by the natural map of group schemes $Z(\breve{G}) \times B \to \breve{J}$. For any $K \in Z(\breve{G})\text{-tors}(C)$ we write $K_J := \breve{\mathfrak{l}}_J(\{K\} \times B) \in \breve{\mathscr{P}}(B)$. Next we define $\mathfrak{l}_J$. To this goal, we first generalize a construction of [BD, §4.1].

Let $\pi : \mathcal{C} \to B$ be a smooth proper relative curve over an affine base B (later on $\mathcal{C} = C \times B$). Let

$$0 \to \Pi(1) \to \tilde{\mathcal{G}} \to \mathcal{G} \to 0$$

be an extension of smooth affine group schemes on $\mathcal{C}$ with Π commutative finite étale. Let $\Pi^\vee = \mathrm{Hom}(\Pi, \mathbb{G}_m)$ be its Cartier dual, which is assumed to be étale as well (in particular, the order of Π is prime to char$k$), and let $\Pi^\vee\text{-tors}(\mathcal{C}/B)$ be the Picard stack over B of $\Pi^\vee$-torsors on $\mathcal{C}$ relative to B. We construct a Picard functor

$$\mathfrak{l}_{\mathcal{G}} : \Pi^\vee\text{-tors}(\mathcal{C}/B) \to \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B))$$

of Picard stacks over B as follows. First, let $\Pi\text{-gerbe}(\mathcal{C}/B)$ denote the Picard stack of Π-gerbes on $\mathcal{C}$ relative to B[1]. Then there is the generalized (or categorical) Chern class map $\tilde{c}_{\mathcal{G}} : \mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B) \to \Pi(1)\text{-gerbe}(\mathcal{C}/B)$ that assigns every B-scheme S and a $\mathcal{G}$-torsor E on $\mathcal{C}_S$, the Picard groupoid of the lifting of E to a $\tilde{\mathcal{G}}$-torsor. We have

Lemma 3.8.1. *The dual of the Picard stack* $\Pi\text{-gerbe}(\mathcal{C}/B)$ *(as defined in §A.2) is canonically isomorphic to* $\Pi^\vee\text{-tor}(\mathcal{C}/B)$.

We follow [BD, §4.1.5] for a "scientific interpretation" of this lemma and refer to [BD, §4.1.2-4.1.4] for the precise construction. As explained in §A.1, the Picard stack $\Pi\text{-gerbe}(\mathcal{C}/B)$ is incarnated by the complex $\tau_{\geq -1} R\pi_*\Pi[2](1)$, and $\Pi^\vee\text{-tor}(\mathcal{C}/B)$ is incarnated by the complex $\tau_{\leq 0} R\pi_*\Pi^\vee[1]$. Let μ'_∞ denote the group of prime-to-p roots of unit. Note that $\pi^!\mu'_\infty \simeq \mu'_\infty[2](1)$. Then by the Verdier duality,

$$R\mathrm{Hom}(R\pi_*\Pi[2](1), \mu'_\infty) \simeq R\pi_* R\mathrm{Hom}(\Pi[2](1), \pi^!\mu'_\infty) \simeq R\pi_*\Pi^\vee.$$

By shifting by [1] and truncating $\tau_{\leq 0}$, one obtains the lemma. As explained in [BD, §4.1.5], working in the framework of derived categories is in not enough to turn the above heuristics into a proof. One can either give a concrete construction as in [BD, §4.1.2-4.1.4] or understand the above argument in the framework of stable ∞-categories.

Therefore, each $K \in \Pi^\vee\text{-tors}(\mathcal{C}/B)$ induces a functor

$$\mathfrak{l}_{\mathcal{G},K} : \mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B) \xrightarrow{\tilde{c}_{\mathcal{G}}} \Pi(1)\text{-gerbes}(\mathcal{C}/B) \xrightarrow{<,K>} \mathrm{B}\mathbb{G}_m$$

or equivalently a line bundle $\mathcal{L}_{\mathcal{G},K}$ on $\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)$ and the assignment $K \to \mathcal{L}_{\mathcal{G},K}$ defines a tensor functor

$$\mathfrak{l}_{\mathcal{G}} : \Pi^\vee\text{-tors}(\mathcal{C}/B) \to \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)),$$

which factors through the n-torsion of $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})(\mathcal{C}/B)$ where n is the order of $\Pi^\vee$.

Note that in the above discussion we do not assume that $\mathcal{G}$ is commutative. But if $\mathcal{G}$ is commutative then $\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)$ has a natural structure of Picard stack over

[1] It is in fact a Picard 2-stack.

B and one can check that $\mathfrak{l}_{\mathcal{G}}$ factor through a tensor functor $\mathfrak{l}_{\mathcal{G}}: \Pi^{\vee}\text{-tors}(\mathcal{C}/B) \to (\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B))^{\vee}$.

Now we assume that $\mathcal{C} = C \times B$, where B is the Hitchin base as before. If $\mathcal{G} = G \times \mathcal{C}$ is a semi simple algebraic group and $\tilde{\mathcal{G}} = G_{sc} \times \mathcal{C}$ is the simply-connected cover, then $\Pi(1) = \Pi_G(1)$ is the fundamental group and $\Pi_G^{\vee}$ is canonical isomorphic to the center $Z(\check{G})$ of $\check{G}$. So in this case $\mathfrak{l}_{\mathcal{G}}$ is a Picard functor

$$\mathfrak{l}_G : Z(\check{G})\text{-tors}(C) \to \mathrm{Pic}(\mathrm{Bun}_G).$$

Similarly, for the extensions $0 \to \Pi_G \to T_{sc} \to T \to 0$, and $0 \to (\Pi_G)_B \to J_{sc} \to J \to 0$, we denote the corresponding functors by $\mathfrak{l}_T$ and $\mathfrak{l}_J$. Given $K \in Z(\check{G})\text{-tors}(C)$ we define $\mathcal{L}_{G,K} := \mathfrak{l}_G(K) \in \mathrm{Pic}(\mathrm{Bun}_G)$, $\mathcal{L}_{J,K} := \mathfrak{l}_J(\{K\} \times B) \in (\mathscr{P})^{\vee}(B)$. The following lemma will be used in §5.6.

Lemma 3.8.2. *Let κ be a square root of ω. Then the pullback of $\mathcal{L}_{G,K}$ along the map $\mathscr{P} \stackrel{\epsilon_\kappa}{\to} \mathrm{Higgs} \stackrel{\mathrm{pr}}{\to} \mathrm{Bun}_G$ is isomorphic to $\mathcal{L}_{J,K}$, i.e., we have $\mathcal{L}_{J,K} \simeq \epsilon_\kappa^* \circ \mathrm{pr}^* \mathcal{L}_{G,K}$.*

Proof. It is enough to show that the composition

$$\mathscr{P} \stackrel{\epsilon_\kappa}{\to} \mathrm{Higgs} \stackrel{\mathrm{pr}}{\to} \mathrm{Bun}_G \stackrel{\tilde{c}_G}{\to} \Pi_G(1)\text{-gerbes}(C)$$

is isomorphic to

$$\mathscr{P} \stackrel{\tilde{c}_J}{\to} \Pi_G\text{-gerbes}(C) \times B \to \Pi_G(1)\text{-gerbes}(C).$$

Let $P \in \mathscr{P}$ and $(E, \phi) := \epsilon_\kappa(P)$. We need to construct a functorial isomorphism between $\tilde{c}_J(P)$ and $\tilde{c}_G(E)$ where $\tilde{c}_J(P)$ (resp. $\tilde{c}_G(E)$) is the $\Pi_G(1)$-gerbe of liftings of P to J_{sc}-torsors (resp. G_{sc}-torsors).

Note that the G-torsor E_κ given by the Kostant section has a natural lifting $\tilde{E}_\kappa \in \mathrm{Bun}_{G_{sc}}$ (it is due to the fact that the cocharacter $2\rho : \mathbb{G}_m \to G$ has a natural lifting to G_{sc}). Thus any lifting $\tilde{P} \in \tilde{c}_J(P)$ defines a lifting $\tilde{E} := \tilde{P} \times^{J_{sc}} \tilde{E}_\kappa \in \mathrm{Bun}_{G_{sc}}$ of $E = P \times^J E_\kappa$ and the assignment $\tilde{P} \to \tilde{E}$ defines a functorial isomorphism between $\tilde{c}_J(P)$ and $\tilde{c}_G(E)$. This finishes the proof. □

We write l_G, l_T, l_J the induced map on the corresponding coarse moduli spaces. The following lemma is a specialization of our construction of the duality given in Lemma 3.8.1.

Lemma 3.8.3. *Let n be a positive integer such that $p \nmid n$. Let $\mathfrak{l} : \check{T}[n]\text{-tors}(C) \to (\mathrm{Bun}_T)^{\vee}[n]$ be the tensor functor given by the extension $0 \to T[n] \to T \stackrel{n}{\to} T \to 0$.[2] Then the induced map $l : H^1(X, \check{T}[n]) \to H^1(X, T[n])^{\vee}$ on the coarse moduli spaces is the same the as map induced by the Poincare duality.*

Now we are ready to state our main result in this subsection.

Proposition 3.8.4. *There is a natural isomorphism of functors $\mathfrak{D}_{cl} \circ \mathfrak{l}_J \simeq \check{\mathfrak{l}}_J$. In particular, we have $\mathfrak{D}_{cl}(\mathcal{L}_{J,K}) \simeq K_J$.*

Proof. The short exact sequence $0 \to Z(\check{G}) \times B \to \check{J} \to \check{J}_{ad} \to 0$ induces an exact sequence

$$0 \to Z(\check{G})\text{-tors}(C) \times B \stackrel{\check{\mathfrak{l}}_J}{\to} \check{\mathscr{P}} \to \check{\mathscr{P}}_{ad}.$$

[2] Recall that we have a canonical isomorphism $\check{T}[n] \simeq (T[n])^{\vee}$.

On the other hand, we claim that the composition

$$Z(\check{G})\text{-tors}(C)\times B \xrightarrow{\mathfrak{l}_J} (\mathscr{P})^\vee \overset{\mathfrak{D}_{cl}}{\simeq} \check{\mathscr{P}} \to \check{\mathscr{P}}_{ad}$$

is zero. From the construction of $\mathfrak{D}_{cl}$, we have the following commutative diagram

$$\begin{array}{ccc} (\mathscr{P})^\vee & \xrightarrow{\mathfrak{D}_{cl}} & \check{\mathscr{P}} \\ \downarrow & & \downarrow \\ (\mathscr{P}_{sc})^\vee & \xrightarrow{\mathfrak{D}_{cl}} & \check{\mathscr{P}}_{ad}. \end{array}$$

Thus above composition can be identified with

$$Z(\check{G})\text{-tors}(C)\times B \xrightarrow{\mathfrak{l}_J} (\mathscr{P})^\vee \to (\mathscr{P}_{sc})^\vee \overset{\mathfrak{D}_{cl}}{\simeq} \check{\mathscr{P}}_{ad}.$$

This map is zero because the map $Z(\check{G})\text{-tors}(C)\times B \xrightarrow{\mathfrak{l}_J} (\mathscr{P})^\vee \to (\mathscr{P}_{sc})^\vee$ is the dual of

$$\mathscr{P}_{sc} \to \mathscr{P} \xrightarrow{\tilde{c}_J} \Pi_G(1)\text{-gerbe}(C)\times B$$

and the later map is zero follows from the construction of $\tilde{c}_J$. This finished the proof of the claim.

By the universal property of the kernel, we see that there is a morphism

$$\mathfrak{i} : Z(\check{G})\text{-tors}(C)\times B \to Z(\check{G})\text{-tors}(C)\times B$$

such that $\mathfrak{D}_{cl}\circ\mathfrak{l}_J \simeq \check{\mathfrak{l}}_J\circ\mathfrak{i}$. We now show that $\mathfrak{i}$ is isomorphic to the identity morphism. As argued in §3.5, we reduce to show that $\mathfrak{i}$ induced the identity map on the coarse moduli space $H^1(C, Z(\check{G}))\times B$.

Let $i : H^1(C, Z(\check{G}))\times B \to H^1(C, Z(\check{G}))\times B$, $l_J : H^1(C, Z(\check{G}))\times B \to P^\vee$ and $\check{l}_J : H^1(C, Z(\check{G}))\times B \to \check{P}$ be the induced maps on the corresponding coarse moduli spaces. Our goal is to show that $i = \mathrm{id}$. Since $\check{l}_J$ is injective (recall that $\check{\mathfrak{l}}_J = \ker(\check{\mathscr{P}} \to \check{\mathscr{P}}_{ad})$), it suffices to show that

$$\check{l}_J \circ (i - \mathrm{id}) : H^1(C, Z(\check{G}))\times B \to \check{P}$$

is zero. As in §3.5, we can prove this fiberwise, and therefore we fix $b \in B^0(k)$. Again, to simplify the notation, in the following discussion we write $\widetilde{C}, J, P, \check{P}$ instead of $\widetilde{C}_b, J_b, P_b, \check{P}_b$, etc.

Let $\check{j}^1 : \check{P} \to H^1(\widetilde{C}, \check{T})$ be the map induced by the morphism $\check{j}^1 : \pi^*\check{J} \to \check{T}$. Then the composition $\check{j}^1 \circ \check{l}_J : H^1(C, Z(\check{G})) \to H^1(\widetilde{C}, \check{T})$ is also injective (note that $\check{j}^1 \circ \check{l}_J$ is induced by the natural map $Z(\check{G}) \to \check{T}$). Thus it is enough to show that $\check{j}^1 \circ \check{l}_J \circ (i - \mathrm{id}) = 0$. Since $D_{cl} \circ l_J = \check{l}_J \circ i$, it is equivalent to show that

$$\check{j}^1 \circ D_{cl} \circ l_J - \check{j}^1 \circ \check{l}_J = 0. \tag{3.8.1}$$

Let us consider the following diagram

$$\begin{array}{ccccc} H^1(C, Z(\check{G})) & \xrightarrow{\check{l}_J} & H^1(C, \check{J}) & \xrightarrow{\check{j}^1} & H^1(\widetilde{C}, \check{T}) \\ \downarrow{\scriptstyle id} & & \uparrow{\scriptstyle D_{cl}} & & \uparrow{\scriptstyle D_{cl}} \\ H^1(C, Z(\check{G})) & \xrightarrow{l_J} & H^1(C, J)^\vee & \xrightarrow{\mathrm{Nm}^\vee} & H^1(\widetilde{C}, T)^\vee \end{array}.$$

The right rectangle of above diagram is commutative, therefore to prove (3.8.1) it is enough to show that the outer diagram is commutative.

Let n be the order of $Z(\check{G})$. Then $\check{j}^1 \circ \check{l}_J$ and $\mathrm{Nm}^\vee \circ \mathfrak{l}_J$ will factor through $H^1(\widetilde{C}, \check{T})[n] \simeq H^1(\widetilde{C}, \check{T}[n])$ and $H^1(\widetilde{C}, T)^\vee[n] \simeq H^1(\widetilde{C}, T[n])^{*3}$. Thus the outer diagram factor as

$$\begin{array}{ccc} H^1(C, Z(\check{G})) & \xrightarrow{\check{j}^1 \circ \check{l}_J} & H^1(\widetilde{C}, \check{T}[n]) \\ \downarrow{\scriptstyle id} & & \uparrow{\scriptstyle D_{cl}} \\ H^1(C, Z(\check{G})) & \xrightarrow{\mathrm{Nm}^\vee \circ \mathfrak{l}_J} & H^1(\widetilde{C}, T[n])^\vee \end{array}.$$

Unraveling the definition of $\mathfrak{l}_J$, one see that $\mathrm{Nm}^\vee \circ l_J$ can be identified with

$$H^1(C, Z(\check{G})) \to H^1(\widetilde{C}, \check{T}[n]) \to H^1(\widetilde{C}, T[n])^\vee$$

where the first map is induced by the natural morphism $Z(\check{G}) \to \check{T}[n]$ and the second map is the map l in Lemma 3.8.3. Since it is known that the duality D_{cl} : $H^1(\widetilde{C}, \check{T}[n]) \simeq H^1(\widetilde{C}, T[n])^\vee$ is the Poincare duality, the commutativity of above diagram follows from Lemma 3.8.3. □

4. Multiplicative one forms

In this section, we establish a technical result. Namely, we show that the pullback of the canonical one form θ_{can} on $T^* \mathrm{Bun}_G$ along $\mathscr{P} \to T^* \mathrm{Bun}_G$ induced by a Kostant section κ is is multiplicative in the sense of §C.2 and therefore is independent of the choice of κ.

4.1. **Lie algebra valued one forms.** In this subsection, we restrict everything to B^0 and therefore omit the subscript ^{0} from the notation. Recall that there is a group scheme $\check{\mathfrak{T}} = \widetilde{C} \times^{\mathrm{W}} \check{T}$ (resp. $\mathfrak{T} = \widetilde{C} \times^{\mathrm{W}} T$) over $[\widetilde{C}/\mathrm{W}]$ and Proposition 2.5.1 says that there is a homomorphism $\check{j}^1 : \pi^* \check{J} \to \check{\mathfrak{T}}$ (resp. $j^1 : \pi^* J \to \mathfrak{T}$) where $\pi : [\widetilde{C}/\mathrm{W}] \to C \times B$ is the projection. It induces the following commutative diagram

$$\begin{array}{ccc} \pi^*(\Omega_{C\times B} \otimes \mathrm{Lie}\check{J}) & \longrightarrow & \pi^*(\Omega_{C\times B/B} \otimes \mathrm{Lie}\check{J}) \\ \downarrow & & \downarrow \\ \Omega_{[\widetilde{C}/W]} \otimes \mathrm{Lie}\check{\mathfrak{T}} & \longrightarrow & \Omega_{[\widetilde{C}/W]/B} \otimes \mathrm{Lie}\check{\mathfrak{T}} \end{array}$$

Note that the arrow in the first arrow admits a canonical splitting. Therefore, the section $(\check{\tau} : B \to B_{\check{J}}) \in \Gamma(C \times B, \Omega_{C\times B/B} \otimes \mathrm{Lie}\check{J})$ induces a section of

$$\theta_{\widetilde{C}} \in \Gamma([\widetilde{C}/\mathrm{W}], \Omega_{[\widetilde{C}/\mathrm{W}]} \otimes \mathrm{Lie}\check{\mathfrak{T}}) = \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \check{\mathfrak{t}})^{\mathrm{W}}. \tag{4.1.1}$$

4.2. **Canonical one forms.** Let us denote by $T^* \mathrm{Bun}_G^0$ the maximal smooth open substack of $T^* \mathrm{Bun}_G$. Then there is a tautological section

$$\theta_{can} : T^* \mathrm{Bun}_G^0 \to T^*(T^* \mathrm{Bun}_G^0).$$

Note that $T^* \mathrm{Bun}_G \times_B B^0 \subset T^* \mathrm{Bun}_G^0$. From now on, we restriction everything to the open part B^0 and therefore will omit ^{0} from the subscript. Note that for a choice

[3] Note that $p \nmid n$.

of the Kostant section κ, we have an isomorphism $\mathscr{P} \simeq T^* \operatorname{Bun}_G$, and therefore we regard θ_{can} as a section $\mathscr{P} \to T^*\mathscr{P}$, denoted by θ_κ. Note that the composition

$$\mathscr{P} \xrightarrow{\theta_\kappa} T^*\mathscr{P} \to T^*(\mathscr{P}/B) \simeq \mathbb{T}_e^*\mathscr{P} \times_B \mathscr{P}$$

is exactly the morphism as in Lemma 2.7.2, and therefore can be identified with $\tau^* \times \mathrm{id}$. Let $\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet \to \mathscr{P}$ be the Abel-Jacobi map. Considering the pull back

$$(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \{\theta_{\kappa,\lambda}\}_{\lambda \in \mathbb{X}_\bullet} \in \Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}})^{\mathrm{W}},$$

where $\theta_{\kappa,\lambda} \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}})$ is the restriction of $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ to $\widetilde{C} \times \{\lambda\}$. A section $\{\alpha_\lambda\}_{\lambda\in\mathbb{X}_\bullet} \in \Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}})$ (resp. $\Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}/B})$) is called $\mathbb{X}_\bullet$-multiplicative if it satisfies $\alpha_{\lambda+\mu} = \alpha_\lambda + \alpha_\mu$, for any $\lambda, \mu \in \mathbb{X}_\bullet$. Clearly, any $\mathbb{X}_\bullet$-multiplicative section $\{\alpha_\lambda\}_{\lambda\in\mathbb{X}_\bullet}$ corresponds to a $\check{\mathfrak{t}}$-valued section $\alpha \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \check{\mathfrak{t}})$ (resp. $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}/B} \otimes \check{\mathfrak{t}})$). The remainder of the section is mainly concerned with the proof of the following result.

Proposition 4.2.1. *The one form* $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ *is* $\mathbb{X}_\bullet$*-multiplicative. Moreover, if we regard* $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ *as a section of* $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \check{\mathfrak{t}})^{\mathrm{W}}$ *we have*

$$(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \theta_{\widetilde{C}}$$

We have the following corollary. Recall the notion of multiplicative sections $\mathscr{P} \to T^*\mathscr{P}$ as defined In §C.2.

Corollary 4.2.2. *The section* θ_κ *is multiplicative. In particular, it is independent of the choice of Kostant section* κ*.*

Proof. Let us denote the section $m^*\theta_\kappa$ as

$$m^*\theta_\kappa : \mathscr{P} \times_B \mathscr{P} \to T^*\mathscr{P} \times_{\mathscr{P}} (\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}),$$

where the first map is the base change of θ_κ along the multiplication $m : \mathscr{P} \times_B \mathscr{P} \to \mathscr{P}$, and the second map is the differential m_d of m. Let us also denote $(\theta_\kappa, \theta_\kappa)$ as

$$(\theta_\kappa, \theta_\kappa) : \mathscr{P} \times_B \mathscr{P} \to (T^*\mathscr{P} \times T^*\mathscr{P})|_{\mathscr{P}\times_B\mathscr{P}} \to T^*(\mathscr{P} \times_B \mathscr{P}).$$

We need to show that $(\theta_\kappa, \theta_\kappa) = m^*\theta_\kappa$.

Consider the following short exact sequence of vector bundles on $\mathscr{P} \times_B \mathscr{P}$

$$0 \to T^*B \times_B (\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}/B) \to 0.$$

As the projection of θ_κ to $T^*(\mathscr{P}/B) = \mathbb{T}_e^*\mathscr{P} \times_B \mathscr{P}$ is identified with $\tau^* \times \mathrm{id}$, it is clear that $(\theta_\kappa, \theta_\kappa) = m^*\theta_\kappa$ in $T^*(\mathscr{P} \times_B \mathscr{P}/B)$. Therefore, their difference can be regarded as a section

$$m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa) \in \Gamma(\mathscr{P} \times_B \mathscr{P}, \mathrm{pr}^*\Omega_B) = (\pi_0(\mathscr{P}) \times \pi_0(\mathscr{P})) \otimes \Gamma(B, \Omega_B).$$

The Abel-Jacobi map $\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet \to \mathscr{P}$ induces a map

$$\mathrm{AJ}^{\mathscr{P},2} : \widetilde{C} \times \mathbb{X}_\bullet \times \mathbb{X}_\bullet \to \mathscr{P} \times_B \mathscr{P}.$$

It is enough to show that the pullback of $m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa)$ in

$$\Gamma((\widetilde{C} \times \mathbb{X}_\bullet \times \mathbb{X}_\bullet), \mathrm{pr}^*\Omega_B) = (\mathbb{X}_\bullet \times \mathbb{X}_\bullet) \otimes \Gamma(B, \Omega_B)$$

vanishes. By Proposition 4.2.1, the one form $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \{\theta_{\kappa,\lambda}\}_{\lambda\in\mathbb{X}_\bullet}$ is $\mathbb{X}_\bullet$-multiplicative, thus for any $\lambda, \mu \in \mathbb{X}_\bullet$ we have

$$(\mathrm{AJ}^{\mathscr{P},2})^*(m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa))|_{\widetilde{C}\times\{\lambda\}\times\{\mu\}} = \theta_{\kappa,\lambda+\mu} - (\theta_{\kappa,\lambda} + \theta_{\kappa,\mu}) = 0$$

This finished the proof. □

Notation. In what follows, we denote the multiplicative one form θ_κ on $\mathscr{P}$ by θ_m.

4.3. **Proof of Proposition 4.2.1: first reductions.** Let $\tilde{\theta}_\kappa$ and $\tilde{\theta}_{\widetilde{C}}$ be the projection of $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ and $\theta_{\widetilde{C}}$ along

$$\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\Omega_{\widetilde{C}})\to\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\Omega_{\widetilde{C}/B}).$$

In the course of the proof of Corollary 4.2.2 we already showed that the projection of θ_κ in $T^*(\mathscr{P}/B)$ is multiplicative. It implies $\tilde{\theta}_\kappa$ is $\mathbb{X}_\bullet$-multiplicative and can be regard as an element in $\Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\check{\mathfrak{t}})^{\mathrm{W}}$. Let us first show that $\tilde{\theta}_\kappa=\tilde{\theta}_{\widetilde{C}}$.

Recall that we introduced a morphism $\iota:\mathrm{Lie}J\to(\mathrm{Lie}J)^*$ (see (2.7.4)). From the definition of ι it is not hard to check that the following diagram commutes

$$\begin{array}{ccc}\pi^*\mathrm{Lie}J & \xrightarrow{\pi^*\iota} & \pi^*(\mathrm{Lie}J)^*\\ \downarrow{\scriptstyle j^1} & & \uparrow{\scriptstyle (j^1)^*}\\ \mathrm{Lie}\mathfrak{T} & \longrightarrow & \mathrm{Lie}\check{\mathfrak{T}},\end{array}$$

where the arrow in the bottom row is the morphism $\mathrm{Lie}\mathfrak{T}\to\mathrm{Lie}\check{\mathfrak{T}}$ induced by the invariant from $(,)$ on $\mathfrak{t}$. It induces

$$\begin{array}{ccc}\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}J) & \xrightarrow{\iota_*} & \Gamma(C\times B,\Omega_{C\times B/B}\otimes(\mathrm{Lie}J)^*)\\ \downarrow & & \uparrow\\ \Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\mathfrak{t})^{\mathrm{W}} & \longrightarrow & \Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\check{\mathfrak{t}})^{\mathrm{W}}\end{array}$$

If we regard τ (resp. τ^*) as section in $\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}J)$ (resp. $\Gamma(C\times B,\Omega_{C\times B/B}\otimes(\mathrm{Lie}J)^*)$), we have $\tau^*=\iota_*(\tau)$. Note the arrow in the right column is an isomorphism[4] and it maps $\tilde{\theta}_\kappa$ to the section τ^*. On the other hand, the section $\tilde{\theta}_{\widetilde{C}}\in\Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\check{\mathfrak{t}})^{\mathrm{W}}$ is equal to the image of τ under the morphisms in the lower left corner of above diagram. Therefore the section $\tilde{\theta}_{\widetilde{C}}$ also maps to τ^* and it implies $\tilde{\theta}_{\widetilde{C}}=\tilde{\theta}_\kappa$.

We have showed that $\theta_{\widetilde{C}}=(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ in $\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\Omega_{\widetilde{C}/B})$. Therefore, their difference can be regraded as a section

$$\theta_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa\in\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\mathrm{pr}^*\Omega_B). \tag{4.3.1}$$

We need to show above section is zero. Let $\widetilde{U}\subset\widetilde{C}$ be the largest open subset such that $\widetilde{U}\to C\times B$ is étale. It is enough to show that $\theta_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{\widetilde{U}\times\mathbb{X}_\bullet}=0$. Note that for $\tilde{x}\in\widetilde{U}$ we have a canonical decomposition $T_{\tilde{x}}\widetilde{C}=T_xC\oplus T_bB$ and by (4.3.1) it suffices to show that $(\theta_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa)|_{T_bB}=0$. As the section $\theta_{\widetilde{C}}$ is induced by the canonical splitting $\Omega_{C\times B/B}\otimes\mathrm{Lie}\check{J}\to\Omega_{C\times B}\otimes\mathrm{Lie}\check{J}$, the restriction of $\theta_{\widetilde{C}}$ to T_bB is zero, so we reduce to show that $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{T_bB}=0$, i.e. for any $\lambda\in\mathbb{X}_\bullet$ and $v\in T_bB$ we have

$$\langle\theta_{\kappa,\lambda},v\rangle=\langle(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{\widetilde{C}\times\{\lambda\}},v\rangle=0. \tag{4.3.2}$$

[4] It is the relative cotangent map of the isogeny $\mathscr{P}\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$.

For the later purpose, we introduce some notations. Let (E_κ, ϕ_κ) be the Higgs field on $C \times B$ obtained by the pullback along the Kostant section κ. For every $\lambda \in \mathbb{X}_\bullet$ let $\mathrm{AJ}^{\mathscr{P},\lambda} : \widetilde{C} \to \mathscr{P}$ denote the corresponding component of the Abel-Jacobi map and let

$$(E_{\tilde{x}}, \phi_{\tilde{x}}) := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J (E_\kappa, \phi_\kappa).$$

We also define

$$a_\lambda : \widetilde{C} \overset{\mathrm{AJ}^{\mathscr{P},\lambda}}{\to} \mathscr{P} \simeq T^* \operatorname{Bun}_G \to \operatorname{Bun}_G .$$

From the definition of θ_κ, we have

$$\langle \theta_{\kappa,\lambda}, v\rangle = \langle (\mathrm{AJ}^{\mathscr{P},\lambda})^* \theta_\kappa, v\rangle = \langle \phi_{\tilde{x}}, a_{\lambda *} v\rangle,$$

where $a_{\lambda *} : T_{\tilde{x}}\widetilde{C} \to T_{E_{\tilde{x}}} \operatorname{Bun}_G \simeq H^1(C, \mathrm{ad}E_{\tilde{x}})$ is the differential of a_λ and the last pairing is induced by the Serre duality $H^0(C, \mathrm{ad}E_{\tilde{x}} \otimes \Omega_C) \simeq H^1(C, \mathrm{ad}E_{\tilde{x}})^*$.

Therefore we reduce to show that the paring $\langle \phi_{\tilde{x}}, a_{\lambda *} v\rangle$ is zero.

4.4. **Proof of Proposition 4.2.1: calculations of differentials.** We shall need several preliminary steps. Recall that there is the E_κ-twist global Grassmannian $\mathrm{Gr}(E_\kappa)$ which classifies the triples (x, E, β) where $x \in C$, E is a G-torsors and $\beta : E_\kappa|_{C-\{x\}} \simeq E|_{C-\{x\}}$. Given $\mu \in \mathbb{X}_\bullet^+$, it makes sense to talk about the closed substack $\mathrm{Gr}_{\leq\mu}(E_\kappa)$, consisting of those $\beta : E_\kappa|_{C-\{x\}} \simeq E|_{C-\{x\}}$ having relative positive $\leq \mu$ (cf. [BD, §5.2.2]). Let $\mathrm{Gr}_\mu(E_\kappa) = \mathrm{Gr}_{\leq\mu}(E_\kappa) - \bigcup_{\lambda<\mu} \mathrm{Gr}_{\leq\lambda}(E_\kappa)$. We have natural projection maps

$$\operatorname{Bun}_G \overset{pr_1}{\leftarrow} \mathrm{Gr}_{\leq\mu}(E_\kappa) \overset{pr_2}{\to} C.$$

For any $x \in C$, let

$$\mathrm{Gr}_x(E_\kappa) := \mathrm{Gr}(E_\kappa) \times_C \{x\},$$

and similarly we have $\mathrm{Gr}_{x,\leq\mu}(E_\kappa)$, $\mathrm{Gr}_{x,\mu}(E_\kappa)$.

Notice that for any $\tilde{x} \in \widetilde{C}$ the J-torsor $\mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \in \mathscr{P}$ has a canonical trivialization over $C - x$ (here x is the image of $\tilde{x}$ in C), thus it induces a canonial isomorphsim $\beta : E_\kappa|_{C-x} \simeq E_{\tilde{x}}|_{C-x}$ (recall that $E_{\tilde{x}} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J E_\kappa$). The assignment $\tilde{x} \to (x, E_{\tilde{x}}, \beta)$ defines a morphism $\tilde{a}_\lambda : \widetilde{C} \to \mathrm{Gr}(E_\kappa)$. We have the following key lemma:

Lemma 4.4.1. *Let $\mu \in \mathbb{X}_\bullet^+$ and $\lambda \in \mathrm{W} \cdot \mu$. The morphism $\tilde{a}_\lambda$ factors through $\mathrm{Gr}_{\leq\mu}(E_\kappa)$ and the following diagram*

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\tilde{a}_\lambda} & \mathrm{Gr}_{\leq\mu}(E_\kappa) \\ & {\scriptstyle a_\lambda}\searrow & \downarrow {\scriptstyle pr_1} \\ & & \operatorname{Bun}_G \end{array}$$

is commutative. Moreover, for any k-points $\tilde{x} \in \widetilde{U}(k)$ we have $\tilde{a}_\lambda(\tilde{x}) \in \mathrm{Gr}_\mu(E_\kappa)(k)$.

The proof is given at the end of this subsection. We also need the following lemma about differential of $\tilde{a}_\lambda$.

Lemma 4.4.2. *Let $\tilde{x} \in \widetilde{U}(k)$, and let $\tilde{a}_\lambda(\tilde{x}) = (x, E_{\tilde{x}}, \beta) \in \mathrm{Gr}_\mu(E_\kappa)(k)$ (by Lemma 4.4.1). For every $v \in T_b B \subset T_{\tilde{x}}\widetilde{C} = T_x C \oplus T_b B$, we have*

$$u := (\tilde{a}_\lambda)_* v \in T_{(E_{\tilde{x}},\beta)} \mathrm{Gr}_{x,\mu}(E_\kappa) \subset T_{(x,E_{\tilde{x}},\beta)} \mathrm{Gr}_\mu(E_\kappa).$$

Proof. The subspace $T_{(E_{\tilde{x}},\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa)$ is equal to $\mathrm{Ker}((pr_2)_* : T_{(x,E_{\tilde{x}},\beta)}\mathrm{Gr}_\mu(E_\kappa) \to T_xC)$. Therefore it is enough to show $(pr_2)_*(\tilde{a}_\lambda)_*v = 0$. Recall that we have the following commutative diagram (not cartesian)

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\tilde{a}_\lambda} & \mathrm{Gr}_{\leq\mu}(E_\kappa) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_2} \\ C \times B & \xrightarrow{pr_C} & C \end{array} .$$

Thus we have $(pr_2)_*(\tilde{a}_\lambda)_*v = (pr_C)_*(\pi_*v) = (pr_C)_*v = 0$. This finishes the proof. ☐

Combining the above two lemmas we obtain that

$$\langle \theta_{\kappa,\lambda}, v\rangle = \langle \phi_{\tilde{x}}, a_{\lambda *}v\rangle = \langle \phi_{\tilde{x}}, (pr_1)_*u\rangle \tag{4.4.1}$$

where $u := (\tilde{a}_\lambda)_*v \in T_{(E_{\tilde{x}},\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa)$. So we need show that the last paring is zero. To calculate it, we need few more notations. For any $x \in C$ we denote by $\mathcal{O}_x$ the completion of the local ring of C at x and F_x its fractional field. Let $\omega_{\mathcal{O}_x}$ (resp. ω_{F_x}) denote the completed regular (resp. rational) differentials on $\mathrm{Spec}\mathcal{O}_x$. We denote by

$$\mathrm{Res}(,) : \mathfrak{g}(\omega_{F_x}) \times \mathfrak{g}(F_x) \to k$$

the residue paring induced by the G-invariant form $(,)$ on $\mathfrak{g}$.

Let us fix γ_κ a trivialization of $E_\kappa \simeq E^0$ on $\mathrm{Spec}\mathcal{O}_x$. Then, for every trivialization γ of E on $\mathrm{Spec}\mathcal{O}_x$, we obtain

$$g = \gamma_\kappa^{-1}\beta\gamma \in G(F_x).$$

In this way, γ_κ induces an isomorphism

$$\mathrm{Gr}_{x,\mu}(E_\kappa) \simeq \mathrm{Orb}_\mu, \quad (E,\beta) \mapsto \gamma_\kappa^{-1}\beta\gamma G(\mathcal{O}_x),$$

where Orb_μ is the $G(\mathcal{O}_x)$-orbit of $\mu \cdot G(\mathcal{O}_x) \in G(F_x)/G(\mathcal{O}_x)$. Under the isomorphism, we have the identification of the tangent spaces

$$T_{(E,\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa) \simeq \mathfrak{g}(\mathcal{O}_x)/(\mathrm{Ad}_g\,\mathfrak{g}(\mathcal{O}_x) \cap \mathfrak{g}(\mathcal{O}_x)).$$

For any $u \in T_{(E,\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa)$ and $\phi \in T_E\,\mathrm{Bun}_G$ the paring $\langle \phi, (pr_1)_*u\rangle$ can be calculated as follows. Let $\tilde{u} \in \mathfrak{g}(\mathcal{O}_x)$ be a lifting of u under the above isomorphism. Let $\phi(\gamma)$ denote the $\phi : \mathrm{Spec}\mathcal{O}_x \to \mathrm{ad}E \otimes \omega_C \xrightarrow{\gamma} \mathfrak{g}(\omega_{F_x})$. Now we have

$$\langle \phi, (pr_1)_*u\rangle = \mathrm{Res}(\phi(\gamma), \mathrm{Ad}_g^{-1}\,\tilde{u}),$$

Specialize to our case $\phi = \phi_{\tilde{x}} = \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J \phi_\kappa$, the following lemma will imply the vanishing of $\langle \phi_{\tilde{x}}, (pr_1)_*u\rangle$, and therefore will finish the proof of (4.3.2).

Lemma 4.4.3. *We have* $\mathrm{Ad}_g\,\phi_{\tilde{x}}(\gamma) \in \mathfrak{g}(\omega_{\mathcal{O}_x})$.

Proof. Indeed, unraveling the definitions, we have $\mathrm{Ad}_g\,\phi(\gamma) = \phi_\kappa(\gamma_\kappa)$, which is regular. ☐

It remains to prove of lemma 4.4.1. Let $\tilde{a}_\lambda : \widetilde{C} \to \mathrm{Gr}(E_\kappa)$ be the morphism constructed as in the Lemma. Since $\widetilde{C}$ is smooth and $\widetilde{U} \subset \widetilde{C}$ is open dense, to prove the lemma, it is enough to show $\tilde{a}_\lambda(\widetilde{U}(k)) \subset \mathrm{Gr}_\mu(E_\kappa)(k)$. Let $\tilde{x} \in \widetilde{U}(k)$ and $\tilde{a}_\lambda(\tilde{x}) = (x, E_{\tilde{x}}, \beta) \in \mathrm{Gr}(E_\kappa)(k)$ be its image, where $E_{\tilde{x}} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J E_\kappa$

and $\beta : E_\kappa|_{C-x} \simeq E_{\tilde{x}}|_{C-x}$ is the isomorphism induced by the canonical section $s \in \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x})(C-x)$. Let

$$rel : \mathrm{Gr}_x(E_\kappa) \to \mathbb{X}_\bullet^+$$

be the relative position map (cf. [BD, §5.2.2]). We have $(E_{\tilde{x}}, \beta) \in \mathrm{Gr}_x(E_\kappa)$ and we need to show that $rel(E_{\tilde{x}}, \beta) = \mu$. For simplicity, we will denote $\mathcal{P} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x})$.

Let Gr_J and Gr_T be the global Grassmannian for the group scheme J and T. By [Yun, lemma 3.2.5], the morphism $j^1 : \pi^* J \to T \times \widetilde{C}$ induces a W-equivariant isomorphism

$$j_{\mathrm{Gr}} : \mathrm{Gr}_J \times_{(C\times B)} \widetilde{U} \simeq \mathrm{Gr}_T \times_C \widetilde{U}$$

of group ind-scheme over $\widetilde{U}$. We denote by $j_{\tilde{x},\mathrm{Gr}} : \mathrm{Gr}_{x,J_b} \simeq \mathrm{Gr}_{x,T}$ the restriction of j_{Gr} to $\tilde{x}$. We have $(\mathcal{P}, s) \in \mathrm{Gr}_{x,J_b}(k)$ (here $s \in \mathcal{P}(C-x)$ is the canonical section) and one can check that $j_{\tilde{x},\mathrm{Gr}}(\mathcal{P}, s) = \lambda \in \mathrm{Gr}_{x,T}(k) \simeq \mathbb{X}_\bullet$. The action of Gr_{x,J_b} on (E_κ, ϕ_κ) defines a map $a_\kappa : \mathrm{Gr}_{x,J_b} \to \mathrm{Gr}_x(E_\kappa)$. We claim that the following diagram commutes

(4.4.2)
$$\begin{array}{ccc} \mathrm{Gr}_{x,J_b}(k) & \xrightarrow{a_\kappa} & \mathrm{Gr}_x(E_\kappa)(k) \\ \downarrow{\scriptstyle j_{\tilde{x},\mathrm{Gr}}} & & \downarrow{\scriptstyle rel} \\ \mathrm{Gr}_{x,T}(k) \simeq \mathbb{X}_\bullet & \longrightarrow & \mathbb{X}_\bullet^+ \simeq \mathbb{X}_\bullet/\mathrm{W} \end{array} .$$

Assuming the claim we see that $rel(E, \beta) = rel(a_\kappa(\mathcal{P}, s))$ is equal to the image of $j_{\tilde{x},\mathrm{Gr}}(\mathcal{P}, s) = \lambda \in \mathbb{X}_\bullet$ in $\mathbb{X}_\bullet^+$. But by assumption $\lambda \in \mathrm{W} \cdot \mu$. This finishes the proof of lemma 4.4.1.

To prove the claim, recall that a trivialization γ_κ of E_κ on $\mathrm{Spec}\mathcal{O}_x$ defines an isomorphism $\mathrm{Gr}_x(E_\kappa) \simeq G(F_x)/G(\mathcal{O}_x)$. Moreover, under the canonical isomorphism $\mathrm{Gr}_{x,J_b}(k) \simeq J_b(F_x)/J_b(\mathcal{O}_x)$, $\mathrm{Gr}_{x,T}(k) \simeq T(F_x)/T(\mathcal{O}_x)$ and $G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) = \mathbb{X}_\bullet^+$, the diagram (4.4.2) can be identified with

$$\begin{array}{ccc} J_b(F_x)/J_b(\mathcal{O}_x) & \longrightarrow & G(F_x)/G(\mathcal{O}_x) \\ \downarrow & & \downarrow{\scriptstyle pr} \\ T(F_x)/T(\mathcal{O}_x) & \longrightarrow & G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x), \end{array}$$

where the the upper arrow is induced by the homomorphism

(4.4.3)
$$J_b \overset{a_{E_\kappa,\phi_\kappa}}{\simeq} \mathrm{Aut}(E_\kappa, \phi_\kappa) \to \mathrm{Aut}(E_\kappa) \overset{\gamma_\kappa}{\simeq} G$$

and the arrow in the left column is induced by the homomorphism $j^1 : \pi^* J \to T \times \widetilde{C}$. Now using the definition of a_{E_κ,ϕ_κ} in (2.6.1) it is not hard to see that the restriction of (4.4.3) to $\mathrm{Spec}\mathcal{O}_x$ can be identified with $J_b \simeq H \hookrightarrow G$, where H is the centralizer of $\phi_\kappa(\gamma_\kappa(x)) \in \mathfrak{g}^{rs}$ in G (it is a maximal torus). To prove the claim, it is enough to show that the restriction of j^1 to $\mathrm{Spec}(\mathcal{O}_x)$[5] is conjugate to the morphism $J_b \simeq H \hookrightarrow G$ by an element in G. To see this, recall that the point $\tilde{x}$ defines Borel subgroup $B_{\tilde{x}}$ containing H. Let $\tilde{g} \in G$ such that $\mathrm{Ad}_{\tilde{g}}((H \subset B_{\tilde{x}})) = (T \subset B)$. Then it follows from the construction of j^1 in [N2, Propsition 4.2.2] that $j^1|_{\mathrm{Spec}\mathcal{O}_x}$ is equal to $J_b \simeq H \overset{\mathrm{Ad}_{\tilde{g}}}{\to} T$. We are done.

[5]Here we identify $\mathrm{Spec}\mathcal{O}_{\tilde{x}} \simeq \mathrm{Spec}\mathcal{O}_x$ and regard j^1 as a map of group schemes over $\mathrm{Spec}\mathcal{O}_x$.

Let D_{Bun_G} be the sheaf of algebra on Higgs'_G in Proposition B.5.1. Let $D^0_{\mathrm{Bun}_G} := D_{\mathrm{Bun}_G}|_{\mathrm{Higgs}'_G\times_{B'}B'^0}$ be the restriction of D_{Bun_G} to the smooth part of the Hitchin fibration. We define $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ to be the category of $D^0_{\mathrm{Bun}_G}$-modules. As we explained in B.5, the category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ is a localization of the category of $\mathcal{D}$-modules on Bun_G and is canonical equivalent to the category of twisted sheaves $\mathrm{QCoh}(\mathscr{D}^0_{\mathrm{Bun}_G})_1$, where $\mathscr{D}^0_{\mathrm{Bun}_G}=\mathscr{D}_{\mathrm{Bun}_G}\times_{B'}B'^0$ and $\mathscr{D}_{\mathrm{Bun}_G}$ is the grebe of crystalline differential operators on Higgs'_G. On the dual side, let $\mathrm{LocSys}_{\check{G}}$ be the stack of de Rham $\check{G}$-local systems on C. Recall that in [CZ], we constructed a fibration

$$h_p : \mathrm{LocSys}_{\check{G}} \to B'$$

from $\mathrm{LocSys}_{\check{G}}$ to the Hitchin base B', which can be regraded as a deformation of the usual Hitchin fibration. We define

$$\mathrm{LocSys}^0_{\check{G}} := \mathrm{LocSys}_{\check{G}}\times_{B'}B'^0.$$

Our goal is to prove the following Theorem:

Theorem 5.0.4. *For a choice of a square root κ of ω_C, we have a canonical equivalence of bounded derived categories*

$$\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)\simeq D^b(\mathrm{Qcoh}(\mathrm{LocSys}^0_{\check{G}}))$$

The proof of above Theorem is divided into two steps. The first step which involves Langlands duality is a twisted version of the classical duality (see §5.2). The second step which does not involve Langlands duality is two abelianisation Theorems (see §5.3) and here we need a choice of square root κ of ω_C. Combining above two steps, our main Theorem follows from a general version of the Fourier-Mukai transform (see §5.4).

5.1. **The $\check{\mathscr{P}}'$-torsor $\check{\mathscr{H}}$.** We first recall that in [CZ], we constructed a $\check{\mathscr{P}}'$-torsor $\check{\mathscr{H}}$. It is defined via the following Cartesian diagram

$$\begin{array}{ccc} \check{\mathscr{H}} & \longrightarrow & \mathrm{LocSys}_{\check{J}^p} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\check{\tau}'} & B'_{\check{J}'} \end{array} \tag{5.1.1}$$

Here $\check{J}^p$ is the pullback of the universal centralizer $\check{J}'$ over $C'\times B'$ along the relative frobenius map $F_{C'\times B'/B'}: C\times B'\to C'\times B'$. This is a group scheme with a canonical connection along C, and therefore it makes sense to talk about the stack $\mathrm{LocSys}_{\check{J}^p}$ of $\check{J}^p$-torsors with flat connections. In addition, it admits a p-curvature map $\mathrm{LocSys}_{\check{J}^p}\to B'_{\check{J}'}$. We refer to [CZ, Appendix] for the generalities.

Recall that there is a Galois description of $\mathscr{P}$ by $\mathrm{Bun}^{\mathrm{W}}_T(\widetilde{C}/B)$. We give a similar description of the $\check{\mathscr{P}}'$-torsor $\check{\mathscr{H}}$ in terms of a $\mathrm{Bun}^{\mathrm{W}}_T(\widetilde{C}/B)'$-torsor. Recall that $\check{\tau}'$ is regarded as a section of $\Omega_{C'\times B'/B'}\otimes \mathrm{Lie}\check{J}'$, which defines a $\check{J}'$-gerbe $\mathscr{D}(\check{\tau}')$ on $C'\times B'$ (see B.4) and according to [CZ, A.10], $\check{\mathscr{H}}$ is isomorphic to $\mathscr{T}_{\mathscr{D}(\check{\tau}')}$ the stack of splittings of $\mathscr{D}(\check{\tau}')$ over B'. Therefore by Lemma 3.3.1 we have

$$\check{\mathscr{H}}|_{B'^0}\simeq \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\check{\tau}')_{\check{T}}}|_{B'^0}, \tag{5.1.2}$$

here $\mathscr{D}(\breve{\tau}')_{\breve{T}} := (\pi^*\mathscr{D}(\breve{\tau}'))^{\breve{j}^1}$ is the $\breve{T}$-gerbe on $\widetilde{C}'$ induced from $\mathscr{D}(\breve{\tau}')$ using maps $\pi : \widetilde{C}' \to C' \times B'$ and $\breve{j}^1 : \pi^*\breve{J}' \to \breve{T}' \times \widetilde{C}'$ (see A.5 for the introduction of induction functor of grebes).

On the other hand, using the definition of $\theta_{\widetilde{C}'} \in \Gamma(\widetilde{C}', \Omega_{\widetilde{C}'} \otimes \breve{\mathfrak{t}}')^{\mathrm{W}}$ in §4.1 one can easily check that $\breve{j}^1_*\pi^*(\breve{\tau}') = \theta_{\widetilde{C}'}$, here $\breve{j}^1_*\pi^*(\breve{\tau}')$ is the $\breve{\mathfrak{t}}'$-value one form induced from $\breve{\tau}'$ using maps π and $\breve{j}^1$. Therefore, by Lemma B.4.1 we see that over B'^0 we have

$$\mathscr{D}(\breve{\tau}')_{\breve{T}} := (\pi^*\mathscr{D}(\breve{\tau}'))^{\breve{j}^1} \simeq \mathscr{D}(\breve{j}^1_*\pi^*(\breve{\tau}')) \simeq \mathscr{D}(\theta_{\widetilde{C}'}). \tag{5.1.3}$$

Hence combining (5.1.2) and (5.1.3) we get the following Galois description of $\breve{\mathscr{H}}$.

Corollary 5.1.1. *There is a canonical isomorphism of $\breve{\mathscr{P}}'$-torsors $\breve{\mathscr{H}}|_{B'^0} \simeq \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\theta_{\widetilde{C}'})}|_{B'^0}$.*

5.2. **Twisted duality.** Let us construct the twisted duality. Let $\theta'_m : \mathscr{P}' \to T^*\mathscr{P}'$ denote the canonical multiplicative one form constructed in §4.2. Let $\mathscr{D}(\theta'_m)$ denote the corresponding $\mathbb{G}_m$-gerbe on $\mathscr{P}'$ obtained by pullback of $\mathscr{D}_{\mathscr{P}}$ on $T^*\mathscr{P}'$ by θ'_m (see B.4). According to C.2, the grebe $\mathscr{D}(\theta'_m)$ is canonically multiplicative. Moreover, according to A.6, the stack of multiplicative splittings of $\mathscr{D}(\theta'_m)$ over B' is a $(\mathscr{P}')^\vee$-torsor $\mathscr{T}_{\mathscr{D}(\theta'_m)}$. Our goal is to prove the following Theorem.

Theorem 5.2.1. *There is a canonical isomorphism of $\mathscr{P}'^\vee \simeq \breve{\mathscr{P}}'$-torsors*

$$\mathfrak{D} : \mathscr{T}_{\mathscr{D}(\theta'_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}.$$

For the rest of this subsection we will restrict everything to B'^0. Recall the Abel-Jacobi map $\mathrm{AJ}^{\mathscr{P}'} : \widetilde{C}' \times \mathbb{X}_\bullet \to \mathscr{P}'$. By Proposition 4.2.1 we have $(\mathrm{AJ}^{\mathscr{P}'})^*\theta'_m = \theta_{\widetilde{C}'}$. Therefore, Lemma B.4.1 implies

$$(\mathrm{AJ}^{\mathscr{P}'})^*\mathscr{D}(\theta'_m) = \mathscr{D}(\theta_{\widetilde{C}'}).$$

Since the Abel-Jacobi map $\mathrm{AJ}^{\mathscr{P}'}$ is W-equivariant, pull back via $\mathrm{AJ}^{\mathscr{P}'}$ defines a functor

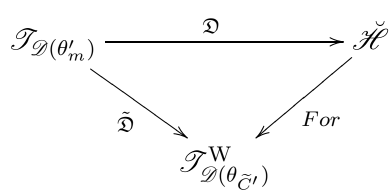
$$\tilde{\mathfrak{D}} : \mathscr{T}_{\mathscr{D}(\theta'_m)} \to \mathscr{T}^{\mathrm{W}}_{\mathscr{D}(\theta_{\widetilde{C}'})}.$$

We claim that $\tilde{\mathfrak{D}}$ canonically lifts to a morphism $\mathfrak{D} : \mathscr{T}_{\mathscr{D}(\theta'_m)} \to \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\theta_{\widetilde{C}'})} \overset{\mathrm{Cor}\ 5.1.1}{\simeq} \breve{\mathscr{H}}$. Let $E \in \mathscr{T}_{\mathscr{D}(\theta'_m)}$ be a tensor splitting of $\mathscr{D}(\theta'_m)$. We thus need to show that the splitting

$$(\tilde{\mathfrak{D}}(E))^\alpha|_{\widetilde{C}'_\alpha} = (\mathrm{AJ}^{\mathscr{P}'})^*E|_{(\widetilde{C}'_\alpha,\breve{\alpha})}$$

admits a canonical isomorphism with the canonical splitting E^0_α of $\mathscr{D}(\theta_{\widetilde{C}'})^\alpha|_{\widetilde{C}'_\alpha} = (\mathrm{AJ}^{\mathscr{P}'})^*\mathscr{D}(\theta'_m)|_{(\widetilde{C}'_\alpha,\breve{\alpha})}$. However, this follows from $\mathrm{AJ}^{\mathscr{P}'}((x,\breve{\alpha}))$ is the unit of $\mathscr{P}'$ for $x \in \widetilde{C}'_\alpha$ and a tensor splitting E of a multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta'_m)$ is canonial isomorphic to the canonical splitting E^0_α of $\mathscr{D}(\theta'_m)$ over the unit. To summarize, we have constructed the following commutative diagram

(5.2.1)



By construction, the morphism $\mathfrak{D}$ is compatible with the $\mathscr{P}'^{\vee} \simeq \breve{\mathscr{P}}'$-action, hence is an equivalence. This finished the proof of Theorem 5.2.1.

5.3. **Abelianisation Theorem.** We need to fix a square root κ of ω_C. Then the Kostant section for $\mathrm{Higgs}'_G \to B'$ induces a map $\epsilon_{\kappa'} : \mathscr{P}' \simeq \mathrm{Higgs}_G^{'reg} \subset \mathrm{Higgs}'_G$, where $\mathrm{Higgs}_G^{'reg}$ is the smooth sub-stack consisting of regular higgs fields. The first abelianisation theorem is the following.

Theorem 5.3.1. *We have a canonical isomorphism $\epsilon_{\kappa'}^*\mathscr{D}_{\mathrm{Bun}_G} \simeq \mathscr{D}(\theta'_m)$, where $\mathscr{D}_{\mathrm{Bun}_G}$ is the $\mathbb{G}_m$-gerbe on Higgs'_G of crystalline differential operators. Moreover, the pullback along the map $\epsilon_{\kappa'}$ defines an equivalence of categories of twisted sheaves*

$$\mathfrak{A}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G^0)) \simeq D^b(\mathrm{QCoh}(\mathscr{D}^0_{\mathrm{Bun}_G}))_1 \overset{\epsilon_{\kappa'}^*}{\simeq} D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1.$$

Proof. By Proposition B.3.3, the restriction of $\mathscr{D}_{\mathrm{Bun}_G}$ to $\mathrm{Higgs}_G^{'reg}$ is isomorphic to the gerbe $\mathscr{D}(\theta'_{can})$ defined by the canonical one form θ'_{can} on $\mathrm{Higgs}_G^{'reg}$. On the other hand, recall that in §4.2 we show that $\epsilon_{\kappa'}^*\theta'_{can} = \theta'_m$ and it implies

$$\epsilon_{\kappa'}^*\mathscr{D}_{\mathrm{Bun}_G} \simeq \epsilon_{\kappa'}^*\mathscr{D}(\theta'_{can}) \simeq \mathscr{D}(\epsilon_{\kappa'}^*\theta_{can}) \simeq \mathscr{D}(\theta'_m).$$

The last statement follows from the fact that the base change of $\epsilon_{\kappa'} : \mathscr{P}' \to \mathrm{Higgs}'_G$ to B'^0 is an isomorphism (see Proposition 2.6.1). □

To state the second abelianisation theorem, recall that in [CZ] we constructed a canonical isomorphism

$$\mathfrak{C} : \breve{\mathscr{H}} \times^{\breve{\mathscr{P}}'} \mathrm{Higgs}'_{\breve{G}} \simeq \mathrm{LocSys}_{\breve{G}}.$$

Moreover, in *loc. cit.* it shows that the choice of the theta characteristic κ induces an isomorphism

$$\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}_{\breve{G}}^{reg} \subset \mathrm{LocSys}_{\breve{G}},$$

here $\mathrm{LocSys}_{\breve{G}}^{reg}$ is the open substack consisting of $\breve{G}$-local systems with regular p-curvature, and we have $\mathrm{LocSys}_{\breve{G}}^{reg}|_{B'^0} = \mathrm{LocSys}_{\breve{G}}^0$. It implies:

Theorem 5.3.2. *For each choice κ of square root of ω_C, we have a canonical isomorphism of $\breve{\mathscr{P}}'$-torsors $\mathfrak{C}_\kappa|_{B'^0} : \breve{\mathscr{H}}|_{B'^0} \simeq \mathrm{LocSys}_{\breve{G}}^0$ and it induces an equivalence of categories*

$$\mathfrak{C}_\kappa^* : D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\breve{G}}^0)) \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})).$$

5.4. **Proof of Theorem 5.0.4.** We deduce our main theorem from the twisted duality and above two abelianisation theorems. By the twisted duality we have an isomorphism of $\mathscr{P}'^{\vee} \simeq \breve{\mathscr{P}}'$-torsors $\mathscr{T}_{\mathscr{D}(\theta'_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}$. Therefore the twisted Fourier-Mukai transform (Theorem A.6.2) implies an equivalence of categories

$$\mathfrak{D} : D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1 \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})).$$

Now combining Theorem 5.3.1 and Theorem 5.3.2 we get the desired equivalence

$$\mathfrak{D}_\kappa = (\mathfrak{C}_\kappa^*)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G^0)) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\breve{G}}^0)).$$

5.5. **A μ_2-gerbe of equivalences.** In this subsection we study how those equivalences $\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$ in Theorem 5.0.4 depend on the choice of the theta characteristics κ. Our discussion is very similar to [FW] and can be regarded as a verification of the predictions of [FW] in our setting.

Let $\omega^{1/2}(C)$ be groupoid of square root of ω_C. The groupoid $\omega^{1/2}(C)$ is a torsor over the Picard category $\mu_2\text{-tors}(C)$ of μ_2-torsors on C. Let **GLC** be the groupoid of equivalences between $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$, i.e. objects in **GLC** are equivalences $\mathbf{E} : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$ and morphisms are isomorphism between equivalences. We first construct an action of $\mu_2\text{-tors}(C)$ on **GLC**.

Let $Z = Z(G)$ be the center of G. We have a map $\alpha : \mu_2 \to Z(G)$ by restricting the co-character $2\rho : \mathbb{G}_m \to G$ to μ_2 (see [BD, §3.4.2]). Thus for each $\chi \in \mu_2\text{-tors}(C)$ and $(E, \nabla) \in \mathrm{LocSys}_G$ we can twist (E, ∇) by χ using the map

$$\mu_2 \to \ Z \to \mathrm{Aut}(E, \nabla)$$

and get a new G-local system $(E \otimes \chi, \nabla_{E\otimes\chi}) \in \mathrm{LocSys}_G$. The assignment $(\chi, E, \nabla) \to (E \otimes \chi, \nabla_{E\otimes\chi})$ defines a geometric action

$$act_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \to \mathrm{LocSys}_G\,.$$

Likewise, there is $act_G : \mu_2\text{-tors}(C) \times \mathrm{Bun}_G \to \mathrm{Bun}_G$. For $\chi \in \mu_2\text{-tors}(C)$, let $a_{\chi,G} : \mathrm{Bun}_G \simeq \mathrm{Bun}_G$ (resp, $b_{\chi,G} : \mathrm{LocSys}_G \simeq \mathrm{LocSys}_G$) be the automorphisms of Bun_G (resp. LocSys_G) given by $a_{\chi,G}(E) := E \otimes \chi$, (resp. $b_{\chi,G}(E, \nabla) = act_G(\chi, E, \nabla)$). They induce auto-equivalences $a^*_{\chi,G}$ and $b^*_{\chi,G}$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}_G))$ respectively. Note that for the definition of $a^*_{\chi,G}$ and $b^*_{\chi,G}$, there is no restriction of the characteristic of k. However, if $\mathrm{char}k = p > |\mathrm{W}|$, we have

Lemma 5.5.1. *1) The equivalence $a^*_{\chi,G}$ preserves the full subcategory $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$.*
2) The equivalence $b^*_{\chi,G}$ preserves the full subcategory $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_G))$.

Proof. This lemma will be clear after we give alternative descriptions of $a^*_{\chi,G}$ and $b^*_{\chi,G}$.

First, recall that in B.3 we introduce a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathrm{Bun}_G}$ over $T^*\mathrm{Bun}'_G$ and the category $\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$ of twisted sheaves on $\mathscr{D}_{\mathrm{Bun}_G}$ such that there is an equivalence of categories between $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)$ and $\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$. Let $f := da'_{\chi,G} : T^*\mathrm{Bun}'_G \simeq T^*\mathrm{Bun}'_G$ be the differential of $a'_{\chi,G}$. The map f preserves the canonical one form θ'_{can}, thus by lemma B.4.1, there is a canonical 1-morphism $M : f^*\mathscr{D}_{\mathrm{Bun}_G} \sim \mathscr{D}_{\mathrm{Bun}_G}$ of grebes on $T^*\mathrm{Bun}'_G$. The 1-morphism M induces an equivalence $M : \mathrm{QCoh}(f^*\mathscr{D}_{\mathrm{Bun}_G})_1 \simeq \mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$ and it is not hard to see that the functor $a^*_{\chi,G}$ is isomorphic to the composition

$$D^b(\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1) \overset{f^*}{\simeq} D^b(\mathrm{QCoh}(f^*\mathscr{D}_{\mathrm{Bun}_G})_1) \overset{M}{\simeq} D^b(\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1). \tag{5.5.1}$$

Recall that the category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ is by definition the category of twisted sheaves on $\mathscr{D}^0_{\mathrm{Bun}_G} = \mathscr{D}_{\mathrm{Bun}_G}|_{B'^0}$. Therefore, Part 1) follows.

To prove Part 2), note that the map $act_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \to \mathrm{LocSys}_G$ can be also described as follow. There is a map of group schemes $(\mu_2)_{C'\times B'} \to Z(G)_{C'\times B'} \to J'$ over $C' \times B'$. It induces a morphism of Picard stack

$$\mathfrak{l}_{\mu_2} : \mu_2\text{-tors}(C) \times B' \to \mathscr{P}', \tag{5.5.2}$$

and the action map act_G can be identified with

$$act_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \overset{\mathfrak{l}_{\mu_2}\times id}{\to} \mathscr{P}' \times_{B'} \mathrm{LocSys}_G \to \mathrm{LocSys}_G \tag{5.5.3}$$

where the last map is the action of $\mathscr{P}'$ on LocSys_G defined in [CZ, Proposition 3.4]. From this description, it is clear that LocSys_G^0 is invariant under the action of $b_{\chi,G}$, and Part 2) follows. □

From now we regard $a^*_{\chi,G}$ and $b^*_{\chi,G}$ as automorphisms of the category $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and $D^b(\mathrm{Qcoh}(\mathrm{LocSys}^0_G))$.

For each $\chi \in \mu_2\,\mathrm{tors}(C)$ and $\mathbf{E} \in \mathbf{GLC}$ we define $\chi \cdot \mathbf{E} := b^*_{\chi,\check{G}} \circ \mathbf{E} \circ a^*_{\chi,G} \in \mathbf{GLC}$. The following lemma follows from the construction of $b^*_{\chi,\check{G}}$ and $a^*_{\chi,G}$.

Lemma 5.5.2. *The functor $\mu_2\text{-tors}(C) \times \mathbf{GLC} \to \mathbf{GLC}$ given by $(\chi, \mathbf{E}) \to \chi \cdot \mathbf{E}$ defines an action of the Picard category $\mu_2\text{-tors}(C)$ on $\mathbf{GLC}$.*

Now let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories acted by a Picard category $\mathscr{G}$. A $\mathscr{G}$-module functor from $\mathscr{C}_1$ to $\mathscr{C}_2$ is a functor $N : \mathscr{C}_1 \to \mathscr{C}_2$ equipped with a functorial isomorphism $N(a \cdot c) \simeq a \cdot N(c)$ satisfying the natural compatibility condition. Here is the main result of this subsection

Proposition 5.5.3. *The assignment $\kappa \to \mathfrak{D}_\kappa$ defines a $\mu_2\text{-tors}(C)$-module functor*

$$\Phi : \omega^{1/2}(C) \to \mathbf{GLC}.$$

Proof. Given $\chi \in \mu_2\text{-tors}(C)$ and $\kappa \in \omega^{1/2}(C)$ we need to specify a functorial isomorphism $\mathfrak{D}_{\chi\cdot\kappa} \simeq \chi \cdot \mathfrak{D}_\kappa$ satisfying the natural compatibility condition. First, observe that the maps $\epsilon_{\kappa'}, \epsilon_{\kappa'_1} : \mathscr{P}' \to \mathrm{Higgs}'_G$ induced by $\kappa, \kappa_1 := \chi \cdot \kappa \in \mu_2\text{-tors}(C)$ differ by a translation of the section $\mathfrak{l}_{\mu_2}(\{\chi\} \times B') \in \mathscr{P}'(B')$, where $\mathfrak{l}_{\mu_2}$ is the map in (5.5.2). Then it follows from the construction of $\mathfrak{A}_\kappa$ and $\mathfrak{C}_\kappa$ in §5.3. that there are canonical functorial isomorphisms $\mathfrak{A}_{\chi\cdot\kappa} \simeq \mathfrak{A}_\kappa \circ a^*_{\chi,G}$ and $\mathfrak{C}^*_\kappa \circ b^*_{\chi,\check{G}} \simeq \mathfrak{C}^*_{\chi\cdot\kappa}$. Therefore we get a functorial isomorphism

$$\mathfrak{D}_{\chi\cdot\kappa} = (\mathfrak{C}^*_{\chi\cdot\kappa})^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_{\chi\cdot\kappa} \simeq b^*_{\chi,\check{G}} \circ (\mathfrak{C}^*_\kappa)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa \circ a^*_{\chi,G} = \chi \cdot \mathfrak{D}_\kappa,$$

and one can check that they satisfies the natural compatibility condition. □

Remark 5.5.4. The above construction suggests that the geometric Langlands correspondence should be a μ_2-gerbe of equivalences between $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}))$. This gerbe is trivial, but not canonically trivialized. One gets a particular trivialization of this gerbe, and hence a particular equivalence $\mathfrak{D}_\kappa$, for each choice of the square root of the canonical line bundle κ on C. A similar μ_2-gerbe also appears in the work of Witten and Frenkel [FW, §10], where the geometric Langlands correspondence is interpreted in terms of gauge theory duality between twisted B-model of $\mathrm{Higgs}_{\check{G}}$ and twisted A-model of Higgs_G.

5.6. **The actions $a^*_{\chi,G}$ and $b^*_{\chi,G}$ as tensoring of line bundles.** In this subsection we show that, under the equivalence $\mathfrak{D}_\kappa$, the geometric actions $a^*_{\chi,G}$ and $b^*_{\chi,G}$ constructed in previous section become functors of tensoring with line bundles.

Recall that in §3.8 we associated to every $Z(\check{G})$-torsor K on C a line bundle $\mathcal{L}_{G,K}$ on Bun_G. For any $\chi \in \mu_2\text{-tors}(C)$ let $K_{G,\chi} := \chi \times^{\mu_2} Z_G \in Z(G)\text{-tors}(C)$ be the induced $Z(G)$-torsor via the canonical map $2\rho : \mu_2 \to Z(G)$. We denote by $\mathcal{L}_{G,\chi}$ and $\mathcal{L}_{\check{G},\chi}$ be the line bundles on Bun_G and $\mathrm{Bun}_{\check{G}}$ corresponding to $K_{\check{G},\chi}$ and $K_{G,\chi}$. Since the line bundle $\mathcal{L}_{G,\chi}$ carries a canonical connection with zero p-curvature, tensoring with $\mathcal{L}_{G,\chi}$ defines an autoequivalence $\mathcal{L}_{G,\chi}\otimes ?$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$.

For any $\kappa\in\omega^{1/2}(C)$ let $\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)\simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check G}))$ be the equivalence in Theorem 5.0.4.

Theorem 5.6.1. *1) The equivalence $\mathfrak{D}_\kappa$ interwines the autoequivalnce $\mathcal{L}_{G,\chi}\otimes ?$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and the autoequivalence $b^*_{\chi,\check G}$ on $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check G}))$ constructed in §5.5.*
2) The equivalence $\mathfrak{D}_\kappa$ interwines the autoequivalnce $a^*_{\chi,G}$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ as in §5.5 and the autoequivalence $\mathcal{L}_{\check G,\chi}\otimes ?$ on $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check G}))$ (here we regard $\mathcal{L}_{\check G,\chi}$ as a line bundle on $\mathrm{LocSys}^0_{\check G}$ via the projection $\mathrm{LocSys}_{\check G}\to\mathrm{Bun}_{\check G}$).

Remark 5.6.2. Similar geometric action and action of tensor multiplication by line bundles on LocSys_G and Higgs_G also appear in the work of Frenkel and Witten [FW, §10.4]. Moreover, the author also predict that the geometric Langlands correspondence should interchange those actions.

Combining Theorem 5.5.3 and Theorem 5.6.1 we have the following:

Corollary 5.6.3. *Let $\kappa_1,\kappa_2\in\omega^{1/2}(C)$. Then we have a natural isomorphism of equivalences*

$$\mathfrak{D}_{\kappa_1}\simeq(\mathcal{L}_{\check G,\chi}\otimes\ ?)\circ\mathfrak{D}_{\kappa_2}\circ(\mathcal{L}_{G,\chi}\otimes\ ?).$$

Here $\chi\in\mu_2\text{-tors}(C)$ such that $\kappa_1=\chi\cdot\kappa_2$ and $\mathcal{L}_{G,\chi}\otimes\ ?$ (resp. $\mathcal{L}_{\check G,\chi}\otimes\ ?$) is the functor of tensoring with the line bundle $\mathcal{L}_{G,\chi}$ (resp. $\mathcal{L}_{\check G,\chi}$).

The rest of this subsection is to prove this theorem.

We first introduce a morphism of Picard stack

$$\tilde{\mathfrak{l}}: Z(G)\text{-tors}(C)\times B'\to\mathrm{Pic}(\mathscr{H})$$

and prove a twisted version of Proposition 3.8.4. We begin with the construction of $\tilde{\mathfrak{l}}$. Let Bun_{J^p} be the Picard stack of J^p-torsors over C. We have the generalized chern class map $\tilde c_{\check J^p}:\mathrm{Bun}_{\check J^p}\to\Pi_{\check G}(1)\text{-gerbe}(X)\times B'$ and a Picard functor $\mathfrak{l}_{\check J^p}$: $Z(G)\text{-tors}(C)\times B'\to\mathrm{Pic}(\mathrm{Bun}_{\check J^p})$. We define

$$\tilde{\mathfrak{l}}: Z(G)\text{-tors}(C)\times B'\stackrel{\mathfrak{l}_{\check J^p}}{\to}\mathrm{Pic}(\mathrm{Bun}_{\check J^p})\to\mathrm{Pic}(\check{\mathscr{H}})$$

where the last map is induced by the restriction map $\check{\mathscr{H}}=\mathrm{LocSys}_{\check J^p}(\tau')\to\mathrm{Bun}_{\check J^p}$.

Recall the morphism $\check{\mathfrak{l}}_{\check J}: Z_G\text{-tors}(C)\times B'\to\mathscr{P}'$ constructed in §3.8. For any $Z(G)$-torsor K over C, we define

$$\mathcal{L}_{\check J^p,K}:=\tilde{\mathfrak{l}}(\{K\}\times B')\in\mathrm{Pic}(\check{\mathscr{H}}).$$

Let K' denote the Frobenius descendent of K (as $C_{et}\simeq C'_{et}$), and let

$$K'_{J'}=\check{\mathfrak{l}}_{\check J}(\{K'\}\times B')\in\mathscr{P}'(B').$$

We will relate $\mathcal{L}_{\check J^p,K}$ and $K'_{J'}$ via the twisted duality. From the definition of θ'_m in §4.2, one can easily check that the restriction of θ'_m to $K'_{J'}$ is zero. Thus the restriction of the $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta'_m)$ to $K'_{J'}$ is canonical trivial and we can regard the structure sheaf $\delta_{K'_{J'}}\in\mathrm{QCoh}(\mathscr{P}')$ as an object in $\mathrm{QCoh}(\mathscr{D}(\theta'_m))_1$. Let $\tilde{\mathcal{L}}_K=\mathfrak{D}(\delta_{K'_{J'}})\in\mathrm{Pic}(\check{\mathscr{H}})$ be the image of $\delta_{K'_{J'}}$ under the twisted duality $\mathfrak{D}$: $D^b(\mathrm{QCoh}(\mathscr{D}(\theta')))_1\simeq D^b(\mathrm{QCoh}(\check{\mathscr{H}}))$.

Lemma 5.6.4. *We have* $\tilde{\mathcal{L}}_K \simeq \mathcal{L}_{\check{J}^p,K}$.

Proof. Let $\breve{\mathfrak{G}} := \mathscr{D}(\theta'_m)^\vee$. We have a short exact sequence of Beilinson 1-motive $0 \to \breve{\mathscr{P}}' \to \breve{\mathfrak{G}} \xrightarrow{p} \mathbb{Z} \to 0$ and $\breve{\mathscr{H}} = p^{-1}(1)$. The construction of duality for torsors in A.6 implies there is a multiplicative line bundle $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}$ on $\breve{\mathfrak{G}}$ such that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{H}}} \simeq \tilde{\mathcal{L}}_K$. Moreover, this line bundle is characterized by the property that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'} \simeq \breve{\mathfrak{D}}_{cl}^{-1}(K'_{J'})$. Observe that we have a natural map $\breve{\mathfrak{G}} \to \mathrm{Bun}_{\check{J}^p}$ of Picard stack[6] such that the composition $\breve{\mathscr{H}} \to \breve{\mathfrak{G}} \to \mathrm{Bun}_{\check{J}^p}$ is the natural inclusion. Thus the morphism $\tilde{\mathfrak{l}} : Z_G\text{-tors}(C) \times B' \to \mathrm{Pic}(\breve{\mathscr{H}})$ factors through a morphism $\tilde{\mathfrak{l}}_{\breve{\mathfrak{G}}} : Z_G\text{-tors}(C) \times B' \to \breve{\mathfrak{G}}^\vee$, and the corresponding multiplicative line bundle $\mathcal{L}_{\breve{\mathfrak{G}},K} := \tilde{\mathfrak{l}}_{\breve{\mathfrak{G}}}(\{K\} \times B') \in \breve{\mathfrak{P}}^\vee(B')$ satisfying $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{H}}} \simeq \mathcal{L}_{\check{J}^p,K}$. It is enough to show that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K} \simeq \mathcal{L}_{\breve{\mathfrak{G}},K}$. From the characterization of $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}$, it is enough to show that $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'} \simeq \breve{\mathfrak{D}}_{cl}^{-1}(K'_{J'})$. But this follows from Proposition 3.8.4 and the fact that $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'}$ is isomorphic to $\mathcal{L}_{\check{J}',K'}$. □

Recall that a choice of $\kappa \in \omega^{1/2}(C)$ defines an isomorphism $\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}_{\check{G}}^{reg}$. More precisely, we have $\mathfrak{C}_\kappa(P,\nabla) = (P \otimes F_C^* E_{\kappa'}, \nabla_{P \otimes F_C^* E_{\kappa'}})$ where $P \otimes F_C^* E_{\kappa'} := P \times^{J^p} F_C^* E_{\kappa'}$ and $\nabla_{P \otimes F_C^* E_{\kappa'}}$ is the product connection.

Lemma 5.6.5. *The pull back of the line bundle* $\mathcal{L}_{\check{G},K}$ *along the map* $\breve{\mathscr{H}} \xrightarrow{\mathfrak{C}_\kappa} \mathrm{LocSys}_{\check{G}} \xrightarrow{\mathrm{pr}} \mathrm{Bun}_{\check{G}}$ *is isomorphic to* $\tilde{\mathcal{L}}_K$. *I.e. we have* $\tilde{\mathcal{L}}_K \simeq \mathfrak{C}_\kappa^* \circ \mathrm{pr}^* \mathcal{L}_{\check{G},K}$.

Proof. The proof is similar to the proof of Lemma 3.8.2. Recall that the line bundles $\mathcal{L}_{\check{G},K}$ and $\tilde{\mathcal{L}}_K \simeq \mathcal{L}_{\check{J}^p,K}$ are induced by the generalized chern class map $\tilde{c}_{\check{G}}$, $\tilde{c}_{\check{J}^p}$. Therefore it is enough to show that for any $(P,\nabla) \in \breve{\mathscr{H}}$ there is a canonical isomorphism $\tilde{c}_{\check{J}^p}(P) \simeq \tilde{c}_{\check{G}}(\mathfrak{C}_\kappa(P))$ of $\Pi_{\check{G}}$-gerbes, where $\mathfrak{C}_\kappa(P) = P \times^{J^p} F_C^* E_{\kappa'}$. Let $\tilde{P} \in \tilde{c}_{\check{J}^p}(P)$ and $\tilde{E}_{\kappa'}$ be the canonical lifting of the Kostant section appearing in Lemma 3.8.2. The G_{sc}-torsor $\tilde{P} \times^{(J^p_{sc})} F_C^* \tilde{E}_{\kappa'}$ is a lifting of $\mathfrak{C}_\kappa(P)$ and the assignment $\tilde{P} \to \tilde{P} \times^{(J^p_{sc})} F_C^* \tilde{E}_{\kappa'}$ defines an isomorphism between $\tilde{c}_{J^p}(P)$ and $\tilde{c}_{\check{G}}(\mathfrak{C}_\kappa(P))$. This finished the proof. □

Now we prove the theorem. Recall that we have $\mathfrak{D}_\kappa = (\mathfrak{C}_\kappa^*)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa$ where $\mathfrak{A}_\kappa$ and $\mathfrak{C}_\kappa^*$ are equivalences constructed in §5.3. It follows from the definition that under the equivalence $\mathfrak{C}_\kappa^*$ the functor $b^*_{\check{G},\chi}$ becomes the functor induced by the geometric action of $K'_{\check{G},\chi} \in Z(\check{G})\text{-tors}(C')$ on $\breve{\mathscr{H}}$[7]. Now Theorem A.6.2 implies, under the equivalence

$$\mathfrak{D} : D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1 \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})),$$

above geometric action becomes the functor of tensoring with the line bundle $\mathcal{L}'_{J,\chi} := \mathfrak{D}_{cl}^{-1}(K'_{\check{G},\chi}) \in (\mathrm{Bun}_{J'})^\vee$. By Lemma 3.8.4 and Lemma 3.8.2, the line bundle $\mathcal{L}'_{J,\chi}$ is equal to the pull back of $\mathcal{L}'_{G,\chi}$ under the map $\mathscr{P}' \xrightarrow{\epsilon'_\kappa} \mathrm{Higgs}'_G \to \mathrm{Bun}'_G$. On the other

[6]We have $\breve{\mathfrak{G}} = \{(n,t) | n \in \mathbb{Z},\ t \in \breve{\mathscr{H}}^{\otimes n}\}$ and $\breve{\mathscr{H}}^{\otimes n}$ is isomorphic to $\mathrm{LocSys}_{\check{J}^p}(n \cdot \tau')$ the base change of $\mathrm{LocSys}_{\check{J}^p} \to B_{\check{J}'}$ along the section $n \cdot \tau' : B' \to B_{\check{J}'}$. Thus there is a natural map $\breve{\mathscr{H}}^{\otimes n} \to \mathrm{Bun}_{\check{J}^p}$ and the map $\breve{\mathfrak{G}} \to \mathrm{Bun}_{\check{J}^p}$ is given by $\breve{\mathfrak{G}} \to \breve{\mathscr{H}}^{\otimes n} \to \mathrm{Bun}_{\check{J}^p}$.

[7]Recall the $K_{\check{G},\chi}$ carries a canonical connection with zero p-curvature and $K'_{\check{G},\chi}$ is its Frobenius descent.

hand, since the equivalence $\mathfrak{A}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G^0)) \simeq D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1$ is induced by pullback along the morphism $\epsilon_\kappa : \mathscr{P} \to \mathrm{Higgs}_G$, an easy exercise shows that under the equivalence $\mathfrak{A}_\kappa$ the functor of tensoring with $\mathcal{L}'_{J,\chi}$ becomes the functor of tensoring with $\mathcal{L}_{G,\chi}$. This implies Part 1).

The proof of part 2) is similar to part 1). Unraveling the definitions of $a^*_{G,\chi}$ and the construction of $\mathfrak{A}_\kappa$, one see that $\mathfrak{A}_\kappa$ interchange the functor $a^*_{G,\chi}$ with the functor of convolution product with $\delta_{K'_{G,\chi}} \in \mathrm{QCoh}(\mathscr{P}')$. Now Theorem A.6.2 implies, under the equivalence $\mathfrak{D}$, above convolution action becomes the functor of tensoring with the line bundle $\tilde{\mathcal{L}}_{K_{G,\chi}} := \mathfrak{D}(K'_{G,\chi}) \in \mathrm{Pic}\,\breve{\mathscr{H}}$. By Lemma 5.6.4 and Lemma 5.6.5, the line bundle $\tilde{\mathcal{L}}_{K_{G,\chi}}$ is isomorphic to the pull back of $\mathcal{L}_{\check{G},\chi}$ under the map $\breve{\mathscr{H}} \overset{\mathfrak{C}_\kappa}{\to} \mathrm{LocSys}_{\check{G}} \overset{\mathrm{pr}}{\to} \mathrm{Bun}_{\check{G}}$. It implies $\mathfrak{C}^*_\kappa \circ (\mathrm{pr}^*\mathcal{L}_{\check{G},\chi} \otimes ?) \simeq (\tilde{\mathcal{L}}_{K_{G,\chi}} \otimes ?) \circ \mathfrak{C}^*_\kappa$.

Appendix A. Beilinson's 1-motive

In this section, we review the duality theory of Beilinson's 1-motives. The main references are [DP, Lau].

A.1. **Picard Stack.** Let us first review the theory of Picard stacks. The standard reference is [Del, §1.4]. Let $\mathcal{T}$ be a given site. Recall that a Picard Stack is a stack $\mathscr{P}$ over $\mathcal{T}$ together with a bifunctor

$$\otimes : \mathscr{P} \times \mathscr{P} \to \mathscr{P},$$

and the associativity and commutative constraints

$$a : \otimes \circ (\otimes \times 1) \simeq \otimes \circ (1 \times \otimes), \qquad c : \otimes \simeq \otimes \circ \mathrm{flip},$$

such that for every $U \in \mathcal{T}$, $\mathscr{P}(U)$ form a Picard groupoid (i.e. symmetrical monoidal groupoid such that every object has a monoidal inverse). The Picard stack is called strictly commutative if $c_{x,x} = \mathrm{id}_x$ for every $x \in \mathscr{P}$. In the paper, Picard stacks will always mean strictly commutative ones.

Let us denote $\mathcal{PS}/\mathcal{T}$ to be the 2-category of Picard stacks over $\mathcal{T}$. This means that if $\mathscr{P}_1, \mathscr{P}_2$ are two Picard stacks over $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{PS}/\mathcal{T}}(\mathscr{P}_1, \mathscr{P}_2)$ form a category. Indeed, $\mathcal{PS}/\mathcal{T}$ is canonically enriched over itself. For $\mathscr{P}_1, \mathscr{P}_2 \in \mathcal{PS}/\mathcal{T}$, we use $\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2)$ to denote the Picard stack of 1-homorphisms from $\mathscr{P}_1$ to $\mathscr{P}_2$ over $\mathcal{T}$ (cf. [Del] §1.4.7). On the other hand, let $C^{[-1,0]}$ be the 2-category of 2-term complexes of sheaves of abelian groups $d : \mathcal{K}^{-1} \to \mathcal{K}^0$ with $\mathcal{K}^{-1}$ injective and 1-morphisms are morphisms of chain complexes (and 2-morphisms are homotopy of chain complexes). Let $\mathcal{K} \in C^{[-1,0]}$. We associate to it a Picard prestack $\mathrm{pch}(\mathcal{K})$ whose U point is the following Picard category

(1) Objects of pch($\mathcal{K}$)(U) are equal to $\mathcal{K}^0(U)$.

(2) If $x, y \in \mathcal{K}^0(U)$, a morphism from x to y is an element $f \in \mathcal{K}^{-1}(U)$ such that $df = y - x$.

Let ch($\mathcal{K}$) be the stackification of pch($\mathcal{K}$). Then a theorem of Deligne says that the functor

$$\mathrm{ch} : \mathrm{C}^{[-1,0]} \to \mathcal{PS}/\mathcal{T}$$

is an equivalence of 2-categories.

Let us fix an inverse functor $()^\flat$ of the above equivalence. So for $\mathscr{P}$ a Picard stack, we have a 2-term complex of sheaves of abelian groups $\mathscr{P}^\flat := \mathcal{K}^{-1} \to \mathcal{K}^0$.

For example, if A is an abelian group in $\mathcal{T}$, then its classifying stack BA is a natural Picard stack and $(BA)^\flat$ can be represented by a 2-term complex quasi-isomorphic to $A[1]$. The following result of Deligne is convenient for computations.

$$(\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2))^\flat \cong \tau_{\leq 0}\, \mathrm{R}\, \underline{\mathrm{Hom}}(\mathscr{P}_1^\flat, \mathscr{P}_2^\flat). \tag{A.1.1}$$

A.2. **The duality of Picard stacks.** Let S be a noetherian scheme. We consider the category Sch/S of schemes over S. We will endow Sch/S with *fpqc* topology in the following discussion.

Definition A.2.1. For a Picard stack $\mathscr{P}$, we define the dual Picard stack as

$$\mathscr{P}^\vee := \underline{\mathrm{Hom}}(\mathscr{P}, B\mathbb{G}_m).$$

Example A.2.2. Let $A \to S$ be an abelian scheme over S. Then by definition $A^\vee := \underline{\mathrm{Hom}}(A, B\mathbb{G}_m) = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ classifies the multiplicative line bundles on A, is represented by an abelian scheme over S, called the dual abelian scheme of A.

Example A.2.3. Let Γ be a finitely generated abelian group over S. By definition, this means locally on S, Γ is isomorphic to the constant sheaf M_S, where M is a finitely generate abelian group (in the naive sense). Recall that the Cartier dual of Γ, denoted by $\mathrm{D}(\Gamma)$ is the sheaf which assigns every scheme U over S the group $\mathrm{Hom}(\Gamma \times_S U, \mathbb{G}_m)$, which is represented by an affine group scheme over S. We claim that $\Gamma^\vee \simeq B\,\mathrm{D}(\Gamma)$. By (A.1.1), it is enough to show that $\mathrm{R}^i\, \underline{\mathrm{Hom}}(\Gamma, \mathbb{G}_m) = 0$ if $i > 0$. By this is clearly since locally on S, Γ is represented by a 2-term complex $\mathbb{Z}_S^m \to \mathbb{Z}_S^n$.

Example A.2.4. Let G be a group of multiplicative type over S, i.e. $G = D(\Gamma)$ for some finitely generated abelian group Γ over S. Let $\mathscr{P} = BG$, the classifying stack of G. We have

$$\mathscr{P}^\vee \simeq \tau_{\leq 0}\, \mathrm{R}\, \underline{\mathrm{Hom}}(BG, B\mathbb{G}_m) \simeq \underline{\mathrm{Hom}}(G, \mathbb{G}_m) \simeq \Gamma.$$

Definition A.2.5. Let $\mathscr{P}$ be a Picard stack. We say that $\mathscr{P}$ is dualizable if the canonical 1-morphism $\mathscr{P} \to \mathscr{P}^{\vee\vee}$ is an isomorphism.

By the above examples, abelian schemes, finitely generated abelian groups, and the classify stacks of groups of multiplicative type are dualizable.

Let $\mathscr{P}$ be a dualizable Picard stack. There is the Poincare line bundle $\mathcal{L}_{\mathscr{P}}$ over $\mathscr{P} \times_S \mathscr{P}^\vee$. Let $D^b(\mathrm{QCoh}(\mathscr{P}))$ denote the bounded derived category of quasicoherent sheaves on $\mathscr{P}$. We define the Fourier-Mukai funtor

$$\Phi_{\mathscr{P}} : D^b(\mathrm{QCoh}(\mathscr{P})) \to D^b(\mathrm{QCoh}(\mathscr{P}^\vee)), \quad \Phi_{\mathscr{P}}(F) = (p_2)_*(p_1^* F \otimes \mathcal{L}_{\mathscr{P}}).$$

Here $p_1 : \mathscr{P} \times_S \mathscr{P}^\vee \to \mathscr{P}$ and $p_2 : \mathscr{P} \times_S \mathscr{P}^\vee \to \mathscr{P}^\vee$ denote the natural projections. It is easy to see in the case when $\mathscr{P}$ is of the form given in the above examples, $\Phi_{\mathscr{P}}$ is an equivalence of categories. Indeed, the case when $\mathscr{P} = A$ follows from the results of Mukai; the case when $\mathscr{P} = \Gamma$ or BG is clear.

It is not clear to us whether $\Phi_{\mathscr{P}}$ is an equivalence for all dualizable Picard stacks. In the following subsection, we select out a particular class of Picard stacks, called the Beilinson's 1-motive (following [DP, DP2]), for which the Fourier-Mukai transforms are equivalences.

A.3. **Beilinson's 1-motives.** Let $\mathscr{P}_1, \mathscr{P}_2$ be two Picard stacks. We say that $\mathscr{P}_1 \subset \mathscr{P}_2$ if there is a 1-morphism $\phi : \mathscr{P}_1 \to \mathscr{P}_2$, which is a fatihful embedding.

Definition A.3.1. We called a Picard stack $\mathscr{P}$ a Beilinson's 1-motive if it admits a two step filtration $W_\bullet \mathscr{P}$:

$$W_{-1} = 0 \subset W_0 \subset W_1 \subset W_2 = \mathscr{P}$$

such that (i) $\mathrm{Gr}_0^W \simeq BG$ is the classifying stack of a group G of multiplicative type; (ii) $\mathrm{Gr}_1^W \simeq A$ is an abelian scheme; and (iii) $\mathrm{Gr}_2^W \simeq \Gamma$ is a finitely generated abelian group.

Lemma A.3.2. *The dual of a Beilinson's 1-motive is a Beilinson's 1-motive and Beilinson's 1-motive are dualizable.*

Proof. This is proved via the induction on the length of the filtration. We use the following fact. Let

$$0 \to \mathscr{P}' \to \mathscr{P} \to \mathscr{P}'' \to 0$$

be a short exact sequence of Picard stacks. Then

$$0 \to (\mathscr{P}'')^\vee \to \mathscr{P}^\vee \to (\mathscr{P}')^\vee$$

with the right arrow surjective if $\mathrm{R}^2 \underline{\mathrm{Hom}}((\mathscr{P}'')^\flat, \mathbb{G}_m) = 0$.

If $\mathscr{P} = W_0\mathscr{P}$, this is given by Example A.2.4. If $\mathscr{P} = W_1\mathscr{P}$, we have the following exact sequence

$$0 \to BG \to \mathscr{P} \to A \to 0.$$

Using the fact that $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m) = 0$ (See [LB1, Remark 6]), we know that $\mathscr{P}$ is also a Beilinson's 1-motive. In general, we have

$$0 \to W_1\mathscr{P} \to \mathscr{P} \to \Gamma \to 0,$$

and the lemma follows from the fact $\underline{\mathrm{Ext}}^2(\Gamma, \mathbb{G}_m) = 0$ (see Example A.2.3). □

Corollary A.3.3. *Let $\mathscr{P}$ be a Beilinson 1-motive, and $\mathscr{P}^\vee$ be its dual. Then $\mathrm{D}(\mathrm{Aut}_{\mathscr{P}}(e)) = \pi_0(\mathscr{P}^\vee)$, where e denotes the unit of $\mathscr{P}$ and π_0 denotes the group of connected components of $\mathscr{P}^\vee$.*

Lemma A.3.4. *Let $\mathscr{P}$ be a Beilinson's 1-motive. Then locally on S,*

$$\mathscr{P} \simeq A \times BG \times \Gamma.$$

Proof. It is enough to prove that

$$\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^1(\Gamma, A) = \underline{\mathrm{Ext}}^1(A, BG) = 0.$$

Clearly, $\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^2(\Gamma, G) = 0$. To see that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$, we can assume that $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Then it follows that $A \xrightarrow{n} A$ is surjective in the flat topology that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$.

To see that $\underline{\mathrm{Ext}}^1(A, BG) = 0$, let $\mathscr{P}$ to the Beilinson's 1-motive corresponding to a class in $\underline{\mathrm{Ext}}^1(A, BG)$. Taking the dual, we have $0 \to A^\vee \to \mathscr{P}^\vee \to \mathrm{D}(G) \to 0$. Therefore, locally on S, $\mathscr{P}^\vee \simeq A^\vee \times \mathrm{D}(G)$, and therefore locally on S, $\mathscr{P}^{\vee\vee} \simeq A \times BG$. □

Now we have the following result.

Theorem A.3.5. *Let $\mathscr{P}$ be a Beilinson's 1-motive. Then the functor $\Phi_{\mathscr{P}}$ is an equivalence of categories.*

Proof. Observe that $\Phi_{\mathscr{P}^\vee}$ is the adjoint functor of $\Phi_{\mathscr{P}}$. Let $\mathcal{M} \in D^b(\mathrm{QCoh}(\mathscr{P}))$ and let $i : \mathcal{M} \to \Phi_{\mathscr{P}^\vee} \circ \Phi_{\mathscr{P}}(\mathcal{M})$ be the adjunction map. We claim that i is an isomorphism. Indeed, using Lemma A.3.4 and the fact that $\Phi_{\mathscr{P}}$ and $\Phi_{\mathscr{P}^\vee}$ commutes with base change, we can assume $\mathscr{P} \simeq A \times BG \times \Gamma$ and the claim follows from the original Fourier-Mukai duality. The same argument shows that $\Phi_{\mathscr{P}^\vee} \circ \Phi_{\mathscr{P}}(\mathcal{M}) \to \mathcal{M}$ is an isomorphism and it implies $\Phi_{\mathscr{P}}$ is an equivalence. □

A.4. **Multiplicative torsors and extensions of Beilinson 1-motives.** Let us return to the general set-up. Let $\mathcal{T}$ be a fixed site and let $\mathscr{P}$ be a Picard stack over $\mathcal{T}$. A torsor of $\mathscr{P}$ is a stack $\mathscr{Q}$ over $\mathcal{T}$, together with a bifunctor

$$\mathrm{Action} : \mathscr{P} \times \mathscr{Q} \to \mathscr{Q},$$

satisfying the following properties:
(i) the bifunctor Action defines a monoidal action of $\mathscr{P}$ on $\mathscr{Q}$;
(ii) For every $V \in \mathcal{T}$, there exists a covering $U \to V$, such that $\mathscr{Q}(U)$ is non-empty.
(iii) For every $U \in \mathcal{T}$ such that $\mathscr{Q}(U)$ is non-empty and let $D \in \mathscr{Q}(U)$, the functor

$$\mathscr{P}(U) \to \mathscr{Q}(U), \quad C \mapsto \mathrm{Action}(C, D)$$

is an equivalence.

In the case when $\mathscr{P}$ is the Picard stack of G-torsors for some sheaf of abelian gorups G, people usually called a $\mathscr{P}$-torsor $\mathscr{Q}$ a G-gerbe.

All $\mathscr{P}$-torsors form a 2-category, denoted by $B\mathscr{P}$, is canonically enriched over itself ([OZ, §2.3]). I.e., given two $\mathscr{P}$-torsors $\mathscr{Q}_1, \mathscr{Q}_2$, $\underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}_1, \mathscr{Q}_2)$ is a natural $\mathscr{P}$-torsor. An object in $\underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}_1, \mathscr{Q}_2)$ induces an equivalence between $\mathscr{Q}_1$ and $\mathscr{Q}_2$. In addition, there is a monoidal structure on $B\mathscr{P}$ making $B\mathscr{P}$ a Picard 2-stacks.

Now, let $\mathscr{P}$ be a Picard stack and let $\mathscr{G}$ be a $\mathscr{P}_1$-torsor over $\mathscr{P}$. Let $m : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$, $e : \mathcal{T} \to \mathscr{P}$ be respectively the multiplication morphism and the unit morphism. Let $\sigma : \mathscr{P} \times \mathscr{P} \to \mathscr{P} \times \mathscr{P}$ be the flip map $\sigma(x, y) = (y, x)$.

Definition A.4.1. A commutative group structure on $\mathscr{G}$ is the following structure:
1. An equivalence $M : \mathscr{G} \boxtimes \mathscr{G} \simeq m^*\mathscr{G}$ of $\mathscr{P}_1$-torsors over $\mathscr{P} \times \mathscr{P}$;
2. A 2-morphism γ between the resulting two 1-morphisms between $\mathscr{G} \boxtimes \mathscr{G} \boxtimes \mathscr{G}$ and $m^*\mathscr{G}$ over $\mathscr{P} \times \mathscr{P} \times \mathscr{P}$, which satisfies the cocycle condition.
3. Observe that $\sigma^*(M)$ is another 1-morphism between $m^*\mathscr{G}$ and $\mathscr{G} \boxtimes \mathscr{G}$. A commutative group structure on $\mathscr{G}$ is a 2-morphism $i : \sigma^* M \simeq M$ such that $i^2 = id$.

Clearly, all $\mathscr{P}_1$-torsors over $\mathscr{P}$ with commutative group structures also form a 2-category.

Definition A.4.2. A multiplicative splitting of a $\mathscr{P}_1$-torsor $\mathscr{G}$ over $\mathscr{P}$ with commutative group structure is a 1-morphism (in the category of all $\mathscr{P}_1$-torsors over $\mathscr{P}$ with commutative group structures): $\mathscr{P} \to \mathscr{G}$.

We have the following lemma.

Lemma A.4.3. *A commutative group structure on $\mathscr{G}$ will make $\mathscr{G}$ into a Picard stack which fits into the following short exact sequence:*

$$0 \to \mathscr{P}_1 \to \mathscr{G} \to \mathscr{P} \to 0.$$

In particular, if $\mathscr{P}$ is a Beilinson 1-motive, and $\mathscr{P}_1 = B\mathbb{G}_m$, then $\mathscr{G}$ is a Beilinson 1-motive and we will called $\mathscr{G}$ a multiplicative $\mathbb{G}_m$-gerbe over $\mathscr{P}$.

A.5. **Induction functor.** Let $\phi : \mathscr{P} \to \mathscr{P}_1$ be a morphism of Picard stacks. Then for each $\mathscr{P}$-torsor $\mathscr{Q}$ we may associate to it a $\mathscr{P}_1$-torsor $\mathscr{Q}^\phi$ with the property that there exists a canonical functor $\mathscr{Q} \to \mathscr{Q}^\phi$, compatible with their $\mathscr{P}$ and $\mathscr{P}_1$-structure via ϕ. For any section E of $\mathscr{Q}$ we denote by E^ϕ the section of $\mathscr{Q}^\phi$ induced by the canonical map $\mathscr{Q} \to \mathscr{Q}^\phi$.

When $\mathscr{P} = \mathrm{Bun}_G$ and $\mathscr{P}_1 = \mathrm{Bun}_H$ are Picard stacks of G, H-torsors for some sheaves of abelian groups G and H, any morphism $\phi : G \to H$ will induce a morphism of Picard stacks $\mathscr{P} \to \mathscr{P}_1$ and for any G-gerbe $\mathscr{Q}$ and section E of $\mathscr{Q}$ we also denote by $\mathscr{Q}^\phi$, E^ϕ to be the induced H-gerbe and the corresponding section.

A.6. **Duality for torsors.** Let $\mathscr{Y}$ be an algebraic stack. Let $\widetilde{\mathscr{Y}}$ be a $\mathbb{G}_m$-gerbe over $\mathscr{Y}$, i.e. $\widetilde{\mathscr{Y}}$ is $B\mathbb{G}_m$-torsor over $\mathscr{Y}$. We called $\widetilde{\mathscr{Y}}$ split if it is isomorphic to $\mathscr{Y} \times B\mathbb{G}_m$. Let $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))$ be the bounded derived category of coherent sheaves on $\widetilde{\mathscr{Y}}$. If $\widetilde{\mathscr{Y}}$ is split, there is a decomposition

$$D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}})) = \oplus_{n\in\mathbb{Z}} D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$$

according to the character of $\mathbb{Z}$. If $\widetilde{\mathscr{Y}}$ is not split we still have a decomposition of $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))$ into direct sum like above. This decomposition is described as following. Let $a : B\mathbb{G}_m \times \widetilde{\mathscr{Y}} \to \widetilde{\mathscr{Y}}$ be the action map. Then $\mathcal{M} \in D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$ if only if $a^*(\mathcal{M}) \in D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$.

Definition A.6.1. The direct summand $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_1$ is called the category of twisted sheaves on $\widetilde{\mathscr{Y}}$.

Now we further assume $\mathscr{Y} = \mathscr{P}$ is a Beilinson's 1-motive oer S and $\widetilde{\mathscr{Y}} = \mathscr{D}$ is a multiplicative $\mathbb{G}_m$-gerbe over $\mathscr{P}$. Let $\mathscr{P}$ and $\mathscr{D}$ as above. Then by Lemma A.4.3 we have the following short exact sequence

$$0 \to B\mathbb{G}_m \xrightarrow{i} \mathscr{D} \xrightarrow{p} \mathscr{P} \to 0 \tag{A.6.1}$$

as Picard stacks. Note that in this case $\mathscr{D}$ is also a Beilinson's 1-motive. Let $\mathscr{D}^\vee$ be the dual Beilinson's 1-motive. It fits into the short exact sequence

$$0 \to \mathscr{P}^\vee \to \mathscr{D}^\vee \xrightarrow{\pi} \mathbb{Z}_S \to 0.$$

Let

$$\mathscr{T}_{\mathscr{D}} = \pi^{-1}(1) \tag{A.6.2}$$

be the $\mathscr{P}^\vee$-torsor associated to above extension. We call $\mathscr{T}_{\mathscr{D}}$ the stack of multiplicative splitting of $\mathscr{D}$. To justify its namely, let us give a description of $\mathscr{T}_{\mathscr{D}}$. By definition the dual of $\mathscr{D}$ is

$$\mathscr{D}^\vee = \underline{\mathrm{Hom}}(\mathscr{D}, B\mathbb{G}_m).$$

An element $s \in \mathscr{D}^\vee$ belongs to $\mathscr{T}_{\mathscr{D}} := (\mathscr{D}^\vee)_1$ if and only if the composition

$$B\mathbb{G}_m \xrightarrow{i} \mathscr{D} \xrightarrow{s} B\mathbb{G}_m$$

is equal to the identity. Equivalently, $s \in \mathscr{T}_{\mathscr{D}}$ gives a splitting of the exact sequence (A.6.1) and according to A.4.2 it is a multiplicative splitting of $\mathscr{D}$.

This construction yields an equivalence of the category of extensions of $\mathbb{Z}_S$ to $\mathscr{P}^\vee$ and the category of $\mathscr{P}^\vee$-torsors. The inverse functor is given as follows: for a $\mathscr{P}^\vee$-torsor $\mathscr{T}$ the corresponding extension is given by $\mathscr{T}^{ext} = \{(n, t) \mid n \in \mathbb{Z}, t \in \mathscr{T}^{\otimes n}\}$.

We have the following

Theorem A.6.2 ([DP2, Trav]). *1) The Fourier-Mukai functor $\Phi_{\mathscr{D}}$ restricts to an equivalence*

$$\Phi_{\mathscr{D}}: D^b(\mathrm{QCoh}(\mathscr{D}))_1 \simeq D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}})).$$

2) On the category $D^b(\mathrm{QCoh}(\mathscr{D}))_1$ we have functors of tensor multiplication by quasi-coherent sheaves on $\mathscr{P}$. On the other hand, the category $D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}}))$ is acted on by convolutions with objects in $D^b(\mathrm{QCoh}(\mathscr{P}^\vee))$. The equivalence $\Phi_{\mathscr{D}}$ intertwines these actions.

3) The commutative group structure on $\mathscr{D}$ defines a convolution product on $D^b(\mathrm{QCoh}(\mathscr{D}))_1$. On the other hand, the category $D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}}))$ has the monoidal structure of tensor multiplication. The equivalence $\Phi_{\mathscr{D}}$ intertwines these structures.

Appendix B. $\mathcal{D}$-module on stacks and Azumaya property

In this section we review some basic facts about $\mathcal{D}$-modules on algebraic stack and Azumaya property of sheaf of differential operators. Standard reference for those materials are [BD] and [BB].

B.1. **Azumaya algebras and twisted sheaves.** Let us begin with a review of the basic definition of Azumaya algebras and the category of twisted sheaves. Let S be a Noetherian scheme. Let $\mathscr{X}$ be an algebraic stack over S. Recall that an Azumaya algebra $\mathcal{A}$ over $\mathscr{X}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-algebras, which is locally in smooth topology isomorphic to $\mathcal{E}nd(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on $\mathscr{X}$. Such an isomorphism between $\mathcal{A}$ and the matrix algebra is called a splitting of $\mathcal{A}$. Given an Azumaya alglebra $\mathcal{A}$ on $\mathscr{X}$, one can associate to it the $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathcal{A}}$ of splittings over $\mathscr{X}$, i.e., for any $U \to S$ we have

(B.1.1) $$\mathscr{D}_{\mathcal{A}}(U) = \{(x, \mathcal{V}, i) | x \in \mathscr{X}(U), i : \mathcal{E}nd(\mathcal{V}) \simeq x^*(\mathcal{A})\}.$$

We will use the following Proposition in the sequel:

Proposition B.1.1 ([DP2], §2.1.2). *Let $\mathcal{A}$ be a sheaf of Azumaya algebra on $\mathscr{X}$. We have the following equivalence of categories*

$$\mathrm{QCoh}(\mathscr{D}_{\mathcal{A}})_1 \simeq \mathcal{A}\text{-mod}(\mathrm{QCoh}(\mathscr{X}))$$

where $\mathcal{A}$-mod(Qcoh($\mathscr{X}$)) is the category of $\mathcal{A}$-modules which is quasi-coherent as $\mathcal{O}_X$-modules.

B.2. **$\mathcal{D}$-module on scheme.** Let X be a scheme smooth over S. Let $\mathcal{D}_{X/S}$ be the sheaf of crystalline differential operators on X, i.e., $\mathcal{D}_{X/S}$ is the universal enveloping $\mathcal{D}$-algebra associated to the relative tangent Lie algebroid $T_{X/S}$. By definition, the category of $\mathcal{D}$-module on X is the category of modules over $\mathcal{D}_{X/S}$ that are quasi-coherent as $\mathcal{O}_X$-modules. We denote by $\mathcal{D}$-mod(X) the category of $\mathcal{D}$-module on X. In the case $p\mathcal{O}_S = 0$, we have the following fundamental observation:

Theorem B.2.1 ([BMR], §1.3.2, §2.2.3). *The center of $(F_{X/S})_*\mathcal{D}_{X/S}$ is isomorphic to $\mathcal{O}_{T^*(X'/S)}$ and there is an Azumaya algebra $D_{X/S}$ on $T^*(X'/S)$ such that*

$$(F_{X/S})_*\mathcal{D}_{X/S} \simeq (\tau_{X'})_* D_{X/S}.$$

where $\tau_{X'}: T^*(X'/S) \to X'$ is the natural projection.

In particular, we have the following:

Corollary B.2.2. *There is a canonical equivalence of categories*

$$\mathcal{D}\text{-mod}(X) \simeq \mathrm{QCoh}(\mathscr{D}_{D_{X/S}})_1$$

where $\mathscr{D}_{D_{X/S}}$ *is the gerbe of splittings of* $D_{X/S}$.

In what follows, the gerbe $\mathscr{D}_{D_{X/S}}$ will be denoted by $\mathscr{D}_{X/S}$ for simplicity.

B.3. **$\mathcal{D}$-module on stack.** Let S be a Noetherian scheme and $p\mathcal{O}_S = 0$. Let $\mathscr{X}$ be a smooth algebraic stack over S. A $\mathcal{D}$-module M on $\mathscr{X}$ is an assignment for each $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$, a $\mathcal{D}_{U/S}$-module M_U and for each morphism $f : V \to U$ in $\mathscr{X}_{sm}$ an isomorphim $\phi_f : f^*M_U \simeq M_V$ which satisfies cocycle conditions. We denote the category of $\mathcal{D}$-modules on $\mathscr{X}$ by $\mathcal{D}\text{-mod}(\mathscr{X})$.

Unlike the case of scheme, in general there does not exist a sheaf of algebras $\mathcal{D}_{\mathscr{X}/S}$ on $\mathscr{X}$ such that the category of $\mathcal{D}$-module on $\mathscr{X}$ is equivalent to the category of modules over $\mathcal{D}_{\mathscr{X}/S}$, and therefore the naive stacky generalization of Theorem B.2.1 is wrong. (However, see Remark B.3.2.) On the other hand, it is shown in [Trav] that the obvious stacky version of Corollary B.2.2 is correct:

Proposition B.3.1. *There exists a* $\mathbb{G}_m$*-gerbe* $\mathscr{D}_{\mathscr{X}/S}$ *on* $T^*(\mathscr{X}'/S)$ *such that the category of twisted sheaves on* $\mathscr{D}_{\mathscr{X}/S}$ *is equivalent to the category of* $\mathcal{D}$*-modules on* $\mathscr{X}$*, i.e., we have*

$$\mathcal{D}\text{-mod}(\mathscr{X}) \simeq \mathrm{QCoh}(\mathscr{D}_{\mathscr{X}/S})_1.$$

Remark B.3.2. It is a theorem of Gabber that on a quasi-projective scheme X, every torsion element in $\mathrm{H}^2_{et}(X, \mathbb{G}_m)$ can be constructed from an Azumaya algebra via (B.1.1). However, this fails for non-separated schemes. A theorem of Töen [Toen] shows that in a very general situation, every $\mathbb{G}_m$-gerbe arises from a derived Azumaya algebra. The derived category of $\mathcal{D}$-modules on $\mathscr{X}$ (which is not the derived category of $\mathcal{D}\text{-mod}(\mathscr{X})$ in general) should be equivalent to the category of modules over a derived Azumaya algebra $D^{dr}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$.

Let us sketch the construction of the $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$. As gerbes satisfy smooth descent, it is enough to supply a $\mathbb{G}_m$-gerbe $(\mathscr{D}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S)\times_{\mathscr{X}'} U'$ for every $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$ and compatible isomorphisms for any $\beta : U \to V$ in $\mathscr{X}_{sm}$. But for any $f : U \to \mathscr{X}$ in $\mathscr{X}_{sm}$ we have

$$(f'_U)_d : T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U' \to T^*(U'/S).$$

We have a $\mathbb{G}_m$-gerbe $\mathscr{D}_{U/S}$ on $T^*(U'/S)$ corresponding to the sheaf of relative differential operators $\mathcal{D}_{U/S}$. We define a $\mathbb{G}_m$-gerbe $(\mathscr{D}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U'$ to be the pull back of $\mathscr{D}_{U/S}$ along $(f'_U)_d$. One can check that these gerbes $(\mathscr{D}_{\mathscr{X}/S})_U$ are compatible under pullbacks, and therefore, they define a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}$.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a schematic morphism between two smooth algebraic stacks. From the above construction, the following lemma clearly follows from its scheme theoretic version.

Lemma B.3.3 ([Trav])**.** *(1) There is a canonical* 1*-morphism of* $\mathbb{G}_m$*-gerbe on* $T^*(\mathscr{Y}'/S) \times_{\mathscr{Y}'} \mathscr{X}'$

$$M_f : (f'_p)^*\mathscr{D}_{\mathscr{Y}/S} \simeq (f'_d)^*\mathscr{D}_{\mathscr{X}/S}.$$

(2) For a pair of morphisms $\mathscr{X} \xrightarrow{g} \mathscr{Z} \xrightarrow{h} \mathscr{Y}$ *and their composition* $f = h \circ g : \mathscr{X} \to \mathscr{Y}$ *, there is a canonical* 1*-morphisms of* $\mathbb{G}_m$*-gerbe on* $T^*(\mathscr{Y}'/S) \times_{\mathscr{Y}'} \mathscr{X}'$

$$M_{g,h} : (f'_p)^*\mathscr{D}_{\mathscr{Y}/S} \simeq (f'_d)^*\mathscr{D}_{\mathscr{X}/S},$$

together with a canonical 2*-morphism between* $M_{h\circ g}$ *and* $M_{g,h}$*.*

(3) We have a canonical 1*-morphism of* $\mathbb{G}_m$*-gerbe on* $T^*(\mathscr{X}'/S)^{sm}$*:*

$$\mathscr{D}_{\mathscr{X}/S}|_{T^*(\mathscr{X}'/S)^{sm}} \simeq \mathscr{D}_{T^*(\mathscr{X}'/S)^{sm}/S}(\theta_{can}) := \theta_{can}^*(\mathscr{D}_{T^*(\mathscr{X}'/S)^{sm}/S}),$$

where $T^*(\mathscr{X}'/S)^{sm}$ *is the maximal smooth open substack of* $T^*(\mathscr{X}'/S)$ *and* θ_{can} : $T^*(\mathscr{X}'/S)^{sm} \to T^*(T^*(\mathscr{X}'/S)^{sm})$ *is the canonical one form.*

Let us discuss a stacky version of [OV, §4.3]. Let $\mathscr{X}/S$ be a proper smooth algebraic stack as above and let $\mathscr{P}ic^\natural(\mathscr{X}/S)$ be the Picard stack of invertible sheaves on $\mathscr{X}$ equipped with a connection (i.e. objects in $\mathscr{P}ic^\natural(\mathscr{X}/S)$ are $\mathcal{D}$-modules on $\mathscr{X}$ whose underlying quasi-coherent sheaves are invertible). Note that $\mathscr{P}ic^\natural(\mathscr{X}/S)$ is represented by an algebraic stack. Indeed, the Picard stack $\mathscr{P}ic(\mathscr{X}/S)$ of invertible sheaves on $\mathscr{X}/S$ is representable (cf. [SB, Theorem 1.1]), and $\mathscr{P}ic^\natural(\mathscr{X}/S) \to \mathscr{P}ic(\mathscr{X}/S)$ is schematic. Let $B'_S = \mathrm{Sect}_S(\mathscr{X}', T^*(\mathscr{X}'/S))$.

Proposition B.3.4. *(1) There is a natural morphism* $\psi : \mathscr{P}ic^\natural(\mathscr{X}/S) \to B'_S$*.*

(2) The pullback of the gerbe $\mathscr{D}_{\mathscr{X}/S}$ *along*

$$\mathscr{X}' \times_S \mathscr{P}ic^\natural(\mathscr{X}/S) \stackrel{\mathrm{id}\times\psi}{\to} \mathscr{X}' \times_S B'_S \to T^*(\mathscr{X}'/S)$$

is canonically trivialized.

Proof. For (1), recall that if $\mathscr{X}$ is a scheme, the morphism ψ is given by the p-curvature map (see [OV, §4.3]). We explain how to generalize this map to stacks. Let $U \to \mathscr{X}$ be a smooth morphism. Then via pullback, we obtain a morphism $\mathscr{P}ic^\natural(\mathscr{X}/S) \to \mathscr{P}ic^\natural(U/S) \to \mathrm{Sect}_S(U', T^*(U'/S))$. By considering further pullbacks to $V = U \times_{\mathscr{X}} U$, we find that the above maps fit into the following commutative diagram

$$\begin{array}{ccc} \mathscr{P}ic^\natural(\mathscr{X}/S) & \longrightarrow & \mathscr{P}ic^\natural(U/S) \\ {\scriptstyle\psi_U}\downarrow & & \downarrow \\ \mathrm{Sect}_S(U', T^*(\mathscr{X}'/S) \times_{\mathscr{X}'} U') & \longrightarrow & \mathrm{Sect}_S(U', T^*(U'/S)). \end{array}$$

These ψ_U's are compatible under pullbacks and define the $\pi : \mathscr{P}ic^\natural(\mathscr{X}/S) \to B'_S$.

For (2), again let $U \to \mathscr{X}$ be a smooth morphism. Note that the pullback of the gerbe $\mathscr{D}_{U/S}$ along $U' \times_S \mathscr{P}ic^\natural(U/S) \to T^*(U'/S)$ is canonically trivialized by the object $F_*(\mathcal{L}, \nabla)$, where $(\mathcal{L}, \nabla)$ is the universal object on $U \times_S \mathscr{P}ic^\natural(U/S)$. Combining this with Lemma B.3.3 and the proof of part (i), this shows that the pullback of $\mathscr{D}_{\mathscr{X}/S}$ along $U' \times_S \mathscr{P}ic^\natural(\mathscr{X}/S) \to \mathscr{X}' \times_S \mathscr{P}ic^\natural(\mathscr{X}/S)$ is canonically trivialized. These trivializations glue together and give a canonical trivialization of $\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}' \times_S \mathscr{P}ic^\natural(\mathscr{X}/S)$. □

B.4. **One forms.** In this subsection we make a digression into the construction of gerbe using one forms. We refer to [CZ, Appendix A.10] for more details. Recall that for any smooth algebraic stack $\mathscr{X}/S$ we can associate to it a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$. Thus giving an one form $\theta : \mathscr{X}' \to T^*(\mathscr{X}'/S)$ we can construct a $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta) := \theta^*\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}'$ by pulling back $\mathscr{D}_{\mathscr{X}/S}$ along θ.

When $\mathscr{X} = X$ is a smooth noetherian schemes above construction can be generalized as following. Let $\mathcal{G}$ be a smooth affine commutative group scheme over X. For any section θ of $\mathrm{Lie}\mathcal{G}' \otimes \Omega_{X'/S}$ we can associate to it a $\mathcal{G}$-gerbe $\mathscr{D}(\theta)$ on X' using the four term exact sequence constructed in *loc. cit.*. In the case $\mathcal{G} = \mathbb{G}_m$, the $\mathbb{G}_m$-gerbe

$\mathscr{D}(\theta)$ is isomorphic to $\theta^*\mathscr{D}_{X,S}$ the pull back of $\mathscr{D}_{X/S}$ along $\theta : X' \to T^*(X'/S)$. We have the following basic functorial properties:

Lemma B.4.1.

(1) *Let $\mathscr{Y}$ be another smooth algebraic stack over S and let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism. Let θ be an one form on $\mathscr{X}'$. There is a canonical equivalence of $\mathbb{G}_m$-gerbes on $\mathscr{Y}'$*

$$f'^*\mathscr{D}(\theta) \simeq \mathscr{D}(f'^*\theta).$$

(2) *When X a smooth noetherian scheme and let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of smooth commutative affine group schemes over X. For any section θ of $\mathrm{Lie}\mathcal{G}' \otimes \Omega_{X'}$ let $\phi'_*\theta$ be the section of $\mathrm{Lie}\mathcal{H}' \otimes \Omega_{X'/S}$ induced by ϕ. There is a canonical equivalence of $\mathcal{H}'$-gerbes on X'*

$$\mathscr{D}(\theta)^{\phi'} \simeq \mathscr{D}(\phi'_*\theta),$$

here $\mathscr{D}(\theta)^{\phi'}$ is the $\mathcal{H}'$-gerbe induced form $\mathscr{D}(\theta)$ using the map ϕ' (see A.5).

B.5. **Azumaya property of differential operators on good stack.** Recall that a smooth algebraic stack $\mathscr{X}$ over S of relative dimension d is called relative good if it satisfied the following equivalent properties:

(1) $\dim(T^*(\mathscr{X}/S)) = 2d$.
(2) $\mathrm{codim}\{x \in \mathscr{X} \,|\, \dim Aut(x) = n\} \geqslant n$ for all $n > 0$.
(3) For any $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$, the complex

$$\mathrm{Sym}(T_{U/\mathscr{X}} \to T_{U/S})$$

has cohomology concentrated in degree 0 and

$$H^0(\mathrm{Sym}(T_{U/\mathscr{X}} \to T_{U/S})) \simeq \mathrm{Sym}(T_{U/S})/T_{U/\mathscr{X}}\,\mathrm{Sym}(T_{U/S}).$$

The following Proposition is proved in [BB] (see also [Trav]).

Proposition B.5.1. *Let $\mathscr{X}$ be a relative good stack. Let $\pi_{\mathscr{X}} : T^*(\mathscr{X}/S) \to \mathscr{X}$ be the natural projection and $\pi_{\mathscr{X}'}$ be its Frobenius twist. Let $T^*(\mathscr{X}'/S)^0$ be the maximal smooth open substack of $T^*(\mathscr{X}'/S)$. Then*

(1) *There is a natural coherent sheaf of algebra $D_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$ such that the restriction of $D_{\mathscr{X}/S}$ to $T^*(\mathscr{X}'/S)^0$ is an Azumaya algebra on $T^*(\mathscr{X}'/S)^0$ of rank $p^{2\dim(\mathscr{X}/S)}$.*
(2) *The $\mathbb{G}_m$-gerbe $\mathscr{D}^0_{\mathscr{X}/S} := \mathscr{D}_{\mathscr{X}/S}|_{T^*(\mathscr{X}'/S)^0}$ is isomorphic to $\mathscr{D}_{D^0_{\mathscr{X}/S}}$ the gerbe of splittings of $D^0_{\mathscr{X}/S}$. In particular, we have*

$$D^0_{\mathscr{X}/S}\text{-mod} \simeq \mathrm{QCoh}(\mathscr{D}^0_{\mathscr{X}/S})_1.$$

Remark B.5.2. By Proposition B.3.1, the category $D^0_{\mathscr{X}/S}$-mod can be thought as a localization of the category of $\mathcal{D}$-modules on $\mathscr{X}$.

Appendix C. Abelian Duality

C.1. **Abelian duality for Beilinson 1-motive.** Assume that S is a scheme and $p\mathcal{O}_S = 0$. Let $\mathscr{A}$ be a Picard stack over S. In this subsection, we denote the base change of $\mathscr{A}$ along $Fr_S : S \to S$ by $\mathscr{A}'$ instead of $\mathscr{A}^{(S)}$. Let $\mathbb{T}^*_e\mathscr{A}'$ be the vector bundle on S, which is restriction of the cotangent bundle (relative to S) of $\mathscr{A}'$ along $e : S \to \mathscr{A}'$. Then there is a canonical isomorphism

$$\mathscr{A}' \times_S \mathbb{T}_e^*\mathscr{A}' \simeq T^*(\mathscr{A}'/S).$$

Therefore, via $\pi_S : T^*(\mathscr{A}'/S) \simeq \mathscr{A}' \times_S \mathbb{T}_e^*\mathscr{A}' \to \mathbb{T}_e^*\mathscr{A}'$, $T^*(\mathscr{A}'/S)$ becomes a Picard stack over $\mathbb{T}_e^*\mathscr{A}'$ and we denote by m_S the multiplication map:

$$m_S : T^*(\mathscr{A}'/S) \times_{\mathbb{T}_e^*\mathscr{A}'} T^*(\mathscr{A}'/S) \to T^*(\mathscr{A}'/S).$$

Recall that it makes sense to talk about gerbes on a Picard stack with commutative group structures (cf. A.4.1).

Lemma C.1.1. *The gerbe $\mathscr{D}_{\mathscr{A}/S}$ on $T^*(\mathscr{A}'/S)$ admits a canonical commutative group structure.*

Proof. Let us sketch the construction of the multiplicative structure M and the 2-morphisms γ and i in A.4.1. The multiplication $m : \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$, which induces

$$\begin{array}{ccc} T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') & \xrightarrow{m_d} & T^*(\mathscr{A}' \times_S \mathscr{A}'/S)\ . \\ \downarrow{\scriptstyle m_p} & & \\ T^*(\mathscr{A}'/S) & & \end{array}$$

Observe that the map $m_d : T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') \to T^*(\mathscr{A}' \times_S \mathscr{A}'/S) \simeq T^*(\mathscr{A}'/S) \times_S T^*(\mathscr{A}'/S)$ induces an isomorphism

$$T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') \simeq T^*(\mathscr{A}'/S) \times_{\mathbb{T}_e^*\mathscr{A}'} T^*(\mathscr{A}'/S) \to T^*(\mathscr{A}'/S) \times_S T^*(\mathscr{A}'/S).$$

Under this isomorphism m_p becomes the multiplication map m_S. Now the canonical 1-morphism between $m_S^*\mathscr{D}_{\mathscr{A}/S}$ and $\mathscr{D}_{\mathscr{A}/S} \boxtimes \mathscr{D}_{\mathscr{A}/S}$ comes from Lemma B.3.3. We have two different factorizations of the multiplicative morphism $\mathscr{A} \times_S \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$ and the 2-morphisms γ comes from the 2-morphisms for corresponding equivalences of Lemma B.3.3. Finally, the 2-morphism $i : \sigma^*M \simeq M$ is constructed by applying Lemma B.3.3 to the morphism $\mathscr{A} \times_S \mathscr{A} \xrightarrow{\sigma} \mathscr{A} \times_S \mathscr{A} \xrightarrow{m} \mathscr{A}$. ∎

Now we assume that $\mathscr{A}$ is a Beilinson 1-motive and is good when regarded as an algebraic stack. Let $\mathscr{A}^\natural := \mathscr{P}ic^\natural(\mathscr{A})$ be the Picard stack of multiplicative invertible sheaves on $\mathscr{A}$ with a connection (cf. [Lau]), and let $\psi_S : \mathscr{A}^\natural \to \mathbb{T}_e^*\mathscr{A}'$ be the p-curvature morphism as given in Proposition B.3.4 (1). By [OV, §4.3], there is a natural action of $T^*(\mathscr{A}'/S)^\vee \simeq (\mathscr{A}')^\vee \times_S \mathbb{T}_e^*\mathscr{A}'$ on $\mathscr{A}^\natural$. Concretely, for any $b : U \to \mathbb{T}_e^*\mathscr{A}'$ objects in $\mathscr{A}^\natural \times_{\mathbb{T}_e^*\mathscr{A}'} U$ consist of multiplicative line bundles on $\mathscr{A} \times_S U$ with connections such that the p-curvatures are equal to b. Then for any $\mathcal{L}' \in (\mathscr{A}')^\vee \times_S U \simeq T^*(\mathscr{A}'/S)^\vee \times_S U$ and $(\mathcal{L}, \nabla) \in \mathscr{A}^\natural \times_{B'_S} U$ we define $\mathcal{L}' \cdot (\mathcal{L}, \nabla) := (F_{\mathscr{A}}^*\mathcal{L}' \otimes \mathcal{L}, \nabla_{F_{\mathscr{A}}^*\mathcal{L}'} \otimes \nabla)$ where $\nabla_{F_{\mathscr{A}}^*\mathcal{L}'}$ is the canonical connection on $F_{\mathscr{A}}^*\mathcal{L}'$ giving by Cartier descent. It also follows from Cartier descent that $\mathscr{A}^\natural$ is a $T(\mathscr{A}'/S)^\vee$-torsor under the action.

On the other hand, recall that for a $\mathbb{G}_m$-gerbe $\mathscr{D}$ with commutative group structure on a Beilinson 1-motive $\mathscr{P}$, we defines the $\mathscr{P}^\vee$-torsor $\mathscr{T}_{\mathscr{D}}$ of multiplicative splittings of $\mathscr{D}$ (cf. A.6).

Proposition C.1.2. *There is a canonical $(T^*(\mathscr{A}'/S))^\vee$-equivariant isomorphism $\mathscr{A}^\natural \to \mathscr{T}_{\mathscr{D}_{\mathscr{A}/S}}$.*

Proof. We sketch the proof. Write $\mathscr{T}_{\mathscr{D}_{\mathscr{A}/S}}$ by $\mathscr{T}_{\mathscr{D}}$ for simplicity. Recall that for $U \to \mathbb{T}^*_e\mathscr{A}'$, $\mathscr{T}_{D_A}(U)$ is the groupoid of splittings of $\mathscr{D}_{\mathscr{A}/S}$ over $U \times_{\mathbb{T}^*_e\mathscr{A}'} T^*(\mathscr{A}'/S)$ which are compatible with the commutative group structure of $\mathscr{D}_{\mathscr{A}/S}$. Note that

$$U \times_{\mathbb{T}^*_e\mathscr{A}'} T^*(\mathscr{A}'/S) \simeq U \times_{\mathbb{T}^*_e\mathscr{A}'} (\mathbb{T}^*_e\mathscr{A}' \times_S \mathscr{A}') \simeq \mathscr{A}' \times_S U,$$

and under this isomorphism, the projection of left hand side to the second factor is identified with

$$\mathscr{A}' \times_S U \to \mathscr{A}' \times_S \mathscr{A}^{\natural} \to T^*(\mathscr{A}'/S).$$

Now by Lemma B.3.4, the pull back of $\mathscr{D}_{\mathscr{A}/S}$ to $\mathscr{A}' \times_S U$ has a canonical splitting $\mathcal{L}_{U,\alpha}$. Moreover, one can easily check that this canonical splitting is compatible with the commutative group structure of $\mathscr{D}_{\mathscr{A}/S}$. Thus the assignment $(U,\alpha) \to \mathcal{L}_{U,\alpha}$ defines a map from $\mathscr{A}^{\natural}$ to $\mathscr{T}_{\mathscr{D}}$ which is compatible with their $T(\mathscr{A}'/S)^{\vee}$-torsor structure, hence an equivalence. □

As a corollary, we obtain the following theorem.

Theorem C.1.3. *Assume that $\mathscr{A}$ is a good Beilinson 1-motive. There is a canonical equivalence of categories*

$$D^b(\mathcal{D}\text{-mod}(\mathscr{A})) \simeq D^b(\mathrm{QCoh}(\mathscr{A}^{\natural})).$$

Proof. This is the combination of Theorem A.6.2 and Proposition B.5.1. □

Remark C.1.4. Note that in [Lau], this theorem is proved for abelian schemes over S of characteristic zero. In fact, Laumon's construction applies to any "good" Beilinson 1-motive over a locally noetherian base. When $p\mathcal{O}_S = 0$, it is easy to see that Laumon's equivalence and the equivalence constructed above are the same.

In particular, let $\theta : \mathscr{A}' \to \mathbb{T}^*\mathscr{A}'$ be a section obtained by base change $\tau : S \to \mathbb{T}^*_e\mathscr{A}'$. Let $\mathscr{D}_{\mathscr{A}/S}(\theta) := \theta^*\mathscr{D}_{\mathscr{A}/S}$. Then $\mathscr{D}_{\mathscr{A}/S,\theta}$ is a $\mathbb{G}_m$-gerbe on $\mathscr{A}'$ equipped with a canonical commutative group structure, and the ${\mathscr{A}'}^{\vee}$-torsor $\mathscr{T}_{\mathscr{D}_{\mathscr{A}/S,\theta}}$ of multiplicative splittings can be identified with $\mathscr{A}^{\natural} \times_{\mathbb{T}^*_e\mathscr{A}',\tau} S$.

C.2. **A variant.** In the main body of the paper, however, we need a variant of the above construction. Let k be an algebraically closed field of characteristic p. For a k-scheme X, we denote by X' its Frobenius base change along $Fr : k \to k$. Let S be a smooth k-scheme. For an S-scheme $X \to S$, we denote by $X^{(S)}$ its base change along $Fr_S : S \to S$. Let $\mathscr{A} \to S$ be a Picard stack with multiplication $m : \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$. The goal of this subsection is to construct certain multiplicative gerbe $\mathscr{D}_{\mathscr{A}}(\theta)$ on $\mathscr{A}'$ (rather than on $\mathscr{A}^{(S)}$ as done at the end of the previous subsection).

Let $\theta : \mathscr{A}' \to T^*\mathscr{A}'$ be a section, where $T^*\mathscr{A}'$ is the cotangent bundle of $\mathscr{A}'$ relative to k. We say θ is multiplicative if the upper right corner of the following diagram is commutative

$$\begin{array}{ccccc}
T^*\mathscr{A}' \times T^*\mathscr{A}' & \longleftarrow & T^*\mathscr{A}' \times T^*\mathscr{A}'|_{\mathscr{A}' \times_{S'} \mathscr{A}'} & \longrightarrow & T^*(\mathscr{A}' \times_{S'} \mathscr{A}') \\
\uparrow{\scriptstyle \theta\times\theta} & & \uparrow{\scriptstyle \theta\times\theta} & & \uparrow{\scriptstyle m_d} \\
\mathscr{A}' \times \mathscr{A}' & \longleftarrow & \mathscr{A}' \times_{S'} \mathscr{A}' & \xrightarrow{m^*\theta} & T^*\mathscr{A}' \times_{\mathscr{A}'} (\mathscr{A}' \times_{S'} \mathscr{A}') \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m_p} \\
 & & \mathscr{A}' & \xrightarrow{\theta} & T^*\mathscr{A}'.
\end{array}$$

Let $\mathscr{D}_{\mathscr{A}}(\theta) = \theta^* \mathscr{D}_{\mathscr{A}}$ be the pullback of $\mathscr{D}_{\mathscr{A}}$ to $\mathscr{A}'$. Then by the same argument as in Lemma C.1.1, we have

Lemma C.2.1. *(See also* [BB, Lemma 3.16]*) Let $\theta : \mathscr{A}' \to T^*\mathscr{A}'$ be a multiplicative section. Then $\mathscr{D}_{\mathscr{A}}(\theta)$ is a $\mathbb{G}_m$-gerbe on $\mathscr{A}'$ with a commutative group structure.*

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Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA
E-mail address: chenth@math.northwestern.edu

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA
E-mail address: xinwenz@math.northwestern.edu