

Quiver Tails and $\mathcal{N} = 1$ SCFTs from M5-branes

Prarit Agarwal,^a Ibrahima Bah,^{b,c} Kazunobu Maruyoshi^d and Jaewon Song^a

^a*Department of Physics, University of California, San Diego
La Jolla, CA 92093, USA*

^b*Department of Physics and Astronomy, University of Southern California
Los Angeles, CA 90089, USA*

^c*Institut de Physique Théorique, CEA/Saclay, 91191 Gif-sur-Yvette, France*

^d*California Institute of Technology, Pasadena, CA 91125, USA*

E-mail: pagarwal@physics.ucsd.edu, bah@usc.edu, maruyosh@caltech.edu,
jsong@physics.ucsd.edu

ABSTRACT: We study a class of four-dimensional $\mathcal{N} = 1$ superconformal field theories obtained by wrapping M5-branes on a Riemann surface with punctures. We identify UV descriptions of four-dimensional SCFTs corresponding to curves with a class of punctures. The quiver tails appearing in these UV descriptions differ significantly from their $\mathcal{N} = 2$ counterpart. We find a new type of object that we call the ‘Fan’. We show how to construct new $\mathcal{N} = 1$ superconformal theories using the Fan. Various dual descriptions for these SCFTs can be identified with different colored pair-of-pants decompositions. For example, we find an $\mathcal{N} = 1$ analog of Argyres-Seiberg duality for the $SU(N)$ SQCD with $2N$ flavors. We also compute anomaly coefficients and superconformal indices for these theories and show that they are invariant under dualities.

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | $\mathcal{N} = 1$ quiver theories of class \mathcal{S} and nilpotent Higgsing | 4 |
| 2.1 | Generic features of $\mathcal{N} = 1$ class \mathcal{S} | 4 |
| 2.2 | Linear quiver theory | 5 |
| 2.3 | Nilpotent Higgsing | 7 |
| 3 | Higgsing, Fan and quiver tails | 9 |
| 3.1 | Description of the Fan | 10 |
| 3.2 | Seiberg duality and Fans | 12 |
| 3.3 | Fan as a quiver tail | 14 |
| 3.4 | Nilpotent Higgsing and Fan | 16 |
| 4 | $\mathcal{N} = 1$ dualities | 18 |
| 4.1 | $\mathcal{N} = 1$ quiver tails | 20 |
| 4.2 | $\mathcal{N} = 1$ analog of Argyres-Seiberg duality | 23 |
| 5 | Anomalies and central charges | 26 |
| 5.1 | Anomalies of the Fan | 26 |
| 5.2 | Anomalies of class \mathcal{S} theories | 28 |
| 6 | Superconformal index | 33 |
| 6.1 | Review | 33 |
| 6.2 | $\mathcal{N} = 1$ Argyres-Seiberg duality | 35 |
| 6.3 | Index of the Fan | 37 |
| 7 | Conclusion | 39 |
| A | The superpotential for the Fan | 40 |
| B | Higgsing $\mathcal{N} = 2$ quiver theories | 43 |

1 Introduction

Six-dimensional $(2, 0)$ theory, as the low energy effective theory on the M5-brane worldvolume, plays a crucial role in studying lower dimensional supersymmetric gauge theories. In particular, a large class of four-dimensional $\mathcal{N} = 2$ superconformal theories, which are called

class \mathcal{S} theories, have been discovered in [1, 2] as a compactification of the (2, 0) theory on a Riemann surface with a partial twist. Class \mathcal{S} theories turn out to be related to various objects in different dimensional theories [3, 4], bridged by the (2,0) theory picture.

$\mathcal{N} = 2$ class \mathcal{S} theories are included in a larger class of theories with $\mathcal{N} = 1$ supersymmetry associated to compactifications of the (2, 0) theory [5]. This latter class, which we will call $\mathcal{N} = 1$ class \mathcal{S} , has been investigated in [5–12] in field theory and in [5, 7, 13, 14] in AdS/CFT. (See [15, 16] for the mass deformed $\mathcal{N} = 2$ class \mathcal{S} theories.) This class of theories flow to the superconformal fixed point in the IR. See also [17–20] for theories in Coulomb and confining phases.

The $\mathcal{N} = 1$ theories of class \mathcal{S} are specified through the following data

- The choice of ‘gauge group’ $\Gamma = A, D, E$.
- A Riemann surface $\mathcal{C}_{g,n}$ of genus g with n punctures called UV curve.
- Two integers p, q with a constraint $p + q = 2g - 2 + n$.

From the M5-brane point of view, the class of theories are obtained by wrapping the M5-branes on $\mathcal{C}_{g,n}$ inside the total space of two line bundles over $\mathcal{C}_{g,n}$. Then, p and q are the degrees of two line bundles.

In addition, we assign data on each punctures. A class of punctures, called the regular colored $\mathcal{N} = 2$ punctures, are specified by the following data:

- For each punctures, the choice of ρ_i which is an embedding of $SU(2)$ into Γ .
- The choice of \mathbb{Z}_2 -valued ‘color’ $\sigma_i = \pm$.

When $\Gamma = A_{N-1}$ which we will focus on, the choice of ρ_i is in one-to-one correspondence with the choice of partition of N or a Young tableaux of N boxes¹ with $N = \sum_k n_k k$. The naming of the puncture stems from the fact that locally these punctures are the same as those of $\mathcal{N} = 2$ theories except that we have a choice of choosing one of two normal directions to the M5-branes.²

A four-dimensional UV theory can be associated to every pair-of-pants decomposition of $\mathcal{C}_{g,n}$. UV theories are in the same class, meaning the theories corresponding to the different pants decompositions of the same $\mathcal{C}_{g,n}$, if they flow to the same non-trivial fixed point in the IR. This provides a nice geometric picture of the duality of $\mathcal{N} = 1$ class \mathcal{S} theories [5, 8, 9, 12] generalizing the well-known Seiberg duality [25].

Among these theories, linear quiver gauge theories form an important subset describing characteristic features of class \mathcal{S} . A linear quiver theory has two tails each of which is

¹Punctures can also be twisted by a outer-automorphism group of Γ . This will affect the choice of ρ_i . We will not consider the twist in this paper.

²While we will not study in this paper, the full classification of $\mathcal{N} = 1$ puncture should be given in terms of $\frac{1}{4}$ -BPS boundary conditions of $\mathcal{N} = 4$ super Yang-Mills theory. (This has been studied recently by [21, 22], which generalizes the work [23, 24].)

composed of a product of gauge groups whose ranks are non-decreasing. In $\mathcal{N} = 2$ theories, the quiver tail has been fully understood to be related to a sphere with a maximal puncture ($N = 1 + 1 + \dots + 1$), a number of minimal punctures ($N = 1 + (N - 1)$), and a generic puncture [1]. The purpose of this paper is to identify the $\mathcal{N} = 1$ version of quiver tails associated with a similar sphere but with colors.

It turns out that the $\mathcal{N} = 1$ quiver tails have an important ingredient, which we will call the Fan. The Fan is composed of a collection of various chiral multiplets coupled by the superpotential with global symmetry $SU(N) \times SU(N') \times \prod_k U(n_k) \times U(1)$. The quiver tail is constructed by gauging some of the global symmetries. When N' is absent, the Fan is shown to be associated to a pair-of-pants whose three punctures are of maximal, minimal, and generic specified by a partition of $N = \sum_k kn_k$. (The color of the former two punctures are the same as that of the pair-of-pants, and are different from that of the generic puncture.)

We obtain the $\mathcal{N} = 1$ quiver tail, and in particular the Fan, by the nilpotent Higgsing which first studied in [26] from the different point of view and in [9, 27] from the class \mathcal{S} point of view. We start from the linear quiver theory where all gauge groups are $SU(N)$, and give the nilpotent vev to the quark bilinear at the end of quiver. This produces the quiver tail. In the $\mathcal{N} = 2$ linear quiver theories, the nilpotent Higgsing propagates to neighboring gauge nodes of the quiver because of the F-term equations [27], which we also discuss in detail in appendix B. On the other hand, if there is $\mathcal{N} = 1$ gauge group in the quiver, the Higgsing stops at that node due to the F-term from the quartic superpotential and does not propagate further. This indicates the characteristic difference of the Higgsing in between $\mathcal{N} = 1$ and in $\mathcal{N} = 2$ theories. We will confirm this from different ways using multiple Seiberg dualities.

The Fan can also be used to construct not only the quiver tail, but more general $\mathcal{N} = 1$ gauge theories in class \mathcal{S} as a new building block. Moreover, the Fan plays a crucial role to study the dualities in class \mathcal{S} theories. As a remarkable example, we find that the Fan coupled to $\mathcal{N} = 1$ vector multiplet appears as the dual description of the $\mathcal{N} = 1$ supersymmetric QCD with $N_f = 2N$ flavors. The precise description is $\mathcal{N} = 1$ $SU(N)$ gauge theory coupled to the Fan and to the T_N theory [1] and a chiral adjoint multiplet, with the superpotential. From the UV curve viewpoint, this duality can be seen as an exchange of the maximal and minimal punctures, and therefore is an $\mathcal{N} = 1$ analog of the Argyres-Seiberg duality [28], which was first discussed in [12] for the case of $SO/Sp/G_2$ gauge theories.

The organization of this paper is as follows. In section 2, we first review the $\mathcal{N} = 1$ linear quiver gauge theories of class \mathcal{S} [11], and the nilpotent Higgsing. In section 3, the Fan is introduced. We will see that the $\mathcal{N} = 1$ quiver tail in which the Fan plays a central role can be obtained by the nilpotent Higgsing of the $\mathcal{N} = 1$ linear quiver gauge theory. In section 4, we consider the application of the Fan to the duality. We first confirm that the Fan appears from the different quiver theory with $\mathcal{N} = 2$ quiver tail by using the multiple Seiberg dualities. We then consider the duality of $\mathcal{N} = 1$ SQCD with $N_f = 2N$ flavors. In section 5, we study the 't Hooft anomaly coefficients of the $\mathcal{N} = 1$ class \mathcal{S} theories, in particular the Fan. We then present formulae of the anomalies in terms of the UV curve. In section 6, we calculate the superconformal index of the class \mathcal{S} theories involving the Fan. This could

be the strongest check of the duality conjecture in section 4. In appendix A, we derive the superpotential of the Fan from nilpotent Higgsing. We also discuss the nilpotent Higgsing in the $\mathcal{N} = 2$ linear quiver theories in appendix B.

2 $\mathcal{N} = 1$ quiver theories of class \mathcal{S} and nilpotent Higgsing

Our main object is the class of quivers, in particular quiver tails, obtained by giving nilpotent vevs to $\mathcal{N} = 1$ linear quiver gauge theories of class \mathcal{S} (See [11] for more details). We first discuss our criteria for constructing $\mathcal{N} = 1$ class \mathcal{S} quiver gauge theories in section 2.1 and then discuss $\mathcal{N} = 1$ linear quiver gauge theories of class \mathcal{S} in section 2.2. We then study the generic feature of the nilpotent Higgsing of the quiver theory in section 2.3, focusing on the difference between the Higgsings in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quiver theories.

2.1 Generic features of $\mathcal{N} = 1$ class \mathcal{S}

There is no complete classification of $\mathcal{N} = 1$ class \mathcal{S} field theories from compactifications of the six-dimensional $(2,0)$ theory. But there are two prevalent features of the existing constructions of class \mathcal{S} theories. In our explorations, we impose these conditions as criteria for class \mathcal{S} . They are are:

Criterion I: R-symmetry $\mathcal{N} = 1$ class \mathcal{S} theories admit R -symmetry $U(1)_+ \times U(1)_-$, which we denote the generators as (J_+, J_-) . This corresponds to the maximal subgroup of the $SO(5)$ R -symmetry of the $(2,0)$ theory that can be preserved after a partial topological twist on a Riemann surface. From the point view of M5-branes on a UV curve, this symmetry corresponds to the rotations of the two line bundles fibered over the UV curve. One combination of this symmetry will become the superconformal R -symmetry and the other will be a global symmetry of the of the four-dimensional $\mathcal{N} = 1$ SCFT. We adopt the notation (j_+, j_-) for the charges of chiral fields and chiral operators under this symmetry.

Another notation for the global symmetry $U(1) \times U(1)$ is (R_0, \mathcal{F}) defined as

$$R_0 = \frac{1}{2}(J_+ + J_-), \quad \mathcal{F} = \frac{1}{2}(J_+ - J_-). \quad (2.1)$$

This latter notation is more convenient when computing central charges and anomalous dimensions. The superconformal R -symmetry is

$$R_{\mathcal{N}=1} = R_0 + \epsilon \mathcal{F}, \quad (2.2)$$

where ϵ is fixed by a-maximization [29].

In order to satisfy the R -symmetry criterion, we impose the condition: All additional $U(1)$ symmetries, F_I , are baryonic; i.e., they cannot mix with the R -symmetry. In the class \mathcal{S} theories, there are flavor symmetries associated to the punctures on the UV curve. These are all baryonic and do not mix with the R -symmetry.

Criterion II: Marginal Coupling For every gauge coupling, there is an associated exactly marginal direction. In the construction of class \mathcal{S} , the number of gauge groups is given by the dimension of the complex structure moduli space of the UV curve. The addition of gauge groups maps to the addition of punctures or handles on the UV curve which then increases the dimension of the conformal manifold. [1, 5, 16].

This condition is not entirely correct if the UV curve has an irregular puncture. For example, one can realize $SU(N)$ gauge theory with $N_f < 2N$ flavors by a three-punctured sphere with irregular punctures. This theory flows to a conformal fixed point with no marginal direction. There is no complex structure deformation associated to this UV curve, nevertheless it has a gauge group. In this paper, we will aim at finding the theory with regular punctures only, when the number of gauge groups is the same as the dimension of complex structure moduli space of the UV curve.

These criteria are surprisingly constraining and generic quiver gauge theories do not satisfy them. They are satisfied in $\mathcal{N} = 1$ class \mathcal{S} linear quivers and all theories constructed so far. As we will find, they are always preserved by nilpotent Higgsing.

2.2 Linear quiver theory

Let us consider a linear quiver theory given as follows. This has ℓ gauge groups labelled as $SU(N)_i$, which can be $\mathcal{N} = 2$ or $\mathcal{N} = 1$. The former is an $\mathcal{N} = 1$ vector multiplet with a chiral adjoint. The gauge nodes, $SU(N)_{i+1}$ and $SU(N)_i$, are linked by hypermultiplets, $H_i = (Q_i, \tilde{Q}_i)$, transforming in the bifundamental representation of $SU(N)_{i+1}$ and $SU(N)_i$. Our conventions are such that (Q_i, \tilde{Q}_i) transforms in $(\mathbf{N} \otimes \tilde{\mathbf{N}}, \tilde{\mathbf{N}} \otimes \mathbf{N})$ of $SU(N)_{i+1} \times SU(N)_i$. The right-most hyper is denoted as H_0 and it transforms in the bifundamental representation of $SU(N)_1 \times SU(N)_0$, where $SU(N)_0$ is a flavor symmetry. The left-most hyper is H_ℓ . It transforms in the bifundamental representation of $SU(N)_{\ell+1} \times SU(N)_\ell$, where $SU(N)_{\ell+1}$ is a flavor symmetry. See figure 1a for the $\ell = 5$ case.

As mentioned above, the theory preserves distinguished anomaly-free $U(1)$ symmetries, $U(1)_+ \times U(1)_-$, whose generators we denote as (J_+, J_-) . The charge of any gaugino under this symmetry is $(1, 1)$. We fix the charge of the matter fields and a theory by giving the sequence $(\sigma_{-1}, \sigma_0, \sigma_1, \dots, \sigma_\ell, \sigma_{\ell+1})$ with $\sigma_i^2 = 1$. Each hypermultiplet H_i also comes with a baryonic $U(1)_i$, whose generators we denote as J_i . The charges of the H_i is given as

$$J_\pm(Q_i) = \frac{1 \pm \sigma}{2}, \quad J_j(Q_i) = \delta_{ij}. \quad (2.3)$$

Note that the J_j charge of the anti-fundamental \tilde{Q}_i has an opposite sign.

Each gauge group can come with an $\mathcal{N} = 2$ or with an $\mathcal{N} = 1$ gauge group. When $\sigma_i = \sigma_{i-1} = \pm 1$, the $SU(N)_i$ gauge group has a chiral field ϕ_i^\mp transforming in the adjoint representation of the gauge group and the superpotential

$$W_i = \sigma_i \text{Tr} \left[\phi_i^\mp (Q_{i-1} \tilde{Q}_{i-1} - \tilde{Q}_i Q_i) \right]. \quad (2.4)$$

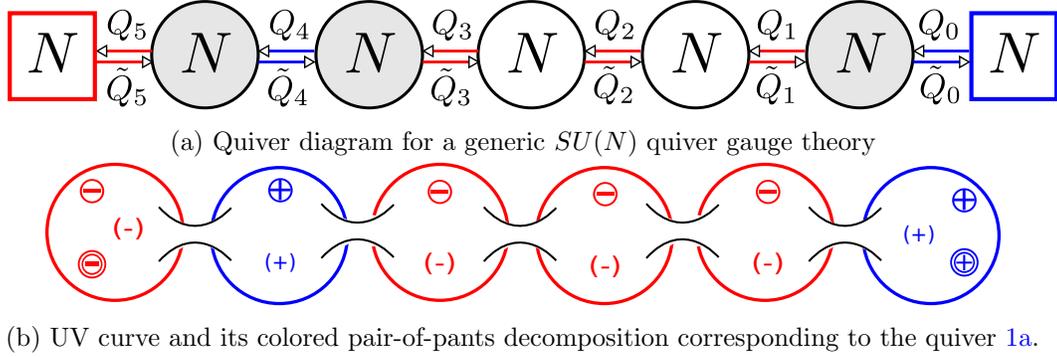


Figure 1: An example of generic $SU(N)$ quiver theory corresponding to the UV curve given by a sphere with two maximal and a number of minimal punctures. The black and white node corresponds to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauge nodes respectively. Note that the colored pair-of-pants mapped to the bifundamentals, and the tubes mapped to the gauge nodes.

For $\sigma_i = -\sigma_{i-1}$, there is no adjoint chiral. Instead, there is a quartic superpotential

$$W_i = \text{Tr} \left(Q_{i-1} \tilde{Q}_{i-1} \tilde{Q}_i Q_i \right) - \frac{1}{N} \text{Tr} (Q_{i-1} \tilde{Q}_{i-1}) \text{Tr} (\tilde{Q}_i Q_i). \quad (2.5)$$

Let us note that these can be uniformly written as

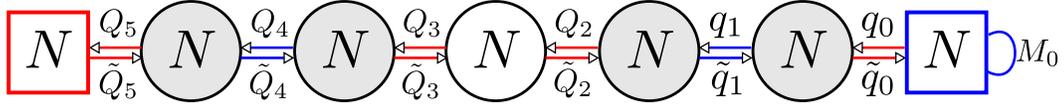
$$W_i = \text{Tr} \left[\tilde{Q}_i Q_i \left(\frac{1 - \sigma_i}{2} \phi_i^+ - \frac{1 + \sigma_i}{2} \phi_i^- \right) + Q_{i-1} \tilde{Q}_{i-1} \left(\frac{1 + \sigma_{i-1}}{2} \phi_i^- - \frac{1 - \sigma_{i-1}}{2} \phi_i^+ \right) \right. \\ \left. + m_i \left(\frac{1 - \sigma_i}{2} \phi_i^- - \frac{1 + \sigma_i}{2} \phi_i^+ \right) \left(\frac{1 - \sigma_{i-1}}{2} \phi_i^- - \frac{1 + \sigma_{i-1}}{2} \phi_i^+ \right) \right], \quad (2.6)$$

where the trace is over the gauge group $SU(N)_i$. Below the energy m_i , some of adjoint fields are integrated out, giving (2.4) or (2.5) depending on σ_i and σ_{i-1} . Then, the total superpotential is given as $W = \sum_{i=1}^{\ell} W_i$.

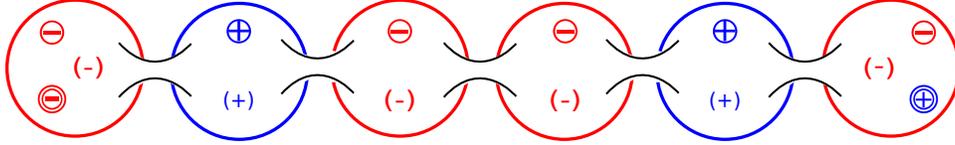
Since the fields $H_{\ell+1}$ and H_{-1} do not exist, and since $SU(N)_{\ell+1}$ and $SU(N)_0$ are flavor groups, the choices σ_{-1} and $\sigma_{\ell+1}$ attaches or turns off adjoint chiral multiplets to the end of hypermultiplets. Namely, if $\sigma_{-1} = \sigma_0 = \pm$, we attach the adjoint ϕ_0^\mp with $W_0 = \text{Tr} \tilde{Q}_0 Q_0 \phi_0^\mp$; if $\sigma_{-1} = -\sigma_0$, we do not have any adjoints. The $U(1)_\pm$ charges of the fields are

$$J_\pm(\phi_i^\pm) = \frac{2 + \sigma_i + \sigma_{i-1}}{2}, \quad J_\pm(\phi_i^\mp) = \frac{2 - \sigma_i - \sigma_{i-1}}{2}. \quad (2.7)$$

Let us now shortly describe the connection with the UV curve picture. This quiver gauge theory is actually in the class \mathcal{S} and associated to the sphere with $\ell + 1$ minimal punctures and two maximal punctures [11]. See figure 1b for illustration. The sphere is decomposed into $\ell + 1$ pairs of pants, each of which has a color. Note that each of the color of the pair-of-pants is the same as the color of the minimal puncture. Therefore locally the pair-of-pants has an $\mathcal{N} = 2$ supersymmetry and corresponds to bifundamental hypermultiplet H_i . The σ_i ($i = 0, 1, \dots, \ell$) is exactly the color of the i -th pair-of-pants. Then the $\mathcal{N} = 1$ vector multiplet



(a) Quiver diagram corresponding to the Seiberg dual of figure 1a.



(b) UV curve and its colored pair-of-pants decomposition corresponding to the quiver 2a.

Figure 2: Seiberg dual of quiver given by figure 1a and its colored pair-of-pants decomposition. Here we dualized the right-most gauge group $SU(N)_1$. Note that the second gauge group $SU(N)_2$ became $\mathcal{N} = 1$ because of the meson dual to $Q_1\tilde{Q}_1$ behaves as an extra adjoint chiral, which generates a mass term for the adjoint chiral. From the UV curve viewpoint, this is represented by that the colors of the second and third pairs of pants are different.

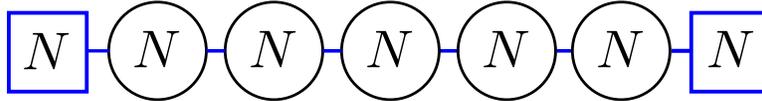


Figure 3: $\mathcal{N} = 2$ linear quiver theory.

appears when two pair-of-pants with different color are connected by a tube; the $\mathcal{N} = 2$ vector multiplet appears when two with the same colors are connected. The σ_{-1} and $\sigma_{\ell+1}$ are associated with the color of the maximal punctures. If the color of the maximal puncture is different from that of the pair-of-pants, an adjoint chiral multiplet is attached. See figures 2a and 2b.

It is important to consider the Seiberg duality in this class of theories. Given a quiver where $SU(N)_i$ gauge group is $\mathcal{N} = 1$ with $\sigma_i = -\sigma_{i-1}$, we can dualize at this node. This will map a linear quiver to another linear quiver since each gauge node satisfies $N_f = 2N_c$. Dualizing at $SU(N)_i$ will have the effect $\sigma_i \rightarrow -\sigma_i$ and $\sigma_{i-1} \rightarrow -\sigma_{i-1}$. From the perspective of UV curve, this is equivalent to exchanging neighboring two minimal punctures of different color and at the same time inverting the color of pair-of-pants, as in figures 2a and 2b. The Seiberg duality preserves the parameters p and q which correspond to the number of pair-of-pants or $\sigma_{i=0,1,\dots,\ell}$'s with + and -, respectively.

2.3 Nilpotent Higgsing

$\mathcal{N} = 2$ Higgsing Before discussing nilpotent Higgsing in $\mathcal{N} = 1$ theories, we summarize the effect in the case of $\mathcal{N} = 2$ theories. We elaborate more in the appendix B. This is also discussed in [27].

Consider a linear quiver theory as in figure 3 with gauge group $G = \prod_{i=1}^{\ell} SU(N)_i$. This is the special case of the quiver introduced in the section 2.2 by setting all the colors of punctures

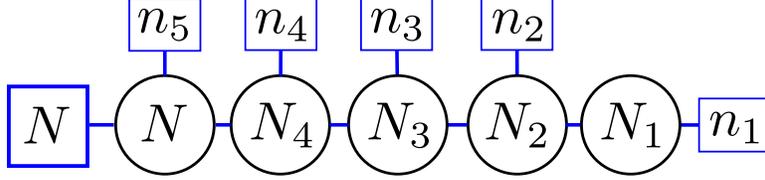


Figure 4: An $\mathcal{N} = 2$ linear quiver theory after Higgsing $N = \sum_{k=1}^5 n_k k$. The ranks of the gauge groups are fixed by conformality condition $2N_i = N_{i-1} + N_{i+1} + n_i$.

and pairs of pants to be the same. From the superpotential (2.4), we get the F-term equation for the ϕ_i

$$F_{\phi_i} = Q_{i-1}\tilde{Q}_{i-1} - \tilde{Q}_i Q_i = 0. \quad (2.8)$$

Now, let us consider a Higgsing of H_0 by giving a nilpotent vev to $\mu_0 = \tilde{Q}_0 Q_0 - \frac{1}{N} \text{Tr} \tilde{Q}_0 Q_0$ to partially close the maximal puncture. For a given partition of $N = \sum_k n_k k$, we give vev $\langle \mu_0 \rangle = \bigoplus_k n_k J_k$, where J_k is the Jordan cell of size k

$$J_k = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}. \quad (2.9)$$

The matrix J_k is the k -dimensional representation of the raising operator $\sigma^+ = \sigma^1 + i\sigma^2$ of $SU(2)$. A crucial observation here is that from the F-term for the adjoint chirals (2.8), the vev of Q 's are propagated to the neighboring node. As it propagates, the operator $\tilde{Q}_i Q_i$ will have smaller rank than that of $\tilde{Q}_{i-1} Q_{i-1}$ until it hits zero at some finite length. See appendix B for a more detailed discussion. From this way, we can explicitly derive the quiver tails corresponding to a given partition of N labelling the puncture, as in figure 4.

Before going to $\mathcal{N} = 1$ theories, let us make a comment on the Higgsing through a diagonal vev such as $Q_0 = \tilde{Q}_0 = \text{diag}(v_1, v_2, 0, \dots, 0)$. It is certainly possible to solve the F-term equation (2.8) by such a diagonal vev for all the bifundamental hypermultiplets $Q_0 = Q_1 = \dots = Q_\ell$. Therefore all the gauge symmetries are broken by the same amount. We will not discuss these cases.

$\mathcal{N} = 1$ Higgsing Suppose every gauge node we described above is replaced by $\mathcal{N} = 1$ gauge nodes. Let us Higgs the theory by giving the vev to μ_0 as before. This time, from the superpotential (2.5) the F-term equation for the Q_i, \tilde{Q}_i

$$F_{Q_i} = Q_{i-1}\tilde{Q}_{i-1}\tilde{Q}_i + \tilde{Q}_i\tilde{Q}_{i+1}Q_{i+1} = 0, \quad (2.10)$$

does not give us a propagating effect to the neighboring node. The F-term can be simply solved by taking all the other Q_i, \tilde{Q}_i to be zero. Therefore, the Higgsing happens completely

locally on the first node. No propagation of vev contrary to the case of $\mathcal{N} = 2$. Generally if we have a number of $\mathcal{N} = 2$ nodes on the right, the propagation continues until it hits the $\mathcal{N} = 1$ node and then stop. In the next section, we will describe how Higgsing creates a $\mathcal{N} = 1$ version of the quiver tail.

In the case of a diagonal vev, the D-term equations for the quiver theories can be solved. The effect of diagonal Higgsing has been thoroughly studied and has been used to test the consistency of the Seiberg duality in $\mathcal{N} = 1$ $SU(N_c)$ SQCD with N_f flavors [25]: the gauge symmetry and the flavor symmetry go down by a same amount, say k . Then the gauge symmetry will be $SU(N_c - k)$ and the flavor symmetry will be $SU(N_f - k)$. On the dual side, the gauge group remains the same, but only the dual quarks become massive and reduces the number of flavors by the same amount k . From the magnetic theory perspective, mass terms for the dual quarks are generated through the superpotential $W = (\langle M \rangle + \delta M) q\tilde{q}$, where $\langle M \rangle$ is of rank k . Once we integrate out the massive (dual) quarks, we generate $M^2 q\tilde{q}$ term in the superpotential which is irrelevant in the IR. The Higgsed theory will have $SU(N_f - k)$ flavor symmetry which is the same as the electric theory.

On the other hand, as we have seen in the $\mathcal{N} = 2$ case, the nilpotent vevs can deform the theory in an interestingly different way. The number of flavors will be reduced, but the superpotential terms generated are quite different from the diagonal Higgsing. Depending on the choice of nilpotent vevs, we can generate various types of flavor symmetry of the form

$$G_F = S \left(\prod_{i=1}^{\ell} U(n_k) \right) . \quad (2.11)$$

We will see how the nilpotent Higgsing works for $\mathcal{N} = 1$ theories in detail in the next section. There will be various seemingly irrelevant terms in the superpotential generated through this procedure. But, we will argue that all of these terms become exactly marginal in the IR SCFT. This kind of operators in the superpotential which looks irrelevant in the UV but not in the IR are called dangerously irrelevant operators. See [30] for example.

3 Higgsing, Fan and quiver tails

In this section, we give an $\mathcal{N} = 1$ version of the quiver tails. First, we define the Fan in section 3.1. Then in section 3.2 we describe its Seiberg duality. Then in section 3.3, we will summarize the $\mathcal{N} = 1$ quiver tail obtained by the nilpotent Higgsing of the linear quiver, where the Fan appears as an important ingredient. Finally in section 3.4 we show that the Fan is indeed obtained by Higgsing the linear quiver with the adjoint fields attached to the end.

3.1 Description of the Fan

The Fan is a collection of free chiral multiplets with certain global symmetries and superpotential. It is labelled by two integers N, N' with $N > N'$ and an ℓ -partition

$$N - N' = \sum_{k=1}^{\ell} kn_k . \quad (3.1)$$

We will refer to ℓ as its size. The matter content is displayed in table 1. We also have a choice of a color, σ ; that we pick to be $\sigma = -1$ for simplicity. The other choice, $\sigma = 1$, corresponds to swapping J_+ and J_- in table 1. It has the global symmetry

$$SU(N) \times SU(N') \times U(1) \times \prod_{i=1}^{\ell} U(n_i) \times U(1)_+ \times U(1)_- . \quad (3.2)$$

Figure 5 is a representation of the Fan with size $\ell = 5$. Each line corresponds to a bifunda-

| | $SU(N)$ | $SU(N')$ | $U(n_i)$ | $U(n_j)$ | $U(1)_B$ | J_+ | J_- |
|--------------------------------|----------------------------|----------------------------|----------------------------|----------------------------|-----------|--------------|-------|
| (Q, \tilde{Q}) | $(\square, \bar{\square})$ | $(\bar{\square}, \square)$ | \cdot | \cdot | $(1, -1)$ | 0 | 1 |
| (Z_i, \tilde{Z}_i) | $(\square, \bar{\square})$ | \cdot | $(\bar{\square}, \square)$ | \cdot | $(1, -1)$ | $1 - i$ | 1 |
| (Y_i, \tilde{Y}_i) | \cdot | $(\square, \bar{\square})$ | $(\bar{\square}, \square)$ | \cdot | \cdot | $i + 1$ | 0 |
| $M_{ii}^{(p)}$ | \cdot | \cdot | adj | \cdot | \cdot | $2(i - p)$ | 0 |
| $(M_{ij}^{(p)}, M_{ji}^{(p)})$ | \cdot | \cdot | $(\square, \bar{\square})$ | $(\bar{\square}, \square)$ | \cdot | $i + j - 2p$ | 0 |

Table 1: The Fan contains many fields organized in representation of the flavor symmetry. The indices i, j range in the interval $[1, \ell]$ and are ordered as $i < j$. The index p labels a tower of fields in the same representation of the flavor symmetry, its range is $0 \leq p \leq i - 1$.

mental hypermultiplet and each loop correspond to an adjoint chiral multiplet.

The Fan appears in quiver gauge theories with the $SU(N) \times SU(N')$ symmetries gauged. When the fan is glued, chiral anomalies at the $SU(N) \times SU(N')$ gauge groups of J_{\pm} must be cancelled. This will restrict the matter content that can appear on either side. The contributions of the Fan to the anomaly coefficient are:

$$SU(N) : \quad \text{Tr} J_+ T^a T^b = 0 , \quad \text{Tr} J_- T^a T^b = N \delta^{ab} , \quad (3.3)$$

$$SU(N') : \quad \text{Tr} J_+ T'^a T'^b = 0 , \quad \text{Tr} J_- T'^a T'^b = (N' - \sum_{i=1}^{\ell} n_i) \delta^{ab} . \quad (3.4)$$

The anomaly at $SU(N)$, when it is gauged with an $\mathcal{N} = 1$ vector multiplet, can be cancelled by coupling the Fan to N $(1, 0)$ fundamental hypermultiplets. When it is gauged with an $\mathcal{N} = 2$ vector, the anomaly is cancelled by coupling N $(0, 1)$ fundamental hypermultiplets to the Fan. This provides $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gluing of the Fan at the $SU(N)$ gauge group.

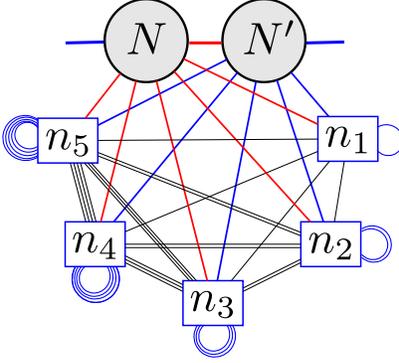


Figure 5: Generic form of the Fan given by (N, N') and the partition $N - N' = \sum_{k=1}^5 kn_k$. Each line corresponds to a bifundamental hypermultiplet and each loop correspond to an adjoint chiral multiplet.

When the $SU(N')$ is gauged with $\mathcal{N} = 1$ vector multiplet, the anomaly at the $SU(N')$ can be cancelled by adding $(N' - \sum_{i=1}^{\ell} n_i)$ $(1, 0)$ fundamental hypermultiplets. Unlike the $SU(N)$ side, when $SU(N')$ is gauged with $\mathcal{N} = 2$ vector multiplet, the anomaly cannot be cancelled with either $(1, 0)$ or $(0, 1)$ hypermultiplets only. We can glue the Fan to an $\mathcal{N} = 2$ quiver tail labelled by a partition of N' . In figure 6a we illustrate the Fan glued to general quivers with $\mathcal{N} = 1$ gluing at the $SU(N)$ gauge group.

Superpotential When the Fan appears in a larger quiver, we can write a superpotential by considering all possible gauge invariant $(2, 2)$ operators that preserve the flavor symmetry. We decompose it into three contributions

$$W_F = W_0 + W_R + W_L \quad (3.5)$$

where W_0 is composed of fields in the Fan only, W_R comes from gluing at $SU(N')$ and W_L comes from gluing at $SU(N)$. Now we describe them.

If we consider the matter content of the Fan, the only superpotential terms we can write are

$$W_0 = \sum_{i=1}^{\ell} \left[\lambda_i^0 \text{Tr} \left(Z_i \tilde{Q} \tilde{Y}_i \right) + \tilde{\lambda}_i^0 \text{Tr} \left(\tilde{Z}_i Q Y_i \right) \right] \quad (3.6)$$

where the λ 's are complex coupling constants.

The next class of operators come from coupling of the quiver tail to the Fan through the $SU(N')$. To write these terms we consider the set of $(2, 0)$ operators, μ' and μ_t , constructed from the $U(n')$ and $SU(M)$ quarks in figure 6a. The superpotential is

$$W_R = \lambda' \text{Tr} \left(Q \tilde{Q} \mu' \right) + \lambda_t \text{Tr} \left(Q \tilde{Q} \mu_t \right). \quad (3.7)$$

The last class operators come from gluing the Fan at the $SU(N)$. To write these terms, we consider the tower operators, $\mu_{\alpha}^{(p)}$, $(\mathcal{M}_{ij}^{(p),\alpha}, \mathcal{M}_{ji}^{(p),\alpha})$, and $\mathcal{M}_{ii}^{(p),\alpha}$. The μ 's are constructed

| | $SU(N)$ | $U(n_i)$ | $U(n_j)$ | J_+ | J_- |
|--|---------|----------------------------|----------------------------|----------|-------|
| $\mu_\alpha^{(p)}$ | adj | \cdot | \cdot | $2p$ | 0 |
| $\mathcal{M}_{ii}^{(p),\alpha}$ | \cdot | adj | \cdot | $2(i-p)$ | 0 |
| $(\mathcal{M}_{ij}^{(p),\alpha}, \mathcal{M}_{ji}^{(p),\alpha})$ | \cdot | $(\square, \bar{\square})$ | $(\bar{\square}, \square)$ | $i+j-2p$ | 0 |

Table 2: Charges of the \mathcal{M} and μ operators used in (3.8).

from fields to the left of the Fan. The \mathcal{M} 's are constructed from the M_{ij} fields of the Fan. Their charges are written in the table 2. When we glue at the $SU(N)$, we obtain the superpotential

$$\begin{aligned}
W_L = & \lambda^\alpha \text{Tr} \left(\mu_\alpha^{(1)} \tilde{Q} Q \right) + \sum_{i=1}^{\ell} \sum_{p=0}^{i-1} \lambda_{i,p}^{\alpha,\beta} \text{Tr} \left(\mu_\alpha^{(p)} \tilde{Z}_i Z_i \mathcal{M}_{ii}^{(p),\beta} \right) \\
& + \sum_{i=1}^{\ell} \sum_{p=0}^{i-1} \lambda_{ij,p}^{\alpha,\beta} \text{Tr} \left(\mu_\alpha^{(p)} \tilde{Z}_i Z_j \mathcal{M}_{ji}^{(p),\beta} \right) + \sum_{i=1}^{\ell} \sum_{p=0}^{i-1} \lambda_{ji,p}^{\alpha,\beta} \text{Tr} \left(\mu_\alpha^{(p)} \tilde{Z}_j Z_i \mathcal{M}_{ij}^{(p),\beta} \right). \quad (3.8)
\end{aligned}$$

To illustrate the \mathcal{M} operators, we consider the set $\mathcal{M}_{ij}^{(p),\alpha}$. The simplest examples in this class are $M_{ik}^{(p_1)} M_{kj}^{(p_2)}$ with $p_1 + p_2 - k = p$ where we trace over the $U(n_k)$ group.

In the case of $\mathcal{N} = 2$ gluing at $SU(N)$, the $\mu_\alpha^{(p)}$ operators are entirely given by the chiral adjoint ϕ in the $\mathcal{N} = 2$ vector multiplet, as $\mu^{(p)} = \phi^p$. The index α is trivial in this case. On the other hand, if we consider $\mathcal{N} = 1$ gluing, then the μ operators are more complicated. To illustrate this, we consider gluing the $\mathcal{N} = 2$ linear quiver in figure 3 with the box N identified with the $SU(N)$ in the Fan gauged with $\mathcal{N} = 1$ vector. In this case, the set $\mu_\alpha^{(p)}$ correspond to the chain operators that can be constructed from the products of the quarks. To give an explicit example, we label the bifundamentals as (Q_a, \tilde{Q}_a) with $a = 1$ corresponding to the one attached to the Fan. The operators, $\mu_\alpha^{(2)}$ are $(\tilde{Q}_1 Q_1)_{\text{adj}}^2$ and $(\tilde{Q}_1 \tilde{Q}_2 Q_2 Q_1)_{\text{adj}}$.

3.2 Seiberg duality and Fans

Under Seiberg duality, a quiver with the Fan maps to another quiver with the Fan. To illustrate this, we consider the quiver in 6a and Seiberg dualize at $SU(N')$ to obtain 6b. Under the duality, the $U(n')$ flavor group is absorbed into the new Fan and thereby increasing its size to $\ell + 1$. We denote the $U(n')$ and $SU(M)$ hypermultiplets as (Q', \tilde{Q}') and (Q_t, \tilde{Q}_t) . We also denote the fields of the new Fan as $(q, \tilde{q}), (z, \tilde{z}), (y, \tilde{y})$ and (m, \tilde{m}) .

- First we need to replace $SU(N')$ with its magnetic dual, $SU(N_f - N')$. The total number of flavors coming into this gauge group is $N_f = N + N'$. This breaks down to N Q 's, $\sum_{i=1}^{\ell} n_i$ Y 's, and $n' + M$ $(1, 0)$ fields. Owing to the anomaly cancellation, this sums up to $N + N'$ flavors.

- The superpotential terms in (3.6) and (3.7) become mass terms under the duality. In the magnetic theory, we replace the meson operators QY_i , $\tilde{Q}\tilde{Y}_i$, $Q\tilde{Q}'$, $Q'\tilde{Q}$, $Q\tilde{Q}_t$, and $Q_t\tilde{Q}$ with their dual chiral superfields. The cubic terms in (3.6) become mass terms for the Z 's while the quartic terms in (3.7) become mass terms for the new chirals. Integrating out the Z 's decouples the $SU(N)$ gauge group from the Fan.
- The chiral superfield dual to $\tilde{Q}Q$ is an adjoint of the first $SU(N)$ group. If we have $\mathcal{N} = 2$ gluing, the first term in equation (3.8) will become a mass term for the chiral adjoint in the vector multiplet. Integrating out the massive chirals yields an $\mathcal{N} = 1$ vector multiplet. On the other hand if the gluing is $\mathcal{N} = 1$, the vector multiplet will become $\mathcal{N} = 2$ with the addition of the chiral fields dual to $\tilde{Q}Q$.
- The cubic superpotential between the chiral adjoint of $SU(M)$ becomes a mass term when we replace the meson \tilde{Q}_tQ_t with its dual chiral superfield. Thus the $SU(M)$ gauge group becomes an $\mathcal{N} = 1$.
- The fields of the Fan in figure 6b come from three different sectors, which are listed as in the table 3. The first set of fields are inherited from the old Fan. And the second set

| New fields | Electric dual |
|--|--|
| $m_{i+1,i+1}^{(p+1)}$ $(m_{i+1,j+1}^{(p+1)}, m_{j+1,i+1}^{(p+1)})$ | $M_{ii}^{(p)}$ $(M_{ij}^{(p)}, M_{ji}^{(p)})$ |
| (q, \tilde{q}) (z_1, \tilde{z}_1) $(z_{i+1}, \tilde{z}_{i+1})$ | (Q_t, \tilde{Q}_t) (Q', \tilde{Q}') (Y_i, \tilde{Y}_i) |
| (y_1, \tilde{y}_1) $(y_{i+1}, \tilde{y}_{i+1})$ $m_{1,1}^{(0)}$ $m_{i+1,i+1}^{(0)}$ $(m_{1,j+1}^{(0)}, m_{j+1,1}^{(0)})$ $(m_{i+1,j+1}^{(0)}, m_{j+1,i+1}^{(0)})$ | $(\text{Tr}_g(q\tilde{Q}_t), \text{Tr}_g(\tilde{q}Q_t))$ $(\text{Tr}_g(Y_i\tilde{Q}_t), \text{Tr}_g(\tilde{Y}_iQ_t))$ $\text{Tr}_g(q\tilde{q})$ $\text{Tr}_g(Y_i\tilde{Y}_i)$ $(\text{Tr}_g(q\tilde{Y}_j), \text{Tr}_g(Y_j\tilde{q}))$ $(\text{Tr}_g(Y_i\tilde{Y}_j), \text{Tr}_g(Y_j\tilde{Y}_i))$ |

Table 3: The set of new fields appear upon dualizing the Fan.

of fields are the dual quarks of the $SU(N')$ gauge group. The last set of fields are the ones dual to the mesons of the old quiver tail.

- The flavor group $U(n')$ is absorbed into the Fan as the first flavor group $U(n'_1)$, and the labeling of the rest is shifted by 1 to $n'_{i+1} = n_i$. This yields the Fan labelled by (N, M) and the partition $N - M = \sum_k kn'_k$.

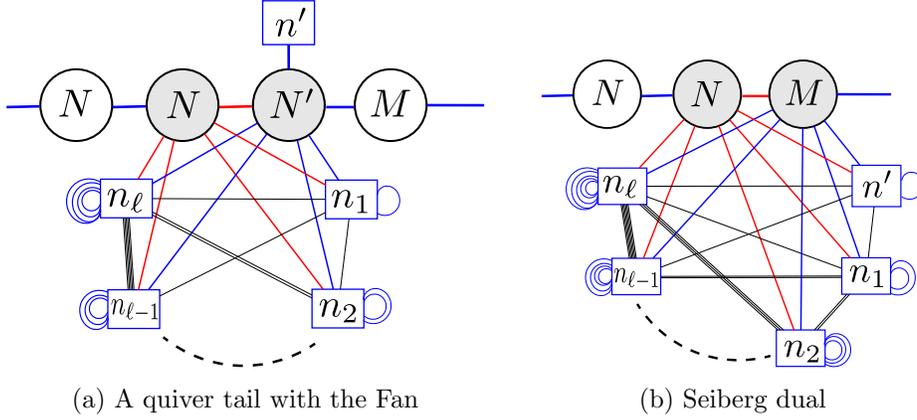


Figure 6: Seiberg dualizing at $SU(N')$ in 6a yields another quiver 6b with the new Fan. The $U(n')$ group is absorbed into the new Fan, labelled by (N, M) and the partition $N - M = \sum_k kn'_k$ with $n'_1 = n', n'_{i+1} = n_i$.

The superpotential of the dual theory is constructed by considering all possible gauge invariant $(2, 2)$ operators that preserve the global symmetry. The same superpotential is reproduced under Seiberg duality.

3.3 Fan as a quiver tail

In this section, we describe how the Fan and quiver tails appear in class \mathcal{S} theories. A quiver tail associated to the partition Y of N is given by a punctured sphere with one maximal, a number of minimal punctures and a puncture labeled by Y . Here Y corresponds to the partition $N = \sum_{k=1}^{\ell} kn_k$.

Starting from the linear quiver given in section 2.2, we can get the quiver tail by Higgsing one of the maximal punctures to Y . When the puncture has the same color as that of the pair-of-pants, this is same as giving a nilpotent vev to the quark bilinear $\mu_0 = (\tilde{Q}_0 Q_0)_{\text{adj}}$ which transforms under the adjoint of $SU(N)$. When the color of the puncture is different from that of the pair-of-pants, we give a vev to the adjoint chiral multiplet. In both cases, the $U(1)_0 \times SU(N)_0$ flavor symmetry of the quiver is broken down to $\left(\prod_{i=1}^{\ell} U(n_i)\right)$.

Now, let us describe the quiver tail associated to the partition above. If the color of the puncture we Higgs is different from that of the pair-of-pants, the theory we obtain is given by attaching the Fan with $(N, N' = 0)$ as in the figure 7.

If the color of the puncture is the same as the pair-of-pants, we proceed as follows.

1. When the neighboring gauge node of Q_0 is $\mathcal{N} = 2$, the flavor node becomes n_1 and the gauge node becomes $N_1 = \sum_{i=1}^{\ell} n_i$. If it is $\mathcal{N} = 1$, then go to step 3.
2. When the next neighboring gauge node is again $\mathcal{N} = 2$, the gauge group becomes $N_2 = N_1 + \sum_{i=2}^{\ell} n_i$, and add n_2 fundamental flavors to it. If it is $\mathcal{N} = 1$, then go to step 3.

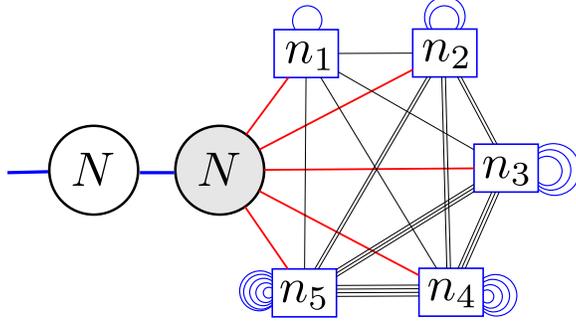
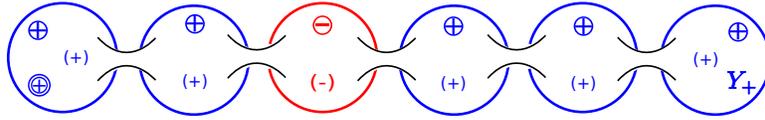
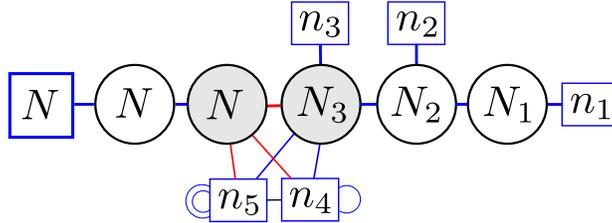


Figure 7: Fan attached to the end of the quiver with $N = 1n_1 + 2n_2 + \dots + 5n_5$ and $N' = 0$.



(a) A Pair-of-pants decomposition corresponding to the quiver tail



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 8: Quiver tail given by the partition $N = 1n_1 + 2n_2 + \dots + 5n_5$. $\mathcal{N} = 2$ Higgsing propagated until we hit $k = 3$. Then the gauge group of the next node became $SU(N)$, and we have the Fan between $SU(N)$ and $SU(N_3)$. The Fan is given by (N, N_3) and the partition $N - N_3 = n_4 + 2n_5$.

3. Proceed until we hit an $\mathcal{N} = 1$ gauge node. In this case, the neighboring gauge node remains to be $SU(N)$, since the Higgsing stops propagating. Suppose we hit the $\mathcal{N} = 1$ node at step k . In this case, the remaining flavor boxes n_i with $k < i < \ell$ should be attached to the gauge node of N_k . Therefore we get the Fan labelled by (N, N_k) with partition $N - N_k = \sum_{m=1}^{\ell-k} mn_{m+k}$.

See figure 8 for the case with $\ell = 5$ and $k = 3$. We see that the Fan serves as a role of gluing $\mathcal{N} = 1$ nodes with different ranks in the quiver tail.

Let us remark on the flavor symmetry of the quiver tail with the Fan. Even though the Fan itself has the flavor symmetry $U(1) \times \prod_k U(n_k)$, the flavor symmetry of the whole quiver tail does not include the overall $U(1)$ piece of $\prod_k U(n_k)$. The global symmetry of the quiver tail associated to the puncture Y does not contain the extra $U(1)$. We can see this directly in the case of figure 7. In this case, we see that the overall $U(1)$ can be identified with $U(1)_B$ symmetry of the Fan.

3.4 Nilpotent Higgsing and Fan

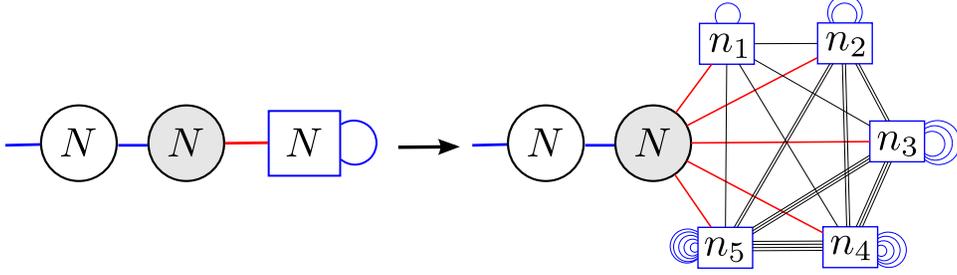


Figure 9: Nilpotent mass to the quarks to get the Fan

In this section, we give a derivation of the Fan for the case when $N' = 0$. Let us now consider the linear quiver theory as in figure 2. It has a chiral adjoint M_0 attached at the flavor $SU(N)$ node. The superpotential is $W = \text{Tr} M_0 \mu_0$, where μ_0 is the quark bilinear $\mu_0 = \tilde{q}_0 q_0 - \frac{1}{N} \text{Tr} \tilde{q}_0 q_0$ with $(J_+, J_-) = (0, 2)$. Here we choose the color of the q 's to be $\sigma = 1$. We Higgs the flavor $SU(N)$ by a nilpotent vev corresponding to the partition $N = \sum_k k n_k$ to M_0 .

Under the $SU(2)$ embedding ρ labelled by the partition of N , the fundamental representation of $SU(N)$ decompose as follows:

$$\mathbf{N} \rightarrow \bigoplus_{i=1}^{\ell} V_{\frac{i-1}{2}} \otimes \mathbf{n}_i, \quad (3.9)$$

where V_j is the spin j representation of $SU(2)$ and \mathbf{n}_i is the fundamental representation of $U(n_i) \subset S[\prod_{i=1}^{\ell} U(n_i)]$. The residual flavor symmetry $S[\prod_{i=1}^{\ell} U(n_i)]$ is given by the commutant of the embedding. The adjoint representation of $SU(N)$ decomposes as

$$\begin{aligned} \text{adj} &\rightarrow \bigoplus_{i,j=1}^{\ell} (V_{\frac{i-1}{2}} \otimes \mathbf{n}_i) \otimes (V_{\frac{j-1}{2}} \otimes \mathbf{n}_j) - V_0 \\ &= \bigoplus_{i < j} \bigoplus_{k=1}^i V_{\frac{j-i+2k-2}{2}} \otimes (\mathbf{n}_i \otimes \bar{\mathbf{n}}_j \oplus \bar{\mathbf{n}}_i \otimes \mathbf{n}_j) \oplus \bigoplus_{i=1}^{\ell} \bigoplus_{k=1}^i V_{k-1} \otimes \mathbf{n}_i \otimes \bar{\mathbf{n}}_i - V_0. \end{aligned} \quad (3.10)$$

This decomposition gives us the quantum numbers of the various elements of the $SU(N)$ -adjoint M .

We now use the decoupling argument of [9]. Due to the vev of M_0 the superpotential is written as

$$W = \mu_{1,-1,1} + \sum_{J,m,f} M_{J,-m,f} \mu_{J,m,f}, \quad (3.11)$$

where $M_{J,m,f}$ is the fluctuation from the vev, and J , m and f labels the spins, σ_3 -eigenvalues and the representations of the flavor symmetry $\prod_i SU(n_i)$ appearing in the decomposition

(3.10). By the presence of the first term the $SU(N)$ current is not conserved anymore, and becomes non-BPS by absorbing the components of μ except for the $m = J$. The components of M which coupled to the absorbed μ will be decoupled and the remaining components are $M_{J,-J,k}$. Namely the $m = -J$ component of each term of (3.10). Also we should note that due to the first term of the superpotential the $U(1)_\pm$ symmetries are shifted as

$$J_+ \rightarrow J_+ - 2\rho(\sigma^3), \quad J_- \rightarrow J_- , \quad (3.12)$$

(or $R_0 \rightarrow R_0 - \rho(\sigma^3)$ and $\mathcal{F} \rightarrow \mathcal{F} - \rho(\sigma^3)$) in order to keep the first term to be $J_+ = J_- = 2$ ($R_0 = 2, \mathcal{F} = 0$).

This gives us the gauge neutral components of the Fan in the low energy theory. We saw that there are i gauge neutral chiral multiplets $(M_{ij}^{(p)}, M_{ji}^{(p)})$, $0 \leq p < i$ transforming as bifundamentals of $U(n_i) \times U(n_j)$, $i \leq j$. We identify these chirals with the component of M (3.10) with $m = -J$ (and $k = i - p$). As a consequence of (3.12), the (J_+, J_-) charges of $(M_{ij}^{(p)}, M_{ji}^{(p)})$ become $(i + j - 2p, 0)$, which indeed match with table 1.

Some elements of the (anti-)quark multiplet transforming in the (anti-)fundamental representation of the $SU(N)$ flavor symmetry become massive due to the Higgsing and will be integrated out. Since $\langle M \rangle = \rho(\sigma^+)$ which is $J = 1$, $m = 1$ component, it implies that the (anti-)quarks Z_i (\tilde{Z}_i) that remain massless are the component with $m = \frac{i-1}{2}$ in $V_{\frac{i-1}{2}} \otimes \mathbf{n}_i$ ($V_{\frac{i-1}{2}} \otimes \bar{\mathbf{n}}_i$). Z_i and \tilde{Z}_i together form a hypermultiplet whose (J_+, J_-) charges are $(1 - i, 1)$ by using (3.12).

The Goldstone superfields In any field theory we expect the spontaneous breaking of global symmetries to be accompanied by the presence of massless Goldstone bosons whose number is equal to the number of broken generators of the global symmetry. In supersymmetric theories these Goldstone bosons will form the scalar components of massless chiral superfields which we will call Goldstone superfields.

However, the number of Goldstone superfields is not necessarily equal to the number of broken generators of the global symmetry. For example, consider the linear quiver of the form as figure 2 with gauge group being $SU(3)$. Upon nilpotent Higgsing (giving a nilpotent vev to M_0) of the $SU(3)$ linear quiver by the partition $3 = 2 + 1$, the $SU(3)$ symmetry gets broken down to $U(1)$. The chiral fields that decouple from the low energy theory are expected to be the Goldstone superfields. But there are only 4 such chiral superfields while the number of broken generators is 7.

The reason behind the discrepancy in this counting is that the scalar in a Goldstone superfield is complex. Thus it might be that a Goldstone superfield is either made up of two Goldstone bosons or a single Goldstone boson that gets paired up with a non-Goldstone scalar. In view of this we see that the number of Goldstone superfields will always be less than or equal to the number broken generators of the global symmetry. The correct number of Goldstone superfields is obtained by observing that the superpotential is holomorphic. This implies we should count the number of broken generators of the complexified global symmetry

[31]. Using this we now show that the number of decoupled chirals indeed matches with the number of expected Goldstone superfields.

In the theories of interest here, we want to consider the breaking of $G = SU(N)$ down to $H = S\left[\prod_{i=1}^{\ell} U(n_i)\right]$. The complexification of G is $\tilde{G} = SL(N, \mathbb{C})$. Since the breaking of global symmetries is achieved through $\langle M \rangle = \rho^+$, we therefore look for generators X of $SL(N, \mathbb{C})$ which satisfy

$$[\rho^+, X] \neq 0. \quad (3.13)$$

Note that any generator of $SL(N, \mathbb{C})$ can be thought of as a complex matrix transforming in the adjoint representation of $SU(N)$. We can therefore label each element of X by its $SU(2) \hookrightarrow SU(N)$ quantum numbers. In fact we can also simultaneously label them by the $S\left(\prod_{i=1}^{\ell} U(n_i)\right)$ symmetries that commute with the $SU(2)$ embedding. The components of X are therefore classified as in (3.10). In terms of $X_{J,m,k}$, we see that (3.13) is satisfied if X has a non-zero component with $m \neq J$. The Goldstone superfield corresponding to such an X will be the quantum fluctuation proportional to $[\rho^+, X]$. These fluctuations therefore correspond to the components in (3.10) that have σ_3 -eigenvalues, $m \neq -J$. This is same as the quantum numbers of the decoupled chiral multiplets. We thus establish a one-to-one correspondence between the expected Goldstone superfields and the decoupled chirals.

4 $\mathcal{N} = 1$ dualities

In this section, we discuss various duality frames for a SCFT associated to a UV curve. In order to give a UV description of the theory, we need to specify a colored pair-of-pants decomposition. Any Riemann surface with negative Euler number can be decomposed in terms of pair-of-pants. We assign \mathbb{Z}_2 -valued colors to each pair-of-pants so that the number of $(+, -)$ -colored pants are the degrees of the normal bundles (p, q) . Different colored pair-of-pants decompositions give rise to different UV descriptions of the same SCFT in the IR. See figure 10 for an example.

Let us assume all the punctures to be maximal for the moment. For a given colored pair-of-pants, we associate the T_N theory of [1]. For each puncture, we have an operator μ_i transforms as the adjoint of $SU(N)_i$. When the puncture has a different color from the pair-of-pants itself, we add chiral field M_i transforming as the adjoint of $SU(N)_i$ and also a superpotential $W = \text{Tr}(M_i \mu_i)$. When we glue two pair-of-pants with the same color, we gauge the flavor symmetry with an $\mathcal{N} = 2$ vector multiplet. When gluing two different colored pair-of-pants, we gauge the flavor symmetry by an $\mathcal{N} = 1$ vector multiplet. See figure 11, which is the UV description corresponding to the pair-of-pants decomposition of figure 10.

Non-maximal punctures can be obtained by Higgsing or partially closing the puncture. Let us call ρ_i to be the $SU(2)$ embedding into Γ that is used to label the punctures. For a puncture having the same color as the pair-of-pants, Higgsing is implemented through giving a nilpotent vev $\rho_i(\sigma^+)$ to the operator μ_i , and for an opposite colored puncture, we give a

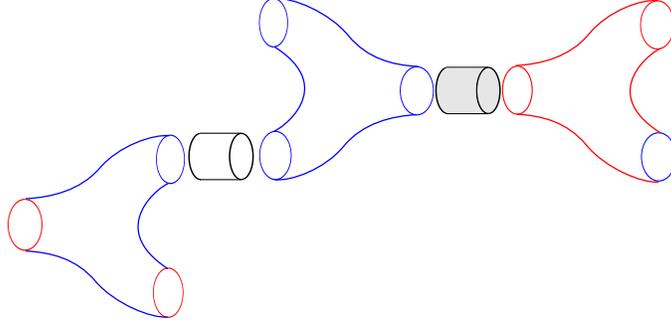


Figure 10: An example of colored pair-of-pants decomposition for $(p, q) = (2, 1)$. The shaded cylinder corresponds to $\mathcal{N} = 1$ vector multiplets and unshaded one correspond to $\mathcal{N} = 2$ vector multiplet. We have 3 punctures of opposite color. There is an adjoint chiral attached to each of them.

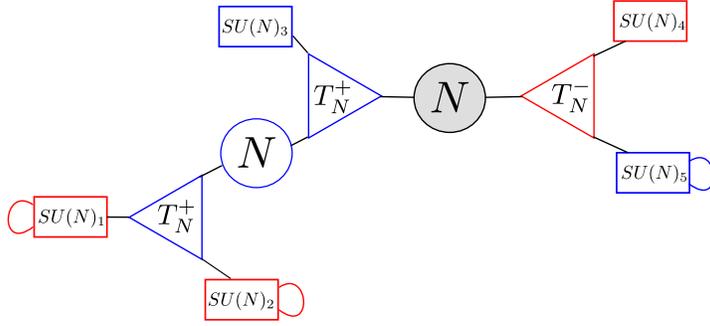


Figure 11: The UV description corresponding to the colored pair-of-pants description of figure 10. Here we assumed all punctures to be maximal.

vev to M_i instead. For example, consider the UV description of figure 11. When we Higgs $SU(N)_3$ and $SU(N)_4$ to minimal punctures, we get the theory as in figure 12. Since we closed

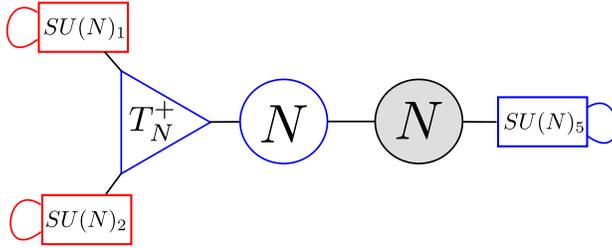


Figure 12: An UV description obtained from partially closing $SU(N)_{3,4}$ punctures to the minimal punctures.

the punctures that has the same color as the pair-of-pants, we can simply use $\mathcal{N} = 2$ results of [1, 32–38] to identify the theory corresponding to the pair-of-pants. This is really the same as choosing $\mathcal{N} = 2$ building block and gluing through the $\mathcal{N} = 1$ or $\mathcal{N} = 2$ vector multiplets.

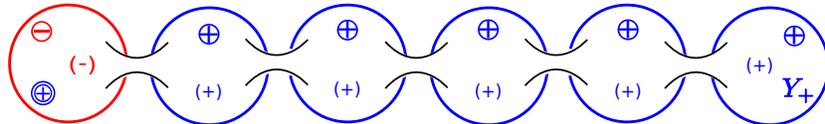
Things are different when we close the punctures with opposite colors. When we close $SU(N)_1$ to minimal puncture, the theory (in this duality frame) is still non-Lagrangian, but can identify decoupled operators and global symmetry [9]. When we close $SU(N)_5$, we give a vev $\rho_5(\sigma^+)$ to the chiral superfield M_5 , from which the quarks acquire nilpotent masses. This theory has a Lagrangian description. As we have seen, this kind of Higgsing yields the Fan labelled by $(N, N' = 0)$ and the partition corresponding to ρ_5 .

We see that there are many different colored pair-of-pants decompositions for a given UV curve. From the 6d perspective, four-dimensional physics in the IR has to be independent from the specific choice of colored pair-of-pants. Therefore we can give equivalent descriptions for the same IR theory from the UV curve and its colored pair-of-pants decompositions. This generalizes the usual Seiberg duality for $\mathcal{N} = 1$ theories and also Argyres-Seiberg-Gaiotto duality of $\mathcal{N} = 2$ class \mathcal{S} theories.

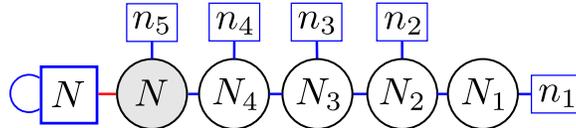
In the rest of this section, we discuss two particular examples. In section 4.1, we study successive application of Seiberg duality of the $\mathcal{N} = 2$ quiver tail connected by an $\mathcal{N} = 1$ gauge node. This illustrates the appearance of the Fan in $\mathcal{N} = 1$ quiver tail. In section 4.2, we discuss duality of $SU(N)$ SQCD with $2N$ fundamental flavors. We find a dual frame involving T_N theory and the Fan, which is similar to the strong coupling dual of $\mathcal{N} = 2$ SQCD discovered by Argyres-Seiberg [28].

4.1 $\mathcal{N} = 1$ quiver tails

Let us consider a UV curve with 5 minimal punctures of + color, 1 minimal puncture of - color, one + colored maximal puncture and one + colored generic puncture labelled by a partition $N = \sum_k kn_k$. We also pick the degrees of normal bundles to be $(p, q) = (5, 1)$. This theory has many different dual frames. We start with a dual frame which resembles more familiar $\mathcal{N} = 2$ quiver tail and then dualize multiple times to see the various dual frames for the $\mathcal{N} = 1$ quiver tail.



(a) A Pair-of-pants decomposition corresponding to the quiver



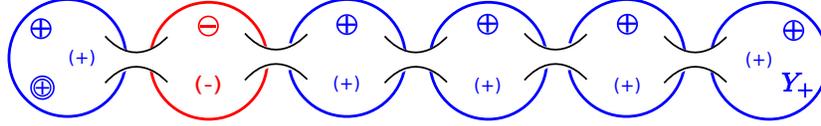
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 13: Quiver tail obtained from $\mathcal{N} = 2$ Higgsing for the partition $N = 1n_1 + 2n_2 + \dots + 5n_5$. The rank of gauge group is fixed by $2N_i = N_{i+1} + N_{i-1} + n_i$.

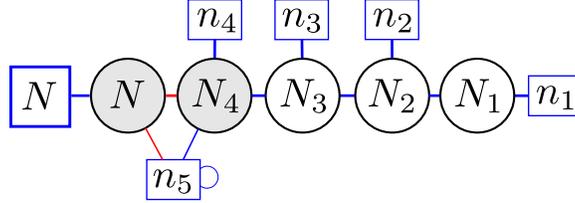
Consider the dual frame given by the colored pair-of-pants decomposition of figure 13a. This is essentially the same as the $\mathcal{N} = 2$ quiver tail, so that we get the 13b. Only the very

last node is gauged via $\mathcal{N} = 1$ vector multiplet.

Now, if we Seiberg dualize the $\mathcal{N} = 1$ node, we get the quiver as shown in figure 14. We see that there is a chiral multiplet dual to the meson formed from the quarks attached at node n_5 . The dual quarks will have the opposite \mathcal{F} charge which is depicted by red. Also, there is an additional blue edge connecting N_4 and n_5 which is the dual to the quark bilinear formed from the $SU(N) \times SU(N_4)$ bifundamental and the fundamental attached at n_5 node in figure 13b. The rest of the dual mesons become massive from the superpotential. In this



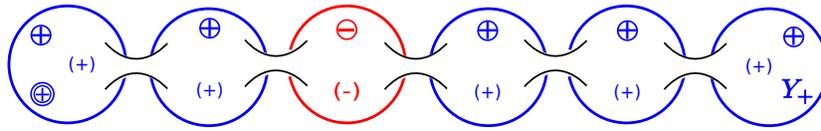
(a) A Pair-of-pants decomposition corresponding to the quiver



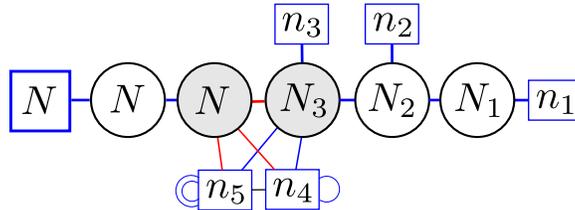
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 14: Quiver tail consists of $\mathcal{N} = 2$ tail of length 4 and the Fan labelled by (N, N_4) and the partition $N - N_4 = 1n_5$.

frame, we see that there is the Fan labelled by (N, N_4) and the partition $N - N_4 = 1 \cdot n_5$, connecting a shorter $\mathcal{N} = 2$ quiver tail of length 4 and the left-hand segment of the quiver. In terms of nilpotent Higgsing of the linear quiver, the propagation of vev is terminated at the $\mathcal{N} = 1$ node N_4 , giving us the Fan that glues to the $SU(N)$ gauge node.



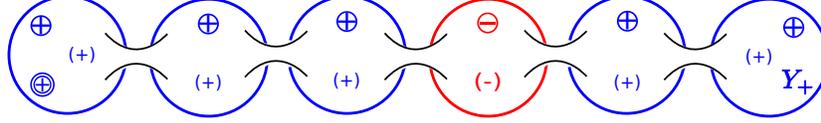
(a) A Pair-of-pants decomposition corresponding to the quiver



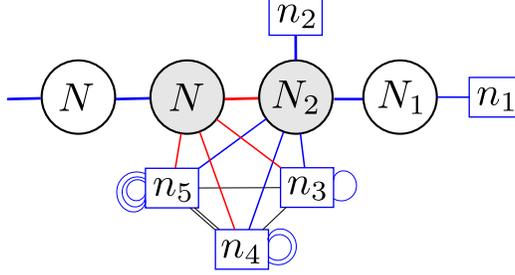
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 15: Quiver tail consists of $\mathcal{N} = 2$ tail of length 3 and the Fan labelled by (N, N_3) and the partition $N - N_3 = 1n_4 + 2n_5$.

Now, we dualize the gauge group $SU(N_4)$ to get the quiver depicted in figure 15. The flavor node n_4 becomes part of the new Fan, which is labelled by (N, N_3) and the partition $N - N_3 = 1n_4 + 2n_5$. We see that there is an extra dual meson attached to the n_5 node.



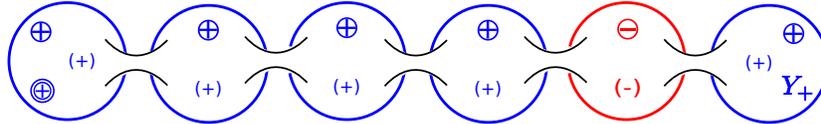
(a) A Pair-of-pants decomposition corresponding to the quiver



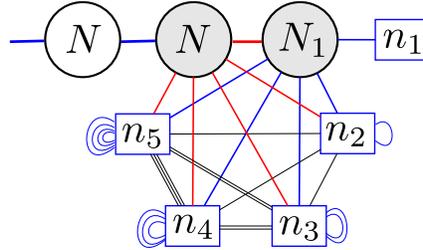
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 16: Quiver tail consists of $\mathcal{N} = 2$ tail of length 2 and the Fan labelled by (N, N_2) and the partition $N - N_2 = 1n_3 + 2n_4 + 3n_5$.

Further dualizing the $SU(N_3)$ node, we get the quiver of figure 16. The flavor node n_3 now becomes the part of the Fan, and we get extra dual mesons for each of the preexisting nodes in the Fan. Note that we also have additional chiral multiplets transforming as the bifundamental of $U(n_4) \times U(n_5)$.



(a) A Pair-of-pants decomposition corresponding to the quiver



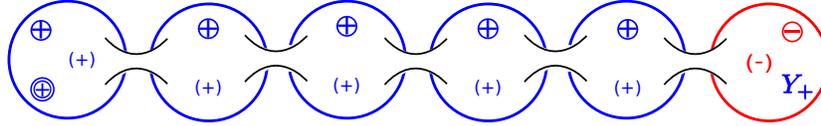
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 17: Quiver tail consists of $\mathcal{N} = 2$ tail of length 1 and the Fan labelled by (N, N_1) and the partition $N - N_1 = 1n_2 + 2n_3 + 3n_4 + 4n_5$.

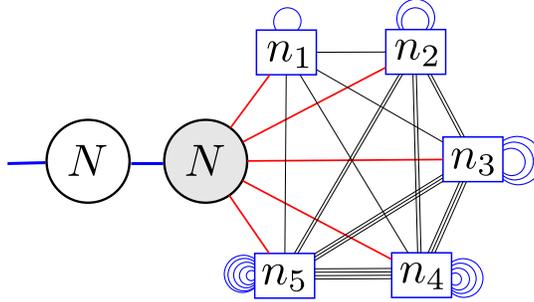
Dualizing once again, we get the quiver tail of figure 17. Once again, the flavor node

n_2 becomes a part of the Fan, and chiral multiplets get added. This quiver tail can also be obtained from starting with the linear quiver and Higgsing $\mu_0 = (\tilde{Q}_0 Q_0)_{\text{adj}}$ directly by a nilpotent vev associated to the partition $N = \sum_k kn_k$. We see that the Higgsing does not propagate beyond N_1 . All the flavor nodes are attached to N_1 and its neighbor N .

Now finally, upon dualizing N_1 gauge node, we get the theory as in the figure 18. This gives us the Fan of size $\ell = 5$ labelled by $(N, 0)$ and the partition $N = \sum_{k=1}^5 kn_k$ attached to the right end of the quiver.



(a) A Pair-of-pants decomposition corresponding to the quiver



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition

Figure 18: Quiver tail consists of maximal Fan of size $\ell = 5$, labelled by $(N, 0)$ and the partition $N = \sum_k kn_k$.

We see that there are many different quiver tail descriptions for a given choice of punctures in $\mathcal{N} = 1$ class \mathcal{S} theories. In the above example, we have only described UV frames that have Lagrangian descriptions. For these cases, all the pair-of-pants have the same color as the minimal puncture inside. In general, one can also consider a dual frame which has a different colored puncture inside its pair-of-pants then the dual frame has a sector with no Lagrangian description. We will discuss such a case in the next section.

4.2 $\mathcal{N} = 1$ analog of Argyres-Seiberg duality

In this section we use the Fan to provide a new dual description of $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors with the quartic coupling (2.5) with $i = 1$. This is the $(\sigma_{-1}, \sigma_0, \sigma_1, \sigma_2) = (-1, 1, -1, 1)$ linear quiver as described in section 2.2. The flavor symmetry of the theory is $SU(N)_1 \times SU(N)_2 \times U(1)_A \times U(1)_B$. We summarize the matter content in table 4 and quiver in figure 19. In this section it is more convenient to use the symmetries R_0 and \mathcal{F} defined in (2.1).

The known dual descriptions of this SQCD are: the Seiberg dual one [25] and the duality frame involving the non-Lagrangian theory found in [9]. These two theories are represented

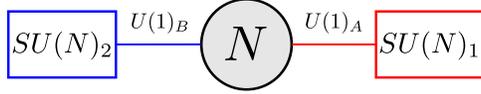


Figure 19: Quiver diagram of $SU(N)$ SQCD with $2N$ flavors

| | $SU(N)_g$ | $SU(N)_1$ | $SU(N)_2$ | $U(1)_{R_0}$ | $U(1)_{\mathcal{F}}$ | $U(1)_A$ | $U(1)_B$ |
|----------------------|----------------------------|----------------------------|----------------------------|--------------|----------------------|-----------|-----------|
| (Q_0, \tilde{Q}_0) | $(\square, \bar{\square})$ | $(\bar{\square}, \square)$ | \cdot | $1/2$ | $-1/2$ | $(1, -1)$ | \cdot |
| (Q_1, \tilde{Q}_1) | $(\bar{\square}, \square)$ | \cdot | $(\square, \bar{\square})$ | $1/2$ | $1/2$ | \cdot | $(1, -1)$ |

Table 4: Charges of matter multiplets in SQCD.

in terms of the curve as exchanges of the punctures, as in figures 20b and 20c. The dual theory we will see in this section corresponds to the figure 20d. The UV description involves the T_N theory [1]. We review relevant details here.

The T_N theory is obtained by compactifying N coincident M5-branes, with $\mathcal{N} = 2$ twist, on a sphere with three maximal punctures. Each puncture carries an $SU(N)$ global symmetry, thereby leading to an $SU(N)^3$ flavor symmetry. It is an $\mathcal{N} = 2$ SCFT and it admits $U(1)_{\mathcal{N}=2} \times SU(2)_R$ R -symmetry. When we describe it as an $\mathcal{N} = 1$ SCFT, we use the $\mathcal{N} = 2$ R -symmetry to write R_0 and \mathcal{F} as

$$R_0 = \frac{1}{2}R_{\mathcal{N}=2} + I_3, \quad \mathcal{F} = -\frac{1}{2}R_{\mathcal{N}=2} + I_3 \quad (4.1)$$

where $R_{\mathcal{N}=2}$ and I_3 are generators of $U(1)_{\mathcal{N}=2}$ and the diagonal $U(1)$ of the $SU(2)_R$ respectively. This theory has chiral operators μ_i (i labels the three $SU(N)$ flavor symmetries) which are the moment maps of the $SU(N)$ flavor symmetries. It also has operators $Q^{(k)}$ transforming in the k -th antisymmetric representation of all three $SU(N)$ symmetries [17, 39, 40].

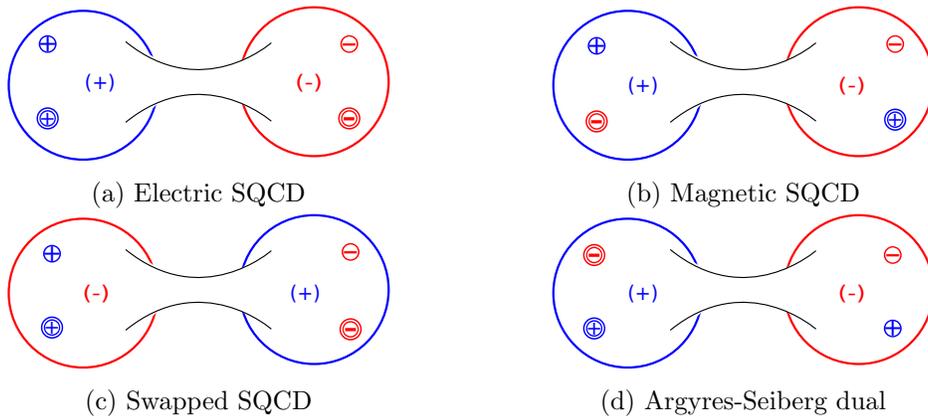


Figure 20: Colored pair-of-pants decompositions of the UV curve corresponding to the SQCD with $SU(N)$ gauge group and $2N$ flavors and its dual descriptions.

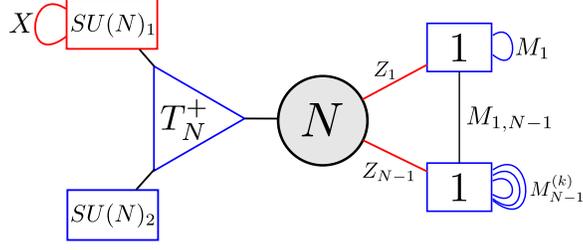


Figure 21: Analog of Argyres-Seiberg dual to the $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors.

Their R_0 and \mathcal{F} charges are

$$R_0(\mu_i) = \mathcal{F}(\mu_i) = 1, \quad R_0(Q^{(k)}) = \mathcal{F}(Q^{(k)}) = \frac{k(N-k)}{2}. \quad (4.2)$$

The results of section 3 tell us that figure 20d represents an $SU(N)$ gauge theory coupled to the Fan with $\sigma = -1$ labelled by $(N, 0)$ and a partition $N = 1 + (N - 1)$, i.e., $\ell = N - 1$, $n_1 = n_{N-1} = 1$ and $n_i = 0$ otherwise. It is coupled to the T_N theory by gauging an $SU(N)$ flavor symmetry. Furthermore, a chiral field X transforming in the adjoint representation of one of $SU(N)_1$ flavor symmetry of the T_N theory is added. $SU(N)_{1,2}$ are the flavor symmetries of the T_N theory which are not gauged.

The dual theory is described by the quiver in figure 21. The matter content is summarized in table 5. For convenience of the discussion, we write fields from the Fan as $M_1 := M_{1,1}^{(0)}$, $M_{N-1}^{(k)} := M_{N-1,N-1}^{(N-1-k)}$ and $z := Z_{N-1}$.

The important data needed in including the T_N in these quivers is its contribution to the anomalies. These are described in section 5. For the purpose of the quiver in 21, the contribution of T_N to the chiral anomalies ($R_0 SU(N)^2, \mathcal{F} SU(N)^2$) is the same as N fundamental $(J_+, J_-) = (1, 0)$ hypermultiplets.

| | $SU(N)_g$ | $SU(N)_1$ | $SU(N)_2$ | $U(1)_{R_0}$ | $U(1)_{\mathcal{F}}$ | $U(1)_1$ | $U(1)_{N-1}$ |
|-------------------------------|----------------------------|-----------|-----------|-----------------|----------------------|-----------|--------------|
| (Z_1, \tilde{Z}_1) | $(\square, \bar{\square})$ | \cdot | \cdot | $1/2$ | $-1/2$ | $(-1, 1)$ | \cdot |
| (z, \tilde{z}) | $(\square, \bar{\square})$ | \cdot | \cdot | $\frac{3-N}{2}$ | $\frac{1-N}{2}$ | \cdot | $(-1, 1)$ |
| M_1 | \cdot | \cdot | \cdot | 1 | 1 | \cdot | \cdot |
| $(M_{1,N-1}, M_{N-1,1})$ | \cdot | \cdot | \cdot | $N/2$ | $N/2$ | $(1, -1)$ | $(-1, 1)$ |
| $M_{N-1}^{(k=1, \dots, N-1)}$ | \cdot | \cdot | \cdot | k | k | \cdot | \cdot |
| X | \cdot | adj | \cdot | 1 | -1 | \cdot | \cdot |

Table 5: Charges of matter multiplets in the dual theory, where $M_1 := M_{1,1}^{(0)}$ and $M_{N-1}^{(k)} := M_{N-1,N-1}^{(N-1-k)}$ and $z = Z_{N-1}$.

Finally, one linear combination of M_1 and $M_{N-1}^{(1)}$ must be projected out. We denote the

combination that survives as \hat{M}_1 . We can then write the superpotential as

$$W_m = \hat{M}_1(\text{Tr}z\mu_g^{N-2}\tilde{z} + \text{Tr}Z_1\tilde{Z}_1) + \sum_{\alpha} \sum_{k=2}^{N-1} \mathcal{M}_{N-1}^{(k),\alpha} \text{Tr}z\mu_g^{N-1-k}\tilde{z} \\ + M_{1,N-1}\text{Tr}Z_1\tilde{z} + M_{N-1,1}\text{Tr}z\tilde{Z}_1 + \text{Tr}\mu_1 X + \text{Tr}Z_1\mu_g\tilde{Z}_1 + \text{Tr}z\mu_g^{N-1}\tilde{z}, \quad (4.3)$$

where μ_1 and μ_g are the moment maps of $SU(N)_g$ and $SU(N)_1$ flavor symmetries respectively. The set of operators, $\mathcal{M}_{N-1}^{(k),\alpha}$ correspond to all possible composite operators with charge $(2k, 0)$.

Note that this is reminiscent of the Argyres-Seiberg duality [28] of $\mathcal{N} = 2$ $SU(3)$ SQCD with six flavors. Indeed, if we consider the analogous UV curve in the $\mathcal{N} = 2$ setting without color assignments, this dual frame is exactly that of Argyres-Seiberg when $N = 3$. The duality presented here is an $\mathcal{N} = 1$ analog of that. It will be interesting to derive this duality through the technique of inherited duality [41, 42].

We identify $U(1)_A$ and $U(1)_B$ of the SQCD as

$$U(1)_A = U(1)_1 + U(1)_{N-1}, \quad U(1)_B = (N-1)U(1)_1 - U(1)_{N-1}. \quad (4.4)$$

It is a straightforward calculation to show that all the anomaly coefficients of the flavor symmetries agree on both sides of the duality. We will see this in section 5. In section 6, we will also see the agreement of the superconformal index of both theories. This will be the strongest check of the duality.

5 Anomalies and central charges

In this section we compute the 't Hooft anomaly coefficients of various objects. In section 5.1, we start with computing those of the Fan introduced in section 3.1. We then interpret the results in terms of a sphere with punctures and give a concise expression for the anomaly coefficients of the class \mathcal{S} theories in general, in section 5.2.

5.1 Anomalies of the Fan

The matter content of the Fan labelled by $(N, N' = 0)$ and a partition $N = \sum_{k=1}^{\ell} n_k$ with $\sigma = -1$ is given in the table 1. One can choose $\sigma = +1$ by swapping J_+ and J_- charges. In evaluating the anomalies, it is useful to write them in terms of

$$N_i = \sum_{k=1}^i n_k k + i \sum_{k=i+1}^{\ell} n_k, \quad (5.1)$$

and to notice the following identity

$$N^2 = 2 \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j - \sum_{i=1}^{\ell} N_i n_i. \quad (5.2)$$

We find the 't Hooft anomaly coefficients of $SU(n_i)$ and $U(1)_i$ for $i \geq 2$ are

$$\begin{aligned}\mathrm{Tr}T_i^2R_0 &= \sigma\mathrm{Tr}T_i^2\mathcal{F} = -\frac{1}{2}N_i, \\ \mathrm{Tr}U_i^2R_0 &= -N_i - i(i-1)n_i, \quad \sigma\mathrm{Tr}U_i^2\mathcal{F} = -N_i + i(i+1)n_i,\end{aligned}\tag{5.3}$$

where T_i and U_i are the generators of $SU(n_i)$ and $U(1)_i$ respectively. The other anomaly coefficients are given by

$$\mathrm{Tr}R_0 = -\sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \quad \sigma\mathrm{Tr}\mathcal{F} = \sum_{i=1}^{\ell} N_i \sum_{j=i+1}^{\ell} n_j + 1,\tag{5.4}$$

$$\mathrm{Tr}\mathcal{F}^a R_0^{3-a} = -\frac{(-\sigma)^a}{4} \sum_{i=1}^{\ell} n_i \left[\sum_{j=1}^i n_j (i^3 j - f_a(i, j)) + \sum_{j=i+1}^{\ell} n_j (i^3 j - f_a(j, i)) \right],\tag{5.5}$$

where

$$f_a(i, j) = \frac{1}{2} \sum_{p=0}^{j-1} (i+j-2p-2)^{3-a} (i+j-2p)^a.\tag{5.6}$$

Writing explicitly,

$$\mathrm{Tr}R_0\mathcal{F}^2 = \frac{1}{4} \sum_{i=1}^{\ell} (N^2 - N_i^2) - \frac{1}{4} \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j,\tag{5.7}$$

$$\sigma\mathrm{Tr}R_0^2\mathcal{F} = \frac{1}{4} \sum_{i=1}^{\ell} (N^2 - N_i^2) + \frac{1}{2}N^2 - \frac{1}{4} \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j.\tag{5.8}$$

We found that the rest can be obtained from

$$\mathrm{Tr}\mathcal{F}^3 = \mathrm{Tr}\mathcal{F} - 3\mathrm{Tr}\mathcal{F}R_0^2, \quad \mathrm{Tr}R_0^3 = \mathrm{Tr}R_0 - 3\mathrm{Tr}\mathcal{F}^2R_0.\tag{5.9}$$

From Linear quiver The above anomalies can also be obtained by a rather indirect way. The idea is to use the duality: as we saw in section 4.1, the Fan was obtained by taking various Seiberg dualities to the linear quiver theories with $\mathcal{N} = 1$ $SU(N)$ gauge theory coupled to $\mathcal{N} = 2$ quiver tail labelled by partitions of N : $N = \sum_{k=1}^{\ell} n_k k$. Thus, let us first focus on this original theory. This theory has gauge symmetry $G = \prod_{k=1}^{\ell} SU(N_k)$ with (5.1). Notice that $N_{\ell} = N$. All the gauge groups except for the ℓ -th one are $\mathcal{N} = 2$. In addition to the bifundamentals, there are n_i fundamental hypermultiplets attached to the $SU(N_i)$ gauge group. The tail has a label $\sigma = \pm 1$ depending on the \mathcal{F} -charge of the matter fields $\mathcal{F} = \sigma/2$. (The \mathcal{F} -charge of the chiral adjoint multiplets of the gauge symmetry is $-\sigma$.) We end the quiver by adding N fundamental hypers with $R_0 = 1/2$ and $\mathcal{F} = -\sigma/2$ to $SU(N_{\ell})$ gauge group. We further attach a chiral multiplet ($R_0 = 1$ and $\mathcal{F} = \sigma$) in the adjoint representation of the $SU(N)$ flavor symmetry of N hypers.

Then, the 't Hooft anomaly coefficients of R_0 and \mathcal{F} of this theory are given as

$$\mathrm{Tr}R_0 = -\ell - \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \quad \mathrm{Tr}R_0^3 = \frac{1}{4}\mathrm{Tr}R_0 + \frac{3}{4} \sum_{i=1}^{\ell} (N_i^2 - 1), \quad (5.10)$$

$$\mathrm{Tr}\mathcal{F} = -\sigma(2 + \mathrm{Tr}R_0), \quad \mathrm{Tr}\mathcal{F}^3 = \frac{1}{4}\mathrm{Tr}\mathcal{F} - \frac{3\sigma}{4} \left[\sum_{i=1}^{\ell} (N_i^2 - 1) - 2(N^2 - 1) \right], \quad (5.11)$$

$$\mathrm{Tr}R_0\mathcal{F}^2 = \frac{1}{4}\mathrm{Tr}R_0 - \frac{1}{4} \sum_{i=1}^{\ell} (N_i^2 - 1), \quad \mathrm{Tr}R_0^2\mathcal{F} = -\sigma \left(\mathrm{Tr}R_0\mathcal{F}^2 + \frac{1}{2}N^2 \right). \quad (5.12)$$

Again we note that they satisfy (5.9).

After repeatedly applying the Seiberg dualities, we end up with an $\mathcal{N} = 2$ linear quiver attached to the Fan. The quiver has ℓ gauge nodes with $SU(N)$ gauge groups linked together by bifundamentals with $R_0 = 1/2$ and $\mathcal{F} = \sigma/2$. All gauge groups are $\mathcal{N} = 2$ vector multiplets except for the one at $k = 1$. The Fan (with $-\sigma$) is attached to this $k = 1$ $\mathcal{N} = 1$ gauge node. By subtracting the contribution of this quiver except for the Fan from (5.10), (5.11) and (5.12), we reproduce the anomaly coefficients (5.4) and (5.5). Note that for $\mathrm{Tr}\mathcal{F}$, $\mathrm{Tr}\mathcal{F}^3$ and $\mathrm{Tr}R_0^2\mathcal{F}$, there are overall sign differences from (5.4) and (5.5). This is because the Fan appeared here is specified by $-\sigma$.

5.2 Anomalies of class \mathcal{S} theories

So far we have computed the anomaly coefficients of the Fan. In the class \mathcal{S} point of view, the Fan with $\sigma = +1$ is associated to a sphere ($p = 1, q = 0$) with a maximal puncture with $\sigma = +1$, a minimal puncture with $\sigma = +1$ and a puncture labeled by Y with $\sigma = -1$ or the opposite choice. Here we will show that the anomaly coefficients can be given in terms of the data of the Riemann sphere and the punctures. By generalizing this observation, we will conjecture that the anomaly coefficients of the class- \mathcal{S} theories can be written down as a sum of contributions from the following:

- Background contribution from the curve: $\mathcal{C}_{g,n}$ with normal bundle $\mathcal{L}(p) \oplus \mathcal{L}(q)$ specified. Here $p + q = 2g - 2 + n$ is imposed.
- Local contributions from each punctures $(\rho, \sigma)_{i=1, \dots, n}$.

If we write the number of punctures with color σ to be n_σ , $n = n_+ + n_-$ is the total number of punctures. We will first summarize the case of $\mathcal{N} = 2$ theories, which have been worked out in full generality by [38], and then give a generalization to the $\mathcal{N} = 1$ theories.

In the $\mathcal{N} = 2$ case, we always set $q = 0$ and $n_- = 0$ so that the total space becomes the cotangent bundle of the Riemann surface $\mathcal{C}_{g,n}$. All the punctures have the same color, thus they are specified entirely by the embedding of $SU(2)$ into Γ labeling the class \mathcal{S} theory. For these $\mathcal{N} = 2$ theories, the number of effective vector multiplets n_v and hypermultiplets n_h can be used to determine the anomaly coefficients of the $\mathcal{N} = 2$ R -symmetries:

$$\mathrm{Tr}R_{\mathcal{N}=2} = \mathrm{Tr}R_{\mathcal{N}=2}^3 = 2(n_v - n_h), \quad \mathrm{Tr}R_{\mathcal{N}=2}I_3^2 = \frac{n_v}{4}. \quad (5.13)$$

The quantities n_v, n_h are well-defined in the case of Lagrangian theories, but it is useful book-keeping device to use for non-Lagrangian theories as well.

For a given punctured Riemann surface, we can separate the contribution from the background Riemann surface and the punctures. For $\Gamma = A_{N-1}$, the background contribution for a genus g Riemann surface with n punctures is given by

$$n_h(\mathcal{C}_{g,n}) = \frac{2}{3}(2g - 2 + n)N(N^2 - 1), \quad (5.14)$$

$$n_v(\mathcal{C}_{g,n}) = \frac{1}{6}(2g - 2 + n)(4N^3 - N - 3). \quad (5.15)$$

Note that the definition of the background contribution is slightly different from the one in the literature by the terms including n . The factor $2g - 2 + n$ is the number of the pairs of pants, and this definition is more convenient to proceed to $\mathcal{N} = 1$ class \mathcal{S} theories.

For a puncture labeled by a Young diagram Y , (called regular punctures)

$$n_h(Y) = \frac{1}{2} \sum_r l_r^2 + \sum_{k=2}^N (2k-1)p_k - \frac{1}{6}(4N^3 - N), \quad (5.16)$$

$$n_v(Y) = \sum_{k=2}^N (2k-1)p_k - \frac{1}{6}(4N^3 - N - 3), \quad (5.17)$$

where p_k labels the structure of the poles at the puncture (which can be read off from Y) [1] and l_r is the length of the r -th row of Y . For example, the maximal puncture has the pole structure $p_{\max} = (0, 1, 2, \dots, N-1)$ and the minimal puncture has $p_{\min} = (0, 1, 1, \dots, 1)$. Note again that the last terms are absent in the definition in the literature. These compensate the changes in the background contributions. In general, one can also have irregular punctures as well, but we will not consider them here. For example, the maximal puncture has

$$n_h(Y_{\max}) = 0, \quad n_v(Y_{\max}) = -\frac{1}{2}(N^2 - 1), \quad (5.18)$$

and the minimal puncture has

$$n_h(Y_{\min}) = -\frac{1}{6}(4N^3 - 6N^2 - 4N), \quad (5.19)$$

$$n_v(Y_{\min}) = -\frac{1}{6}(4N^3 - 6N^2 - N + 3). \quad (5.20)$$

By summing altogether, n_h and n_v are

$$n_h = n_h(\mathcal{C}_{g,n}) + \sum_i n_h(Y_i), \quad n_v = n_v(\mathcal{C}_{g,n}) + \sum_i n_v(Y_i). \quad (5.21)$$

Also, the flavor central charge of an $\mathcal{N} = 2$ theory is defined by

$$k\delta^{ab} = -2\text{Tr}R_{\mathcal{N}=2}T^aT^b \quad (5.22)$$

where T^a is the generator of the flavor symmetry.

We now define the $\mathcal{N} = 1$ version of n_h and n_v . Let σ_i be the sign of the i -th puncture. They are given by

$$\hat{n}_h = \hat{n}_h(\mathcal{L}^{p,q}) + \sum_i \sigma_i n_h(Y_i), \quad \hat{n}_v = \hat{n}_v(\mathcal{L}^{p,q}) + \sum_i \sigma_i n_v(Y_i), \quad (5.23)$$

where

$$\hat{n}_h(\mathcal{L}^{p,q}) = \frac{2}{3}(p-q)N(N^2-1), \quad (5.24)$$

$$\hat{n}_v(\mathcal{L}^{p,q}) = \frac{1}{6}(p-q)(4N^3 - N - 3). \quad (5.25)$$

Since we are considering $\mathcal{N} = 1$ theories, \hat{n}_h and \hat{n}_v do not have the interpretation of the effective numbers of hyper and vector multiplets. However, we continue to use these letters. In terms of these, our proposal for the 't Hooft anomaly coefficients are as follows:

$$\text{Tr}R_0 = n_v - n_h, \quad \text{Tr}R_0^3 = n_v - \frac{n_h}{4}, \quad (5.26)$$

$$\text{Tr}\mathcal{F} = -(\hat{n}_v - \hat{n}_h), \quad \text{Tr}\mathcal{F}^3 = -\hat{n}_v + \frac{\hat{n}_h}{4}, \quad (5.27)$$

$$\text{Tr}R_0\mathcal{F}^2 = -\frac{n_h}{4}, \quad \text{Tr}R_0^2\mathcal{F} = \frac{\hat{n}_h}{4}, \quad (5.28)$$

where n_h and n_v are (5.21) with $2g - 2 + n = p + q$.

In an $\mathcal{N} = 1$ theory which can be obtained from the $\mathcal{N} = 2$ one, we identify the R -symmetries as [43]

$$R_0 = \frac{1}{2}R_{\mathcal{N}=2} + I_3, \quad \mathcal{F} = -\frac{1}{2}R_{\mathcal{N}=2} + I_3. \quad (5.29)$$

With these, the above anomaly coefficients (without hats) can be obtained by using (5.13). Then we changed n_v and n_h into \hat{n}_v and \hat{n}_h for the anomalies involving odd power of \mathcal{F} . We are proposing these formulae, however, for the theories which do not necessarily have the $\mathcal{N} = 2$ origin, like the Fan.

Let us check these formulae are indeed correct for a few theories.

Fan The Fan with $\sigma = +1$ is associated with a sphere with $p = 1$ and $q = 0$ and three punctures, maximal, minimal and the one specified by Y . Therefore, we get from (5.21) and (5.23),

$$n_v = \sum_{k=2}^N (2k-1)p_k - \frac{1}{6}(4N^3 - 3N^2 - N), \quad n_v - n_h = -\frac{1}{2}(N^2 + \sum_r l_r^2),$$

$$\hat{n}_v = -\sum_{k=2}^N (2k-1)p_k + \frac{1}{6}(4N^3 + 3N^2 - N - 6), \quad \hat{n}_v - \hat{n}_h = -\frac{1}{2}(N^2 + 2 - \sum_r l_r^2).$$

It is straightforward to see that the anomaly coefficients obtained by substituting these into (5.28) agree with the ones from the direct computation (5.4) and (5.5), by using the identities $\sum_r l_r^2 = \sum_{i=1}^{\ell} N_i n_i$, $\sum_{k=2}^N (2k-1)p_k = \frac{N}{6}(4N^2 - 3N - 1) - \sum_i (N^2 - N_i^2)$, and (5.2).

$SU(N)$ SQCD with $N_f = 2N$ Now let us try to see how the formulae work in other class \mathcal{S} theories. A simple example is SQCD with $N_f = 2N$ considered in section 4.2 which is associated with a sphere with two maximal punctures with $\sigma = +1$ and $\sigma = -1$ and two minimal punctures with $\sigma = +1$ and $\sigma = -1$ and also with $p = q = 1$. The anomalies are given by

$$\mathrm{Tr}R_0 = -N^2 - 1, \quad \mathrm{Tr}R_0^3 = \frac{N^2}{2} - 1, \quad \mathrm{Tr}R_0\mathcal{F}^2 = -\frac{N^2}{2}, \quad (5.30)$$

$$\mathrm{Tr}\mathcal{F} = \mathrm{Tr}\mathcal{F}^3 = \mathrm{Tr}R_0^2\mathcal{F} = 0. \quad (5.31)$$

These can also be computed directly from the matter content of the SQCD as in the table 4.

For completeness, let us compute the anomaly coefficients of non-Abelian symmetry. For the gauge symmetry, we have $\mathrm{Tr}R_0T_g^2 = \mathrm{Tr}\mathcal{F}T_g^2 = 0$ indicating the vanishing exact beta function and anomaly-free $U(1)_{\mathcal{F}}$. The anomalies which involves $SU(N)$ flavor symmetries are as follows:

$$\mathrm{Tr}R_0T_1^2 = \mathrm{Tr}R_0T_2^2 = \mathrm{Tr}\mathcal{F}T_1^2 = -\mathrm{Tr}\mathcal{F}T_2^2 = -\frac{N}{2}, \quad (5.32)$$

where $T_{1,2}$ are the generators of $SU(N)_{1,2}$. Since there is no non-baryonic $U(1)$ symmetry, the $U(1)_{R_0}$ is the true R -symmetry in the IR.

Linear quiver We have computed in the previous section the 't Hooft anomaly coefficients of the linear quiver with $\mathcal{N} = 2$ tail (5.10), (5.11) and (5.12). Let us reproduce these results from our formulae. The quiver (we fix $\sigma = 1$) is associated with a sphere with $p = \ell$ and $q = 1$ and $\ell + 1$ minimal punctures with $\sigma = +1$, one maximal puncture with $\sigma = +1$ and a puncture specified by Y with $\sigma = -1$. It is easy to get

$$n_v = \sum_{i=1}^{\ell} (N_i^2 - 1), \quad n_v - n_h = -\ell - \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \quad (5.33)$$

$$\hat{n}_v = \sum_{i=1}^{\ell} (N_i^2 - 1) - 2N^2 + 2, \quad \hat{n}_h = -\ell + 2 - \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j. \quad (5.34)$$

These reproduce (5.10), (5.11) and (5.12).

$\mathcal{N} = 1$ gauging Let us consider a pair of class \mathcal{S} theories, \mathcal{T}_1 and \mathcal{T}_2 , each of which has an $SU(N)$ flavor symmetry. Let the colors of the maximal punctures be different and \mathcal{T}_1 and \mathcal{T}_2 be associated to a pair-of-pants decompositions where each color of the maximal puncture is the same as that of the pair-of-pants to which the puncture attached. Then let us think of gluing these punctures. This corresponds to the $\mathcal{N} = 1$ gauging of the diagonal $SU(N)$ symmetry of two $SU(N)$ flavor symmetries of \mathcal{T}_1 and \mathcal{T}_2 . The resulting theory is again in class \mathcal{S} .

The 't Hooft anomaly coefficients of the resulting theory are written as the sum of those of \mathcal{T}_1 and \mathcal{T}_2 , and of $\mathcal{N} = 1$ vector multiplet. The anomalies of the latter can be computed as

$$\mathrm{Tr}R_0 = \mathrm{Tr}R_0^3 = N^2 - 1, \quad \mathrm{Tr}\mathcal{F} = \mathrm{Tr}\mathcal{F}^3 = \mathrm{Tr}R_0^2\mathcal{F} = \mathrm{Tr}R_0\mathcal{F}^2 = 0. \quad (5.35)$$

These can be obtained from our formulae. Indeed from the Riemann surface point of view, the $\mathcal{N} = 1$ gauging corresponds to subtracting two maximal punctures with different signs. Thus, we have $\delta n_v = N^2 - 1$, $\delta \hat{n}_v = \delta n_h = \delta \hat{n}_h = 0$. These reproduce (5.35).

$\mathcal{N} = 2$ gauging Instead, let us consider the gauging by an $\mathcal{N} = 2$ vector multiplet. Namely, consider \mathcal{T}_1 and \mathcal{T}_2 with maximal punctures whose colors are the same. \mathcal{T}_i is associated to pants decomposition where the colors of the maximal puncture and of the pair-of-pants to which the puncture is attached are the same. Let us suppose the color is $\sigma = +$. In this case the 't Hooft anomalies are the sum of those of \mathcal{T}_1 , \mathcal{T}_2 and of $\mathcal{N} = 2$ vector multiplet where the gauge adjoint chiral field has $R_0 = -\mathcal{F} = 1$. The latter contributes to the anomalies as

$$\mathrm{Tr} R_0 = \mathrm{Tr} R_0^3 = -\mathrm{Tr} \mathcal{F} = -\mathrm{Tr} \mathcal{F}^3 = N^2 - 1, \quad \mathrm{Tr} R_0^2 \mathcal{F} = \mathrm{Tr} R_0 \mathcal{F}^2 = 0. \quad (5.36)$$

Again this can be obtained from the formulae with $\delta n_v = \delta \hat{n}_v = N^2 - 1$ and $\delta n_h = \delta \hat{n}_h = 0$.

A theory coupled to an adjoint Let us consider the Riemann surface with a maximal puncture such that $\sigma_{Y_{max}}$ is different from the sign of the background. In [9], it was noticed that this represents a theory (associated to the same Riemann surface where the maximal puncture has the same sign as the bulk) coupled to a chiral multiplet M transforming in the adjoint representation of the $SU(N)$ flavor symmetry associated to the maximal puncture, by the superpotential $W = \mathrm{Tr} \mu M$ where μ is the moment map of the $SU(N)$. The charges of M are $R_0 = 1$ and $\mathcal{F} = \sigma_{Y_{max}}$. (When the Riemann surface is a sphere with two maximal and a minimal punctures, this boils down to the Fan with Y is maximal.) Let us see this is consistent with our formula.

Suppose that the sign of the background is $+1$ and $\sigma_{Y_{max}} = -1$. Compared to the case where $\sigma_{Y_{max}} = +1$, \hat{n}_v increases by $\delta \hat{n}_v = N^2 - 1$, while n_v , n_h and \hat{n}_h are kept intact. Therefore the contribution of changing $\sigma_{Y_{max}}$ from $+1$ to -1 to the anomaly coefficients is

$$\delta \mathrm{Tr} \mathcal{F} = 1 - N^2, \quad \delta \mathrm{Tr} \mathcal{F}^3 = 1 - N^2. \quad (5.37)$$

while other coefficients remain to be the same. These are exactly the contribution of a chiral multiplet in the adjoint representation of $SU(N)$ with $\mathcal{F} = -1$ and $R_0 = 1$.

$\mathcal{N} = 1$ Argyres-Seiberg dual theory The dual theory is an $\mathcal{N} = 1$ $SU(N)$ gauge theory coupled to the Fan specified by Y_{min} and to the T_N theory where a adjoint chiral multiplet is attached to a maximal puncture. By the class \mathcal{S} interpretation of the anomaly coefficients, it is almost trivial to see that the anomalies of this dual theory agree with those of the SQCD, because they are represented by the same Riemann surface. Actually, we already show above that the anomaly coefficients of the Fan satisfies the formulae, and that attaching an adjoint field is interpreted as changing the sign of the puncture. Also, the anomalies of the T_N theory itself are written by using (5.21): $n_v^{T_N} = \frac{2N^3}{3} - \frac{3N^2}{2} - \frac{N}{6} + 1$, $n_h^{T_N} = \frac{2N^3}{3} - \frac{2N}{3}$.

For the anomaly coefficients of non-Abelian symmetries, we use the result of the contribution of the Fan and the T_N theory ($k^{T_N} = 2N$) to the flavor central charge (5.22). It is easy to show that these cancel for $\mathrm{Tr} R T_g^2$ and $\mathrm{Tr} \mathcal{F} T_g^2$. For the anomalies involving the flavor $SU(N)$, the Fan part does not contribute, thus reproducing (5.32) upon using (5.22).

6 Superconformal index

In this section, we compute the superconformal indices of the $\mathcal{N} = 1$ class \mathcal{S} theories. The two-dimensional generalized TQFT structure ensures the invariance of index under various duality we described. The generalized TQFT structure of the $\mathcal{N} = 1$ class \mathcal{S} theories has been shown in [8], generalizing the $\mathcal{N} = 2$ case studied in [4, 40, 44, 45]. In [9, 12], the prescription of adding adjoint chiral field for oppositely colored punctures has been shown to be consistent with the generalized TQFT structure of the $\mathcal{N} = 1$ class \mathcal{S} theories. In this section, we show that the matter content and charges of the Fan can be obtained by assuming the TQFT structure.

6.1 Review

The superconformal index for $\mathcal{N} = 1$ theories are defined as

$$I(\mathfrak{p}, \mathfrak{q}, \xi; \mathbf{z}) = \text{Tr}(-1)^F \mathfrak{p}^{j_1+j_2+R_0/2} \mathfrak{q}^{j_2-j_1+R_0/2} \xi^{\mathcal{F}} \prod_i z_i^{F_i}, \quad (6.1)$$

where j_1, j_2 are the Cartans of the Lorentz group $SU(2)_1 \times SU(2)_2$ and F_i are generators of flavor symmetries. Strictly speaking, R_0 in the index has to be the exact R -charge in the IR. However in our case, we can simply keep it as R_0 , as long as we keep the fugacity ξ turned on. After determining the amount of mixing through a-maximization, we can simply replace $\xi \rightarrow \xi(\mathfrak{p}\mathfrak{q})^{\epsilon/2}$ to get the correct R -charge $R = R_0 + \epsilon\mathcal{F}$.

A good thing about the superconformal index is that it can be computed purely in terms of the matter content in the UV. The contribution for a chiral multiplet in a representation Λ of certain flavor or gauge group is given by

$$I_{\text{chi}}(\mathfrak{p}, \mathfrak{q}, \xi; \mathbf{z}) = \text{PE} \left[\frac{(\mathfrak{p}\mathfrak{q})^{R_0/2} \xi^{\mathcal{F}} \chi_{\Lambda}(\mathbf{z}) - (\mathfrak{p}\mathfrak{q})^{1-R_0/2} \xi^{-\mathcal{F}} \chi_{\bar{\Lambda}}(\mathbf{z})}{(1-\mathfrak{p})(1-\mathfrak{q})} \right], \quad (6.2)$$

where $\chi_{\Lambda}(z)$ is the character of the representation Λ . The R_0 is the R_0 -charge of the scalar in the chiral multiplet.

The chiral multiplet index (6.2) can be written in terms of elliptic Gamma function as

$$I_{\text{chi}}(\mathfrak{p}, \mathfrak{q}, \xi; \mathbf{z}) = \prod_{v \in \Lambda} \Gamma((\mathfrak{p}\mathfrak{q})^{R_0/2} \xi^{\mathcal{F}} \mathbf{z}^v; \mathfrak{p}, \mathfrak{q}), \quad (6.3)$$

where Λ is the weight lattice of the representation. We use the notation $\mathbf{z}^v = \prod_i z_i^{v_i}$. Also,

$$\Gamma(z; \mathfrak{p}, \mathfrak{q}) = \prod_{m, n \geq 0} \frac{1 - z^{-1} \mathfrak{p}^{m+1} \mathfrak{q}^{n+1}}{1 - z \mathfrak{p}^m \mathfrak{q}^n}, \quad (6.4)$$

is the elliptic Gamma function. For a vector multiplet, it contributes

$$I_{\text{vec}}^{\mathcal{N}=1}(\mathfrak{p}, \mathfrak{q}; \mathbf{z}) = ((\mathfrak{p}; \mathfrak{p})(\mathfrak{q}; \mathfrak{q}))^r \prod_{\alpha \in \Delta(G)} \Gamma(\mathbf{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})^{-1}, \quad (6.5)$$

where $\Delta(G)$ is the set of all roots for the gauge group G and r being the rank of G and $(z; q) \equiv \prod_{m=0}^{\infty} (1 - zq^m)$ is the q -Pochhammer symbol.

Generalized TQFT structure of the index The superconformal index of a class \mathcal{S} theory given by a UV curve can be written in terms of pair-of-pants (or three-punctured sphere) and cylinders connecting them. For a pair-of-pants (or three-punctured sphere) with maximal punctures with colors σ_i , we can write the index as

$$I_{(\sigma, \sigma_i)}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \sum_{\lambda} C_{\lambda}^{\sigma}(\mathfrak{p}, \mathfrak{q}, \xi) \psi_{\lambda}^{\sigma_1}(\mathbf{a}_1) \psi_{\lambda}^{\sigma_2}(\mathbf{a}_2) \psi_{\lambda}^{\sigma_3}(\mathbf{a}_3) , \quad (6.6)$$

for the σ -colored pair-of-pants. Here the sum is over the representations λ of Γ labeling the class \mathcal{S} theory. The functions $C_{\lambda}^{\sigma}(\mathfrak{p}, \mathfrak{q}, \xi)$ and $\psi_{\lambda}^{\sigma_i}(\mathbf{a}_i)$ are called the ‘structure constant’ and the wave function of the TQFT respectively. We omitted its dependence on $(\mathfrak{p}, \mathfrak{q}, \xi)$. One of the key relation we use for the wave function is

$$\psi_{\lambda}^{\sigma}(\mathbf{a}) = M^{\sigma}(\mathbf{a}) \psi_{\lambda}^{-\sigma}(\mathbf{a}) , \quad (6.7)$$

where

$$M^{\sigma}(\mathbf{a}) = \text{PE} \left[\frac{(\xi^{\sigma} - \xi^{-\sigma}) \sqrt{\mathfrak{p}\mathfrak{q}}}{(1 - \mathfrak{p})(1 - \mathfrak{q})} \chi_{\text{adj}}(\mathbf{a}) \right] = \Gamma(t_{\sigma}; \mathfrak{p}, \mathfrak{q})^{N-1} \prod_{i \neq j} \Gamma(t_{\sigma} a_i a_j^{-1}; \mathfrak{p}, \mathfrak{q}) , \quad (6.8)$$

where $t_{\sigma} = \xi^{\sigma} \sqrt{\mathfrak{p}\mathfrak{q}}$. It was shown in [8] that this wave function is essentially determined by the $\mathcal{N} = 2$ counterpart, which is given by an eigenfunction of elliptic Ruijsenaars-Schneider model [45]. More precisely, the relation between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ version of the wave function is given as

$$\psi_{\lambda}^{\sigma}(\mathbf{a}; \mathfrak{p}, \mathfrak{q}, \xi) = \psi_{\lambda}(\mathbf{a}; \mathfrak{p}, \mathfrak{q}, t = t_{\sigma}) . \quad (6.9)$$

Also, the structure constant can be simply fixed from that of the $\mathcal{N} = 2$ counterpart as $C_{\lambda}^{\sigma}(\mathfrak{p}, \mathfrak{q}, \xi) = C_{\lambda}(\mathfrak{p}, \mathfrak{q}, t = \xi^{\sigma} \sqrt{\mathfrak{p}\mathfrak{q}}) = \psi_{\lambda}(t_{\sigma}^{\rho}; \mathfrak{p}, \mathfrak{q}, t = t_{\sigma})^{-1}$ where ρ is the Weyl vector of the group Γ .

The wave function can be written as $\psi_{\lambda}^{\sigma}(\mathbf{a}) = K^{\sigma}(\mathbf{a}) \Psi_{\lambda}(\mathbf{a})$ where $K^{\sigma}(\mathbf{a})$ is a prefactor which does not depend on λ and $\Psi_{\lambda}(\mathbf{a})$ is another function which depends on the representations λ of the group Γ . The prefactor is given by

$$K^{\sigma}(\mathbf{a}) = \text{PE} \left[\frac{\xi^{\sigma} \sqrt{\mathfrak{p}\mathfrak{q}} - \mathfrak{p}\mathfrak{q}}{(1 - \mathfrak{p})(1 - \mathfrak{q})} \chi_{\text{adj}}(\mathbf{a}) \right] . \quad (6.10)$$

Note that the function Ψ_{λ} does not depend on color σ . In terms of these functions, we can write the index for a three-punctured sphere as

$$I_{(\sigma, \sigma_i)}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \frac{K^{\sigma_1}(\mathbf{a}_1) K^{\sigma_2}(\mathbf{a}_2) K^{\sigma_3}(\mathbf{a}_3)}{K^{\sigma}(t_{\sigma}^{\rho})} \sum_{\lambda} \frac{\Psi_{\lambda}(\mathbf{a}_1) \Psi_{\lambda}(\mathbf{a}_2) \Psi_{\lambda}(\mathbf{a}_3)}{\Psi_{\lambda}(t_{\sigma}^{\rho})} . \quad (6.11)$$

When we glue the pair-of-pants by a gauge group, we integrate over the gauge fugacities with vector multiplet measure. Since we have

$$\int [da] I_{\text{vec}}^{\sigma\sigma'}(p, q; \mathbf{z}) \psi_{\lambda}^{\sigma}(\mathbf{z}) \psi_{\lambda'}^{\sigma'}(\mathbf{z}) = \delta_{\lambda\lambda'} , \quad (6.12)$$

where $I_{\text{vec}}^{\sigma\sigma'}(\mathbf{p}, \mathbf{q}; \mathbf{z})$ is $\mathcal{N} = 2$ vector multiplet when $\sigma = \sigma'$ and $\mathcal{N} = 1$ otherwise, we can write the superconformal index for any UV curve with colored full punctures as

$$I(\mathbf{a}_i, \mathbf{b}_j; \mathbf{p}, \mathbf{q}, \xi) = \sum_{\lambda} \frac{\prod_{i=1}^{n_+} \psi_{\lambda}^+(\mathbf{a}_i) \prod_{j=1}^{n_-} \psi_{\lambda}^-(\mathbf{b}_j)}{(\psi_{\lambda}^+(t_+^{\rho}))^p (\psi_{\lambda}^-(t_-^{\rho}))^q}, \quad (6.13)$$

where (n_+, n_-) are the number of punctures of each colors, and (p, q) are the degrees of the normal bundles satisfying $2g - 2 + (n_+ + n_-) = p + q$ and $\mathbf{a}_i, \mathbf{b}_j$ are the flavor fugacities. As we see clearly, the index only depends on the topological data.

Now, if we choose the punctures to be non-maximal, we replace the fugacities in the wave function appropriately. The prescription is to replace

$$\Psi_{\lambda}^{\sigma}(\mathbf{a}) \rightarrow \Psi_{\lambda}^{\sigma}(\mathbf{u}t_{\sigma}^{\Lambda}), \quad K^{\sigma}(\mathbf{a}) \rightarrow K_{\Lambda}^{\sigma}(\mathbf{u}) = \text{PE} \left[\sum_j \frac{t_{\sigma}^{1+j} - \mathbf{p}\mathbf{q}t_{\sigma}^j}{(1-\mathbf{p})(1-\mathbf{q})} \chi_{R_j}(\mathbf{u}) \right], \quad (6.14)$$

for a puncture labelled by the $SU(2) \hookrightarrow \Gamma$ embedding Λ which decomposes $\text{adj} \rightarrow \bigoplus_j R_j \otimes V_j$ where R_j is the representation of the commutant of $\Lambda(SU(2))$ in Γ and V_j is the spin- j representation of $SU(2)$. The notation $\mathbf{u}t^{\Lambda}$ means replacing the flavor fugacity appropriately in accordance with the broken flavor symmetry. See [46] for a detailed discussion on this notation and its physical meaning. We will give an example in the section 6.2, and then the full expression in 6.3.

6.2 $\mathcal{N} = 1$ Argyres-Seiberg duality

The agreement of the index for the Argyres-Seiberg duality can be checked using the TQFT language as done in [9, 12]. In the SQCD frame as in figure 19 or figure 20a, the index can be written as

$$I(\mathbf{x}_1, \mathbf{x}_2, a, b) = \frac{K_{\star}^{-}(a)K^{-}(\mathbf{x}_1)K_{\star}^{+}(b)K^{+}(\mathbf{x}_2)}{K_{\varnothing}^{-}K_{\varnothing}^{+}} \sum_{\lambda} \frac{\Psi_{\lambda}(at_{\star}^{-})\Psi_{\lambda}(\mathbf{x}_1)\Psi_{\lambda}(bt_{\star}^{+})\Psi_{\lambda}(\mathbf{x}_2)}{\Psi_{\lambda}(t_{\varnothing}^{-})\Psi_{\lambda}(t_{\varnothing}^{+})}, \quad (6.15)$$

where \star denotes the embedding associated to the minimal puncture. Here, all the $+$ colored contributions are coming from the functions with $+$ labels and vice versa since the color of the pair-of-pants is the same as the punctures. Here we denote fugacities of the flavor symmetry $SU(3)_1 \times SU(3)_2 \times U(1)_A \times U(1)_B$ to be $\mathbf{x}_1, \mathbf{x}_2, a, b$ respectively.

Now, we need to show that this index is the same in the Argyres-Seiberg frame as in the figure 20d. There, we have punctures with different color from the pair-of-pants. On the left-side of the figure, we have maximal punctures with each color and thus an adjoint chiral field N . On the right-hand side, we have two minimal punctures with each color, corresponding to an adjoint field but with a nilpotent vev imposed, giving a number of components to survive according to the $SU(2)$ embedding labelled by the puncture. In the case of generic ρ being $\text{adj} \rightarrow \bigoplus_j R_j \otimes V_j$, this contribution to the index is

$$M_{\rho}^{\sigma}(\mathbf{u}) = \text{PE} \left[\sum_j \frac{t_{\sigma}^{1+j} - \mathbf{p}\mathbf{q}t_{\sigma}^{-1-j}}{(1-\mathbf{p})(1-\mathbf{q})} \chi_{R_j}(\mathbf{u}) \right]. \quad (6.16)$$

This is coming from the shift of R -charge $R_0 \rightarrow R_0 + 2\rho(\sigma_3)$ under the Higgsing. Thus M_\star^+ represents the components appearing in the dual frame.

The index in the Argyres-Seiberg dual frame can be written as

$$M^-(\mathbf{x}_1)M_\star^+(b)\frac{K_\star^-(a)K^-(bt_+^\star)K^+(\mathbf{x}_1)K^+(\mathbf{x}_2)}{K_\emptyset^+K_\emptyset^-}\sum_\lambda\frac{\Psi_\lambda(at_-^\star)\Psi_\lambda(bt_-^\star)\Psi_\lambda(\mathbf{x}_1)\Psi_\lambda(\mathbf{x}_2)}{\Psi_\lambda(t_+^\emptyset)\Psi_\lambda(t_-^\emptyset)}. \quad (6.17)$$

The first two terms are coming from the additional fields in the dual theory and the signs of K^σ s are determined by the color of the pair-of-pants. From the identity [9]

$$M_\Lambda^\sigma(\mathbf{u})K^{-\sigma}(\mathbf{u}t_\sigma^\Lambda) = K_\Lambda^\sigma(\mathbf{u}), \quad (6.18)$$

we see that the (6.15) and (6.17) are equal. This shows consistency of the TQFT description of the superconformal index of class \mathcal{S} theories.

This agreement from the TQFT was quite formal, and works for any kind of puncture. We should be able to calculate this index in the Argyres-Seiberg dual frame using the matter content we find in the previous section. This can be done by looking at the index of the unhiggsed theory and Higgsing to get the Argyres-Seiberg frame. Let us consider the Argyres-Seiberg frame before Higgsing the dual meson M , which is realized by two maximal punctures on the left with each colors, and one maximal puncture with $+$ color and one minimal puncture with $-$ color as in the figure 22. This realizes T_N theory with one of $SU(N)$ gauged by

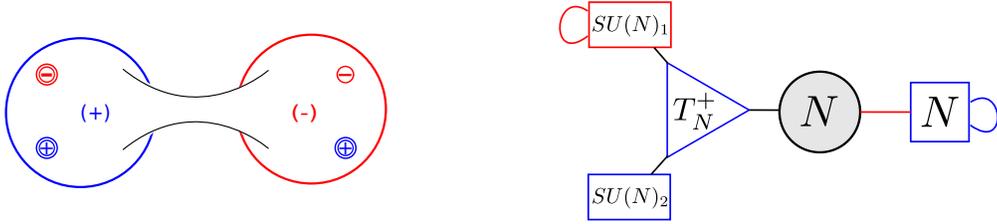


Figure 22: Unhiggsed SQCD in the Argyres-Seiberg frame

$\mathcal{N} = 1$ vector multiplet, and N fundamentals attached to it. We also have an adjoint field N associated to one of $SU(N)$ flavor symmetries on the T_N side, and another adjoint field M attached to the fundamentals. The index of this theory can be written as

$$I(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}, b) = \oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \Delta(\mathbf{z}) I_{\text{vec}}^{\mathcal{N}=1}(\mathbf{z}) I_{T_N}^+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}) \prod_{i,j=1}^N \Gamma(t_-^{\frac{1}{2}}(z_i y_j b)^\pm) \quad (6.19)$$

$$\times \left(\Gamma(t_-)^{N-1} \prod_{i \neq j} \Gamma(t_- x_{1,i} x_{1,j}^{-1}) \right) \left(\Gamma(t_+)^{N-1} \prod_{i \neq j} \Gamma(t_+ y_i y_j^{-1}) \right),$$

where we used the short-hand notation $\Gamma(z) = \Gamma(z; \mathbf{p}, \mathbf{q})$, and $\prod_{i=1}^N z_i = 1$ is assumed. The symbol I_{T_N} refers to the index of the T_N block and $I_{\text{vec}}^{\mathcal{N}=1}$ is the $\mathcal{N} = 1$ vector multiplet contribution to the index and $\Delta(\mathbf{z})$ is the Haar measure of the gauge group. The last term

in the first line is the contribution from the fundamental quarks with $R_0 = \frac{1}{2}, \mathcal{F} = -\frac{1}{2}$. The second line corresponds to the contributions from the fields N and M respectively.

Now, upon Higgsing, we specialize the fugacity \mathbf{y} with the ones determined from the $SU(2)$ embedding $N \rightarrow (N-1)+1$. In our case, we put $\mathbf{y} = (at_+^{\frac{N}{2}-1}, at_+^{\frac{N}{2}-2}, \dots, at_+^{1-\frac{N}{2}}, a^{-N+1})$. Then the last term in the first line of (6.19) becomes

$$\prod_{i=1}^N \left[\left(\prod_{m=1}^{N-1} \Gamma((\mathbf{p}\mathbf{q})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a (\xi \sqrt{\mathbf{p}\mathbf{q}})^{\frac{N}{2}-m} b)^{\pm}) \right) \Gamma((\mathbf{p}\mathbf{q})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a^{-N+1} b)^{\pm}) \right], \quad (6.20)$$

where the terms in the parenthesis can be written as

$$\prod_{m=1}^{N-1} \Gamma((\mathbf{p}\mathbf{q})^{\frac{1+N-2m}{4}} \xi^{-\frac{-1+N-2m}{2}} z_i ab) \prod_{m'=1}^{N-1} \Gamma((\mathbf{p}\mathbf{q})^{\frac{1-N+2m'}{4}} \xi^{-\frac{-1-N+2m'}{2}} (z_i ab)^{-1}). \quad (6.21)$$

Due to the identity $\Gamma(z; \mathbf{p}, \mathbf{q}) \Gamma(\frac{\mathbf{p}\mathbf{q}}{z}; \mathbf{p}, \mathbf{q}) = 1$, all the terms with $m = m' - 1$ are cancelled. The only remaining terms are the ones with $(m, m') = (N-1, 1)$. Therefore, (6.20) can be written as

$$\prod_{i=1}^N \Gamma((\mathbf{p}\mathbf{q})^{\frac{3-N}{4}} \xi^{\frac{1-N}{2}} (z_i ab)^{\pm}) \Gamma((\mathbf{p}\mathbf{q})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a^{-N+1} b)^{\pm}), \quad (6.22)$$

which is the contribution from the quarks of $(J_+, J_-) = (2-N, 1), (0, 1)$ or $(R_0, \mathcal{F}) = (\frac{3-N}{2}, \frac{1-N}{2}), (1, -1)$. We see that the index can be used to extract the matter content and the charges of the Higgsed theory.

Contribution from M upon Higgsing is determined through the minimal $SU(2)$ embedding

$$\text{adj} \rightarrow \left(\bigoplus_{m=1}^{N-1} V_{m-1}^0 \right) \oplus V_{\frac{N-2}{2}}^{-N} \oplus V_{\frac{N-2}{2}}^N, \quad (6.23)$$

where the supersubscript means the charge of the commuting $U(1)$. From this, we get

$$M_{\star}^+(a) = \text{PE} \left[\sum_{m=1}^{N-1} \frac{(\mathbf{p}\mathbf{q})^{\frac{m}{2}} \xi^m - (\mathbf{p}\mathbf{q})^{1-\frac{m}{2}} \xi^{-m}}{(1-\mathbf{p})(1-\mathbf{q})} + \frac{(\mathbf{p}\mathbf{q})^{\frac{N}{4}} \xi^{\frac{N}{2}} - (\mathbf{p}\mathbf{q})^{1-\frac{N}{4}} \xi^{-\frac{N}{2}}}{(1-\mathbf{p})(1-\mathbf{q})} (a^N + a^{-N}) \right]. \quad (6.24)$$

From here, we see that we have mesons with $(J_+, J_-) = (2m, 0)$ or $(R_0, \mathcal{F}) = (m, m)$ with $m = 1, \dots, N-1$ and two mesons with $(J_+, J_-) = (N, 0)$ or $(R_0, \mathcal{F}) = (N/2, N/2)$ which are exactly the same as that of our result in the section 4.2.

6.3 Index of the Fan

We can repeat the similar procedure for a generic Fan as in the section 6.2. Consider a partition of N given by $\sum_{k=1}^{\ell} kn_k$ labelled by a Young diagram Y . For this partition, the flavor fugacity for the puncture is given as

$$\begin{aligned} \mathbf{u} t_{\sigma}^{\Lambda} &= (\mathbf{u}_1 t_{\sigma}^{\Lambda_1}, \mathbf{u}_2 t_{\sigma}^{\Lambda_2}, \dots, \mathbf{u}_{\ell} t_{\sigma}^{\Lambda_{\ell}}), \\ \mathbf{u}_k t_{\sigma}^{\Lambda_k} &= (\mathbf{u}_k t_{\sigma}^{\frac{k-1}{2}}, \mathbf{u}_k t_{\sigma}^{\frac{k-3}{2}}, \dots, \mathbf{u}_k t_{\sigma}^{\frac{1-k}{2}}), \end{aligned} \quad (6.25)$$

where $\mathbf{u}_k = (u_{k,1}, u_{k,2}, \dots, u_{k,n_k})$ is a n_k -dimensional vector for the $U(n_k)$ fugacities. We also impose the condition $\prod_{k=1}^{\ell} \prod_{i=1}^{n_k} u_{k,i} = 1$ so that the flavor symmetry is $S \left[\prod_{k=1}^{\ell} U(n_k) \right]$.

Plugging (6.25) into the index formula for N fundamental quarks, we get

$$\prod_{\alpha,\beta=1}^N \Gamma(\xi^{-1/2}(\mathbf{pq})^{1/4}(z_{\alpha}y_{\beta}b)^{\pm}) \rightarrow \prod_{\alpha=1}^N \prod_{k=1}^{\ell} \prod_{i=1}^{n_k} \prod_{m=1}^k \Gamma(\xi^{-1/2}(\mathbf{pq})^{1/4}(z_{\alpha}u_{k,i}t_{+}^{\frac{k-2m+1}{2}}b)^{\pm}), \quad (6.26)$$

where we assumed that the Fan is of the type $\sigma = -$ as in the previous example for simplicity. As in the section 6.2, we see cancellations among upon Higgsing. The above equation can be written as

$$\prod_{\alpha=1}^N \prod_{k=1}^{\ell} \prod_{i=1}^{n_k} \left[\prod_{m=1}^k \Gamma(\xi^{\frac{k-2m}{2}}(\mathbf{pq})^{\frac{2+k-2m}{4}}z_{\alpha}u_{k,i}) \prod_{m'=1}^k \Gamma(\xi^{\frac{-2+2m'-k}{2}}(\mathbf{pq})^{\frac{2m'-k}{4}}(z_{\alpha}u_{k,i})^{-1}) \right]. \quad (6.27)$$

We can see that the terms with $m' = m + 1$ is cancelled so that only terms with $m = k$, $m' = 1$ contribute. Therefore, we get

$$I^{\text{quarks}}(\mathbf{z}, \mathbf{u}) = \prod_{\alpha=1}^N \prod_{k=1}^{\ell} \prod_{i=1}^{n_k} \Gamma(\xi^{-\frac{k}{2}}(\mathbf{pq})^{\frac{2-k}{4}}(z_{\alpha}u_{k,i}b)^{\pm}), \quad (6.28)$$

which is the contribution from the quarks of desired charges $(J_+, J_-) = (1-k, 1)$ or $(R_0, \mathcal{F}) = (\frac{2-k}{2}, -\frac{k}{2})$.

The contribution from the adjoint fields are given as (6.16). In the current case, the adjoint representation will decompose in to the form written as (3.10). Therefore, the index for the resulting components can be written as

$$M_Y^{\sigma}(\mathbf{u}) = \prod_{i < j} \prod_{k=1}^i \text{PE} \left[\frac{t_{\sigma}^{\frac{1}{2}(j-i+2k)} - (\mathbf{pq})t_{\sigma}^{-\frac{1}{2}(j-i+2k)}}{(1-\mathbf{p})(1-\mathbf{q})} \left(\chi_{R_i}(\mathbf{u}_i)\chi_{\bar{R}_j}(\mathbf{u}_j) + \chi_{\bar{R}_i}(\mathbf{u}_i)\chi_{R_j}(\mathbf{u}_j) \right) \right] \\ \times \left(\prod_{i=1}^{\ell} \prod_{k=1}^i \text{PE} \left[\frac{t_{\sigma}^k - (\mathbf{pq})t_{\sigma}^{-k}}{(1-\mathbf{p})(1-\mathbf{q})} \chi_{\text{adj}}^{U(n_i)}(\mathbf{u}_i) \right] \right) \times \text{PE} \left[\frac{t_{\sigma} - \mathbf{pq}t_{\sigma}^{-1}}{(1-\mathbf{p})(1-\mathbf{q})} \right]^{-1}, \quad (6.29)$$

where the first term is coming from the bifundamentals of $U(n_i) \times U(n_j)$ and the second term is coming from the adjoints of $U(n_i)$ and the last piece takes care of the traceless condition. One can rearrange the first term by taking $i \rightarrow k - p$ so that the R -charges are given by $(R_0, \mathcal{F}) = \frac{1}{2}(i + j - 2p, i + j - 2p)$ with $p = 0, \dots, \min(i, j) - 1$. Likewise, the second term gives the adjoint fields of charge $(R_0, \mathcal{F}) = (i - p, i - p)$ with $p = 0, \dots, i - 1$ which agrees with the charges of the table 1.

Therefore, we find all the matter fields and charges as given in the table 1 for $N' = 0$ case. Now, the index can be written in a contour integral form as

$$I(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}, \mathbf{u}) = M^{-}(\mathbf{x}_1) \int \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \Delta(\mathbf{z}) I_{\text{vec}}(\mathbf{z}) I_{T_N}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}) I^{\text{quarks}}(\mathbf{z}, \mathbf{u}) M_Y^{\sigma}(\mathbf{u}), \quad (6.30)$$

where $I^{\text{quarks}}(\mathbf{z}, \mathbf{u})$ and $M_Y^{\sigma}(\mathbf{u})$ are given by (6.28) and (6.29) respectively, representing the components of the Fan.

Contour of the index integral Let us make one comment on the integration contour of equation (6.17) and (6.30). Normally, for the purpose of evaluating the superconformal index, it is assumed that $|\mathbf{p}|, |\mathbf{q}| < 1$ and all the flavor fugacities to be unimodular $|a| = 1$. But if we use this prescription in the current case, we may hit a pole along the contour of integration. Therefore we need to find a good contour to get away with this situation, because the usual contour of integration is not well-defined. Higgsing gives an answer to this question. Prior to Higgsing, all the flavor fugacities were assumed to be unimodular. Typically, the poles are of the form $z = a(\mathbf{pq})^{r/2}\mathbf{p}^m\mathbf{q}^n$ with $m, n \in \mathbb{Z}$ and R -charge of the chiral multiplet being $r > 0$. Therefore we pick all the poles with $m, n \geq 0$.

But when we Higgs, the flavor fugacities are dressed with \mathbf{p}, \mathbf{q} and quite often it contributes negative powers in \mathbf{pq} . This will push some of the poles with $m = n = 0$ to be outside of the unit circle. As we have seen in the previous paragraph, due to the cancellation among the integrands, some of the poles are gone and the remaining poles under the Higgsing are those coming from the quarks in the Fan. Note that all the Higgsed flavor fugacities ut_σ^Λ have to be unimodular. Even though superficially the poles appear to be outside of the unit circle, it is actually $x(\mathbf{pq})^{\frac{r-1}{2}}$ that has modulus 1 with x being the flavor fugacity. Therefore, we have to include all the poles of the form $z = x(\mathbf{pq})^{r/2}$ even for negative or zero r .

7 Conclusion

We studied nilpotent Higgsing in $\mathcal{N} = 1$ linear quivers of class \mathcal{S} . In the case of $\mathcal{N} = 2$ theories such Higgsings yield regular punctures that can be associated to quiver tails labelled by partitions of N . Surprisingly, In $\mathcal{N} = 1$ linear quivers, such Higgsing yields a new type of quiver dubbed as the Fan. This object is labelled by two integers N and N' , and a partition of $N - N'$. We provide further evidence of the Fan by ‘‘Higgsing’’ the superconformal index.

Armed with the Fan, we construct many new SCFTs. These provide various field theoretic descriptions of M5-branes wrapped on punctured Riemann surfaces. Under Seiberg duality, quivers with Fans will transform to new quivers with different Fans. Geometrically, this corresponds to different colored pair-of-pants decomposition of Riemann surface. Using the Fan, we find a new dual frame of $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors which is analogous to the Argyres-Seiberg duality. This dual frame is described by a T_N theory coupled to the Fan and chiral multiplets.

In our discussion, we only considered the UV curve with locally $\mathcal{N} = 2$ regular punctures. In $\mathcal{N} = 1$ class \mathcal{S} theories, one could have much more general punctures. In terms of generalized Hitchin system [10], we only considered the case where only one of two Hitchin fields become singular at a given puncture. It should be possible to consider the case where two Hitchin fields have singularities at the same point. This will yield genuinely $\mathcal{N} = 1$ punctures that we expect to be associated with a variation of the Fans. This is work in progress.

We hope to find an intersecting brane realization of these new SCFTs in type IIA string theory, which can be uplifted to M-theory. It will also be interesting to find a gravity dual

description of the Fan and its variations in M-theory by using the system of [13]. This is also work in progress.

In this paper, we have not studied the detailed phase structure of the theory. The spectral curve approach from the generalised Hitchin system as done in [18, 19] should be useful. It would be also interesting to identify the Fan for the D, E -type theories of class \mathcal{S} , also possibly with outer-automorphism twists.

Acknowledgments

We would like to thank Rodrigo Alonso, Nikolay Bobev, Abhijit Gadde, Ken Intriligator, Jack Kearney, John McGreevy, Yuji Tachikawa, Hagen Triendl, Nick Warner, Brian Wecht and Dan Xie for various helpful discussions. We especially thank Nikolay Bobev who participated in the early stage of this work. IB is grateful for the hospitality and work space provided by the UCSD Physics Department. KM, JS thank the hospitality of Simons Center for Geometry and Physics and the organizers of the 2014 Summer Simons Workshop in Mathematics and Physics. PA, IB, JS thank the hospitality of KITP and the organizers of the program New Methods in Nonperturbative Quantum Field Theory. PA thanks the organizers of PiTP-2014 for their kind hospitality. IB and JS also thank the Perimeter Institute for Theoretical Physics for the hospitality. KM would like to thank the organizers of Kavli IPMU-FMSP workshop “Supersymmetry in Physics and Mathematics” for hospitality. KM would like to thank Nagoya University, Osaka City University, Rikkyo University, University of Tokyo, Komaba and Hongo, and Yukawa Institute for Theoretical Physics, for hospitality. JS thanks the organizers of “Exact Results in SUSY Gauge Theories in Various Dimensions” at CERN and also CERN-Korea Theory Collaboration funded by National Research Foundation (Korea), for the hospitality and support. The work of PA and JS is supported by DOE grant DOE-FG03-97ER40546. IB is supported in part by the DOE grant DE-FG03-84ER-26-40168, ANR grant 08-JCJC-0001-0, and the ERC Starting Grants 240210-String-QCD-BH, and 259133-ObservableString. The work of KM is supported by a JSPS fellowship for research abroad.

A The superpotential for the Fan

We now derive the superpotential that is obtained after integrating out the massive modes in section 3.4. Before integrating these out, the superpotential is given by

$$W_1 = \text{Tr} q_0 \rho^+ \tilde{q}_0 + \text{Tr} q_0 M \tilde{q}_0 + \text{Tr} \tilde{\mu}_1 q_0 \tilde{q}_0 , \quad (\text{A.1})$$

where $\rho^\pm = \rho(\sigma^\pm)$. Here we write only those terms in the superpotential that are relevant to Higgsing. Recall that ρ^+ here, is also the raising operator for the $SU(2)$ embedding specified by the partition of N . Also, $\tilde{\mu}_1$ is the quark bilinear given by $\tilde{\mu}_1 = \tilde{q}_1 q_1 - \frac{1}{N} \text{Tr} q_1 \tilde{q}_1$.

Let P and \tilde{P} , be the projection matrices that project on to the massive modes of q_0 and \tilde{q}_0 respectively i.e.

$$\begin{aligned}\chi &= q_0 P , \\ \tilde{\chi} &= \tilde{P} \tilde{q}_0 ,\end{aligned}\tag{A.2}$$

where χ and $\tilde{\chi}$ represent the massive chiral fields. It is easy to check that

$$\begin{aligned}\tilde{P} &= \rho^- \rho^+ , \\ P &= \rho^+ \rho^- .\end{aligned}\tag{A.3}$$

These projection operators satisfy $\tilde{P}\tilde{P} = \tilde{P}$ and $PP = P$ as is expected. The massless modes are given by $Z = q_0(\mathbb{1} - P)$ and $\tilde{Z} = q_0(\mathbb{1} - \tilde{P})$.

The superpotential in (A.1) can now be expanded in terms of the massive and massless modes, such that the equation of motion for χ can be written as

$$\rho^+ \tilde{\chi} + M \tilde{\chi} + \tilde{\chi} \tilde{\mu}_1 + M \tilde{Z} + \tilde{Z} \tilde{\mu}_1 = 0 .\tag{A.4}$$

Note that since $\tilde{\mu}_1$ in the above equation is contracted through the color indices, therefore it can be treated as a scalar multiplier in the above equation. This equation of motion can be simplified by multiplying it on the left with ρ^- reducing it to the following form

$$\tilde{\chi} + \rho^- M \tilde{\chi} + \rho^- \tilde{\chi} \tilde{\mu}_1 + \rho^- M \tilde{Z} + \rho^- \tilde{Z} \tilde{\mu}_1 = 0 .\tag{A.5}$$

The solution for $\tilde{\chi}$ is

$$\tilde{\chi} = (\mathbb{1} - A)^{-1} A \tilde{Z} ,\tag{A.6}$$

where

$$A = -(\rho^- M + \tilde{\mu}_1 \rho^-) .\tag{A.7}$$

Recall that here we are treating $\tilde{\mu}_1$ as a scalar multiplier and will appropriately contract it using its color indices at a later stage. Notice that A is a nilpotent matrix such that $A^\ell = 0$. This follows from the fact that $A^\ell \propto (\rho^-)^\ell (M + \tilde{\mu}_1 \mathbb{1})^\ell$ and $(\rho^-)^\ell = 0$ since it is the lowering operator of $SU(2) \hookrightarrow SU(N)$. Here we have also used the commutation relation $[\rho^-, M] = 0$ which is due to the elements of M being in the lowest weight state of their respective $SU(2)$ representations. Thus

$$\tilde{\chi} = \sum_{n=1}^{\ell-1} A^n \tilde{Z} .\tag{A.8}$$

Substituting this back in (A.1) we find that the low energy superpotential is

$$W_{\text{eff}} = \text{Tr} Z \tilde{Z} \tilde{\mu}_1 + \text{Tr} Z M \tilde{Z} + \sum_{n=1}^{\ell-1} \text{Tr} Z M A^n \tilde{Z} + \sum_{n=1}^{\ell-1} \text{Tr} Z A^n \tilde{Z} \tilde{\mu}_1 .\tag{A.9}$$

An example for the $SU(6)$ quiver As an example of our previous derivation, let us study the nilpotent Higgsing of the linear quiver with $SU(6)$ symmetries. Consider the partition $6 \rightarrow 3 + 2 + 1$. This implies

$$\langle M_0 \rangle = \rho^+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.10})$$

The components of the (anti-)quark matrices can be written as

$$q_0 = \begin{pmatrix} \chi_1 & \chi_2 & Z_3 & \chi_3 & Z_2 & Z_1 \end{pmatrix} \quad \text{and} \quad \tilde{q}_0 = \begin{pmatrix} \tilde{Z}_3 \\ \tilde{\chi}_1 \\ \tilde{\chi}_2 \\ \tilde{Z}_2 \\ \tilde{\chi}_3 \\ \tilde{Z}_1 \end{pmatrix}.$$

with $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3, \tilde{Z}_1, \tilde{Z}_2$, and \tilde{Z}_3 being row vectors, each of which corresponds to an anti-fundamental of $SU(6)_1$; similarly, $\chi_1, \chi_2, \chi_3, Z_1, Z_2$, and Z_3 are column vectors, each of which corresponds to a fundamental of $SU(6)_1$. The vev for M gives mass to $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3, \chi_1, \chi_2$ and χ_3 . The fluctuations M (around the vev ρ^+) that stay coupled to the theory are found by using the argument in [9]. These are

$$M = \begin{pmatrix} M_{33}^2 & 0 & 0 & 0 & 0 & 0 \\ M_{33}^1 & M_{33}^2 & 0 & M_{32}^1 & 0 & 0 \\ M_{33}^0 & M_{33}^1 & M_{33}^2 & M_{32}^0 & M_{32}^1 & M_{31}^0 \\ M_{23}^1 & 0 & 0 & M_{22}^1 & 0 & 0 \\ M_{23}^0 & M_{23}^1 & 0 & M_{22}^0 & M_{22}^1 & M_{21}^0 \\ M_{13}^0 & 0 & 0 & M_{12}^0 & 0 & -(3M_{33}^2 + 2M_{22}^1) \end{pmatrix}. \quad (\text{A.11})$$

Upon integrating out the massive chiral fields and including the fluctuations (A.11), the effective superpotential becomes

$$\begin{aligned} W_{\text{eff}} = & \text{Tr} \tilde{\mu}_1 Z_1 \tilde{Z}_1 - 3 \text{Tr} Z_1 M_{33}^2 \tilde{Z}_1 - 2 \text{Tr} Z_1 M_{22}^1 \tilde{Z}_1 + \text{Tr} Z_2 M_{22}^0 \tilde{Z}_2 + \text{Tr} Z_3 M_{33}^0 \tilde{Z}_3 \\ & - \text{Tr} Z_2 \tilde{Z}_2 (\tilde{\mu}_1)^2 - 2 \text{Tr} Z_2 M_{22}^1 \tilde{Z}_2 \tilde{\mu}_1 - \text{Tr} Z_2 (M_{22}^1)^2 \tilde{Z}_2 + \text{Tr} Z_3 (M_{33}^2)^3 \tilde{Z}_3 \\ & + 3 \text{Tr} Z_3 (M_{33}^2)^2 \tilde{Z}_3 \tilde{\mu}_1 + 3 \text{Tr} Z_3 M_{33}^2 \tilde{Z}_3 (\tilde{\mu}_1)^2 + \text{Tr} Z_3 \tilde{Z}_3 (\tilde{\mu}_1)^3 - 2 \text{Tr} Z_3 M_{33}^1 \tilde{Z}_3 \tilde{\mu}_1 \\ & - 2 \text{Tr} Z_3 M_{33}^1 M_{33}^2 \tilde{Z}_3 - \text{Tr} Z_3 M_{23}^1 M_{32}^1 \tilde{Z}_3 + \text{Tr} Z_1 M_{12}^0 \tilde{Z}_2 + \text{Tr} Z_1 M_{13}^0 \tilde{Z}_3 \\ & + \text{Tr} Z_2 M_{21}^0 \tilde{Z}_1 + \text{Tr} Z_2 M_{23}^0 \tilde{Z}_3 - \text{Tr} Z_2 M_{22}^1 M_{23}^1 \tilde{Z}_3 - \text{Tr} Z_2 M_{33}^2 M_{23}^1 \tilde{Z}_3 - 2 \text{Tr} Z_2 M_{23}^1 \tilde{Z}_3 \tilde{\mu}_1 \\ & + \text{Tr} Z_3 M_{31}^0 \tilde{Z}_1 + \text{Tr} Z_3 M_{32}^0 \tilde{Z}_2 - \text{Tr} Z_3 M_{22}^1 M_{32}^1 \tilde{Z}_2 - \text{Tr} Z_3 M_{33}^2 M_{32}^1 \tilde{Z}_2 - 2 \text{Tr} Z_3 M_{32}^1 \tilde{Z}_2 \tilde{\mu}_1 \\ & + \text{Tr} \mu_2 \phi + \text{Tr} \tilde{\mu}_2 \phi. \end{aligned} \quad (\text{A.12})$$

This matches exactly with what one would write for the Fan corresponding to the partition $6 \rightarrow 3 + 2 + 1$.

B Higgsing $\mathcal{N} = 2$ quiver theories

Consider the linear quiver in $\mathcal{N} = 2$ class \mathcal{S} theories of type A_{N-1} with the gauge group

$$G = \prod_{i=1}^{N-1} SU(N)_i . \quad (\text{B.1})$$

The matter content of the theory consist of hypermultiplets $H_i = (Q_i, \tilde{Q}_i)$ of $SU(N)_i \times SU(N)_{i+1}$. In addition to this we also have N hypermultiplets $H_0 = (Q_0, \tilde{Q}_0)$ transforming in the fundamental representation of $SU(N)_1$ and N hypermultiplets $H_{N-1} = (Q_{N-1}, \tilde{Q}_{N-1})$ transforming in the fundamental representation of $SU(N)_{N-1}$. Thus at each of the quiver there is an $SU(N)$ flavor symmetry acting on the hypermultiplets H_0 and H_{N-1} respectively. We denote the flavor symmetry of H_0 by $SU(N)_0$ and that of H_{N-1} by $SU(N)_N$.

In order to avoid introducing too many indices labeling the symmetries under which Q_i and \tilde{Q}_i transform, we will treat them as $N \times N$ matrices such that $Q_i \tilde{Q}_i$ will be an invariant of $SU(N)_i$ while $\tilde{Q}_i Q_i$ will be an invariant of $SU(N)_{i+1}$. Thus the superpotential of this quiver will be given by

$$W = \sqrt{2} \sum_{i=1}^{N-1} \text{Tr} \left(\tilde{Q}_{i-1} \Phi_i Q_{i-1} - Q_i \Phi_i \tilde{Q}_i \right) . \quad (\text{B.2})$$

We now wish to consider an $SU(N)$ linear quiver and Higgsing its leftmost full puncture down to a puncture given by the Young's tableau corresponding to the following partition of N

$$N = n_1 + 2n_2 + \dots + \ell n_\ell . \quad (\text{B.3})$$

This breaks $SU(N)_0$ down to $S[U(n_1) \times U(n_2) \times \dots U(n_\ell)]$. The corresponding vev for $\mu_0 = \tilde{Q}_0 Q_0 - \frac{1}{N} \text{Tr} \tilde{Q}_0 Q_0$ that does the job for us is given by

$$\langle \mu_0 \rangle = \oplus^{n_1} J_1 \oplus^{n_2} J_2 \oplus^{n_3} \dots \oplus^{n_\ell} J_\ell , \quad (\text{B.4})$$

where J_k is the Jordan cell of size k . This can then be decomposed into the following vevs for Q_0 and \tilde{Q}_0 :

$$\langle \tilde{Q}_0 \rangle = \oplus^{n_1} J_1 \oplus^{n_2} J_2 \oplus^{n_3} \dots \oplus^{n_\ell} J_\ell \quad (\text{B.5})$$

and

$$\langle Q_0 \rangle = \oplus^{n_1} J_1 \oplus^{n_2} (J_1 \oplus I_1) \oplus^{n_3} \dots \oplus^{n_\ell} (J_1 \oplus I_{\ell-1}) . \quad (\text{B.6})$$

Here I_k is the identity matrix of size k . It is straight forward to see that this breaks $SU(N)_1$ down to $SU(n_1 + n_2 + \dots + n_k)$. The D-term constraints are trivially satisfied while the F-term for Φ_1 gives us

$$Q_0 \tilde{Q}_0 - \frac{1}{N} \text{Tr} Q_0 \tilde{Q}_0 = \tilde{Q}_1 Q_1 - \frac{1}{N} \text{Tr} Q_1 \tilde{Q}_1 . \quad (\text{B.7})$$

This chiral ring relation then forces us to have

$$\langle \tilde{Q}_1 Q_1 \rangle = \oplus^{n_1+2n_2} J_1 \oplus^{n_3} (J_1 + J_2) \oplus^{n_4} \dots \oplus^{n_\ell} (J_1 \oplus J_{\ell-1}) , \quad (\text{B.8})$$

which decomposes into

$$\langle \tilde{Q}_1 \rangle = \oplus^{n_1+2n_2} J_1 \oplus^{n_3} (J_1 + J_2) \oplus^{n_4} \dots \oplus^{n_\ell} (J_1 \oplus J_{\ell-1}) , \quad (\text{B.9})$$

and

$$\langle Q_1 \rangle = \oplus^{n_1+2n_2} J_1 \oplus^{n_3} (J_1 \oplus J_1 \oplus I_1) \oplus^{n_4} \dots \oplus^{n_\ell} (J_1 \oplus J_1 \oplus I_{\ell-2}) , \quad (\text{B.10})$$

thereby breaking $SU(N)_2$ down to $SU(n_1 + 2n_2 + 2n_3 + \dots + 2n_k)$. Application of chiral ring relation at each node then gives us the general pattern of the vevs, which are found to be

$$\begin{aligned} \langle \tilde{Q}_{i-1} Q_{i-1} \rangle &= \oplus^{n_1+2n_2+\dots+in_i} J_1 \oplus^{n_{i+1}} \dots \oplus^{n_k} (\oplus^{i-1} J_1 \oplus J_{k-i+1}) \\ &\quad \oplus^{n_{k+1}} \dots \oplus^{n_\ell} (\oplus^{i-1} J_1 \oplus J_{\ell-i+1}) , \end{aligned} \quad (\text{B.11})$$

such that

$$\begin{aligned} \langle \tilde{Q}_{i-1} \rangle &= \oplus^{n_1+2n_2+\dots+in_i} J_1 \oplus^{n_{i+1}} \dots \oplus^{n_k} (\oplus^{i-1} J_1 \oplus J_{k-i+1}) \\ &\quad \oplus^{n_{k+1}} \dots \oplus^{n_\ell} (\oplus^{i-1} J_1 \oplus J_{\ell-i+1}) , \end{aligned} \quad (\text{B.12})$$

and

$$\begin{aligned} \langle Q_{i-1} \rangle &= \oplus^{n_1+2n_2+\dots+in_i} J_1 \oplus^{n_{i+1}} \dots \oplus^{n_k} (\oplus^i J_1 \oplus I_{k-i}) \\ &\quad \oplus^{n_{k+1}} \dots \oplus^{n_\ell} (\oplus^i J_1 \oplus I_{\ell-i}) . \end{aligned} \quad (\text{B.13})$$

To check that these vevs do satisfy (B.11) we use the rules that $J_k \cdot (J_1 \oplus I_{k-1}) = J_k$ and $(J_1 \oplus I_{k-1}) \cdot J_k = J_1 \oplus J_{k-1}$. The structure of these vevs imply that $SU(N)_i$ gets broken down to $SU(n_1 + 2n_2 + 3n_3 + \dots + in_i + in_{i+1} \dots + in_k)$. Also $SU(N)_{\ell-1}$ gets broken down to $SU(N - n_\ell)$ while all the gauge groups from $SU(N)_\ell$ onwards remain unbroken. Thus we see that the gauge symmetry of the low energy theory obtained after Higgsing is given by

$$G' = \prod_{i=1}^{\ell-1} SU(N_i) \times \prod_{j=1}^{N-\ell} SU(N)_j , \quad (\text{B.14})$$

where $N_i = n_1 + 2n_2 + 3n_3 + \dots + in_i + in_{i+1} \dots + in_\ell$. Apart from hypermultiplets H_i transforming as the bifundamental of $SU(N_{i-1}) \times SU(N_i)$, there will be m_i fundamentals at the gauge group $SU(N_i)$. Superconformality requires that

$$m_i + N_{i-1} + N_{i+1} = 2N_i , \quad (\text{B.15})$$

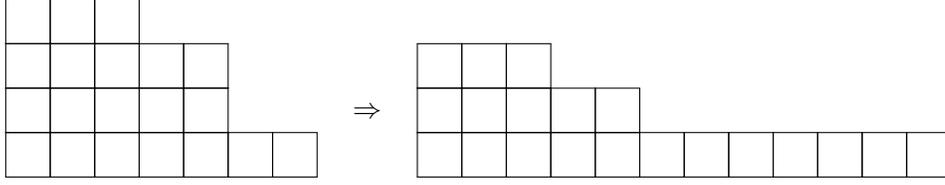


Figure 23: Collapsing of a Young tableau

which then leads to $m_i = n_i$. This is coherent with the fact that the flavor symmetry of the Higgsed puncture corresponds to the symmetry associated with the additional n_i fundamentals attached to $SU(N_i)$.

Notice that the vev $\langle \mu_i \rangle = \langle \tilde{Q}_i Q_i \rangle - \frac{1}{N} \langle \text{Tr} \tilde{Q}_i Q_i \rangle$ can be understood as the vev corresponding to partitioning N as $N = (N_{i-1} + n_i) + 2n_{i+1} + \dots + (\ell - i + 1)n_\ell$. The section of the quiver tail from the i -th node onward can then be thought of as being obtained from a linear $SU(N)$ -quiver whose left puncture has been Higgsed according to this partition. This implies that the propagation of vevs along the tail can also be neatly encoded into the process of collapsing the Young's tableau at each step. Thus if we start with the partition $N = n_1 + 2n_2 + \dots + \ell n_\ell$, then the Young's tableau at the next step in the quiver tail is obtained in the following manner: We remove the highest box from each column of boxes in the tableau. The boxes that were removed are stacked against the residual tableau in a single row. For example if we consider the partition $20 = 1 + 1 + 3 + 3 + 4 + 4 + 4$, then at the next step in the quiver tail, its tableau collapses into the partition as described in figure 23.

The massive and massless matter fields In order to obtain the number of fundamentals at the i -th node of the tail, we had invoked superconformality of the low energy theory, however, we should be able to derive this without resorting to an a priori assumption that the low energy theory is superconformal. To do this we now focus on the various matter fields that get massive in the process of giving vevs. Once again we consider the case of partial Higgsing (given by the partition of N , as in (B.3)) of a full-puncture of the $SU(N)$ linear quiver. We will make use of the following rules of decomposition:

$$\begin{aligned}
SU(N)_i &\rightarrow SU(N_i) \\
N &\rightarrow N_i \oplus^{(N-N_i)} \mathbf{1} , \\
\text{adj} &\rightarrow \text{adj} \oplus^{(N-N_i)} N_i \oplus^{(N-N_i)} \bar{N}_i \oplus^{(N-N_i)^2} \mathbf{1} .
\end{aligned} \tag{B.16}$$

Also note that H_{i-1} transforms as a bifundamental of $SU(N)_{i-1} \times SU(N)_i$ and can be decomposed into irreducible representations of $SU(N_{i-1}) \times SU(N_i)$ as

$$\begin{aligned}
SU(N)_{i-1} \times SU(N)_i &\rightarrow SU(N_{i-1}) \times SU(N_i) \\
Q_{i-1} : (\bar{N}, N) &\rightarrow (\bar{N}_{i-1}, N_i) \oplus^{(N-N_i)} (\bar{N}_{i-1}, \mathbf{1}) \oplus^{(N-N_{i-1})} (\mathbf{1}, N_i) \oplus^{(N-N_i)(N-N_{i-1})} (\mathbf{1}, \mathbf{1}) , \\
\tilde{Q}_{i-1} : (N, \bar{N}) &\rightarrow (N_{i-1}, \bar{N}_i) \oplus^{(N-N_i)} (N_{i-1}, \mathbf{1}) \oplus^{(N-N_{i-1})} (\mathbf{1}, \bar{N}_i) \oplus^{(N-N_i)(N-N_{i-1})} (\mathbf{1}, \mathbf{1}) .
\end{aligned}$$

From (B.16) we see that upon Higgsing $SU(N)_i \rightarrow SU(N_i)$ via vevs for H_{i-1} and H_i , the vector multiplets of $SU(N)_i$ that end up getting a mass will need to eat $2(N - N_i)$ chiral multiplets transforming as the N_i -dimensional representation of $SU(N_i)$. There are $(N - N_{i-1})$ such chirals in H_{i-1} and $(N - N_i)$ such chirals in H_i . Thus we are left behind with $2(N - N_i) - (N - N_{i-1}) - (N - N_i) = n_i$ chiral super fields that transform as fundamentals of $SU(N_i)$. We will similarly be left with n_i chiral multiplets transforming as the anti-fundamental of $SU(N_i)$. These will together give us n_i hypers transforming in the fundamental of $SU(N_i)$. We also end up eating some of the singlets. The number of singlet hypers that are left behind (these are the hypers that decouple from the rest of the quiver) is then given by

$$\sum_{i=1}^k (N - N_i)(N_i - N_{i-1}) \quad \text{where } N_0 = 0 . \quad (\text{B.17})$$

These decoupled hypers are the Goldstone multiplets that we expect upon spontaneously breaking the global symmetry. It can be easily checked that the number of the Goldstone chiral superfields in these hypers is same as the number of generators of the complexified $SU(N)$ that are broken by $\langle \mu \rangle$ i.e. the Goldstone chiral superfields are in one-to-one correspondence with the generators X of $SL(N, \mathbb{C})$ which obey

$$[X, \langle \mu_0 \rangle] \neq 0 . \quad (\text{B.18})$$

Apart from these there will of course be massless hypers that transform as bifundamentals of $SU(N_{i-1}) \times SU(N_i)$. We thus obtained the desired low energy quiver.

As an explicit example of the above pattern of massive and massless matter fields, we consider an $SU(4)$ linear quiver and Higgs its left full-puncture down to a simple puncture. We give appropriate vevs to H_0 and H_1 , Higgsing $SU(4)_1 \times SU(4)_2$ down to $SU(2) \times SU(3)$. The decomposition of vector multiplets into irreps. of the low energy gauge symmetry is given by

$$\begin{aligned} SU(4)_1 \times SU(4)_2 &\rightarrow SU(2) \times SU(3) \\ V_1 : (\text{adj}, 1) &\rightarrow (\text{adj}, 1) \oplus (2, 1) \oplus (2, 1) \oplus (\bar{2}, 1) \oplus (\bar{2}, 1) \oplus^4 (1, 1) , \\ V_2 : (1, \text{adj}) &\rightarrow (1, \text{adj}) \oplus (1, 3) \oplus (1, \bar{3}) \oplus (1, 1) , \end{aligned} \quad (\text{B.19})$$

while the hypers H_0 and H_1 decompose as

$$\begin{aligned} SU(4)_1 \times SU(4)_2 &\rightarrow SU(2) \times SU(3) \\ (Q_0)_i : (4, 1) &\rightarrow (2, 1) \oplus^2 (1, 1) , \\ (\tilde{Q}_0)_i : (\bar{4}, 1) &\rightarrow (\bar{2}, 1) \oplus^2 (1, 1) , \\ Q_1 : (\bar{4}, 4) &\rightarrow (\bar{2}, 3) \oplus^2 (1, 3) \oplus (\bar{2}, 1) \oplus^2 (1, 1) , \\ \tilde{Q}_1 : (4, 4) &\rightarrow (2, \bar{3}) \oplus^2 (1, \bar{3}) \oplus (2, 1) \oplus^2 (1, 1) , \end{aligned} \quad (\text{B.20})$$

The various chiral multiplets that get eaten via Higgsing are: 4 copies transforming as $(2, 1)$, 4 copies of $(\bar{2}, 1)$, 2 copies each of $(1, 3)$ and $(1, \bar{3})$ and 10 copies of $(1, 1)$. We are thus left behind

with a chiral multiplet for each of $(2, 1)$, $(\bar{2}, 1)$, $(2, \bar{3})$ and $(\bar{2}, 3)$ along with 10 chirals which are singlets and hence decouple from the rest of the theory. These can then be organized as a hyper transforming in the fundamental of $SU(2)$, another hyper transforming as the bifundamental of $SU(2) \times SU(3)$ and 5 decoupled hypers.

References

- [1] D. Gaiotto, “N=2 dualities,” *JHEP* **1208** (2012) 034, [arXiv:0904.2715 \[hep-th\]](#).
- [2] D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” [arXiv:0907.3987 \[hep-th\]](#).
- [3] L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” *Lett.Math.Phys.* **91** (2010) 167–197, [arXiv:0906.3219 \[hep-th\]](#).
- [4] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, “S-duality and 2d Topological QFT,” *JHEP* **1003** (2010) 032, [arXiv:0910.2225 \[hep-th\]](#).
- [5] I. Bah, C. Beem, N. Bobev, and B. Wecht, “Four-Dimensional SCFTs from M5-Branes,” *JHEP* **1206** (2012) 005, [arXiv:1203.0303 \[hep-th\]](#).
- [6] I. Bah and B. Wecht, “New N=1 Superconformal Field Theories In Four Dimensions,” *JHEP* **1307** (2013) 107, [arXiv:1111.3402 \[hep-th\]](#).
- [7] I. Bah, C. Beem, N. Bobev, and B. Wecht, “AdS/CFT Dual Pairs from M5-Branes on Riemann Surfaces,” *Phys.Rev.* **D85** (2012) 121901, [arXiv:1112.5487 \[hep-th\]](#).
- [8] C. Beem and A. Gadde, “The $N = 1$ superconformal index for class S fixed points,” *JHEP* **1404** (2014) 036, [arXiv:1212.1467 \[hep-th\]](#).
- [9] A. Gadde, K. Maruyoshi, Y. Tachikawa, and W. Yan, “New N=1 Dualities,” *JHEP* **1306** (2013) 056, [arXiv:1303.0836 \[hep-th\]](#).
- [10] D. Xie, “M5 brane and four dimensional $N = 1$ theories I,” *JHEP* **1404** (2014) 154, [arXiv:1307.5877 \[hep-th\]](#).
- [11] I. Bah and N. Bobev, “Linear Quivers and N=1 SCFTs from M5-branes,” [arXiv:1307.7104 \[hep-th\]](#).
- [12] P. Agarwal and J. Song, “New N=1 Dualities from M5-branes and Outer-automorphism Twists,” *JHEP* **1403** (2014) 133, [arXiv:1311.2945 \[hep-th\]](#).
- [13] I. Bah, “Quarter-BPS AdS_5 solutions in M-theory with a T^2 bundle over a Riemann surface,” *JHEP* **1308** (2013) 137, [arXiv:1304.4954 \[hep-th\]](#).
- [14] I. Bah, M. Gabella, and N. Halmagyi, “Punctures from Probe M5-Branes and N=1 Superconformal Field Theories,” [arXiv:1312.6687 \[hep-th\]](#).
- [15] K. Maruyoshi, M. Taki, S. Terashima, and F. Yagi, “New Seiberg Dualities from N=2 Dualities,” *JHEP* **0909** (2009) 086, [arXiv:0907.2625 \[hep-th\]](#).
- [16] F. Benini, Y. Tachikawa, and B. Wecht, “Sicilian gauge theories and N=1 dualities,” *JHEP* **01** (2010) 088, [arXiv:0909.1327 \[hep-th\]](#).

- [17] K. Maruyoshi, Y. Tachikawa, W. Yan, and K. Yonekura, “N=1 dynamics with T_N theory,” *JHEP* **1310** (2013) 010, [arXiv:1305.5250 \[hep-th\]](#).
- [18] G. Bonelli, S. Giacomelli, K. Maruyoshi, and A. Tanzini, “N=1 Geometries via M-theory,” *JHEP* **1310** (2013) 227, [arXiv:1307.7703](#).
- [19] D. Xie and K. Yonekura, “Generalized Hitchin system, Spectral curve and $\mathcal{N} = 1$ dynamics,” *JHEP* **1401** (2014) 001, [arXiv:1310.0467 \[hep-th\]](#).
- [20] K. Yonekura, “Supersymmetric gauge theory, (2,0) theory and twisted 5d Super-Yang-Mills,” *JHEP* **1401** (2014) 142, [arXiv:1310.7943 \[hep-th\]](#).
- [21] A. Hashimoto, P. Ouyang, and M. Yamazaki, “Boundaries and Defects of N=4 SYM with 4 Supercharges, Part I: Boundary/Junction Conditions,” [arXiv:1404.5527 \[hep-th\]](#).
- [22] A. Hashimoto, P. Ouyang, and M. Yamazaki, “Boundaries and Defects of N=4 SYM with 4 Supercharges, Part II: Brane Constructions and 3d N=2 Field Theories,” [arXiv:1406.5501 \[hep-th\]](#).
- [23] D. Gaiotto and E. Witten, “Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory,” *J.Statist.Phys.* **135** (2009) 789–855, [arXiv:0804.2902 \[hep-th\]](#).
- [24] D. Gaiotto and E. Witten, “S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory,” *Adv.Theor.Math.Phys.* **13** (2009) 721, [arXiv:0807.3720 \[hep-th\]](#).
- [25] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” *Nucl.Phys.* **B435** (1995) 129–146, [arXiv:hep-th/9411149 \[hep-th\]](#).
- [26] J. J. Heckman, Y. Tachikawa, C. Vafa, and B. Wecht, “N = 1 SCFTs from Brane Monodromy,” *JHEP* **1011** (2010) 132, [arXiv:1009.0017 \[hep-th\]](#).
- [27] Y. Tachikawa, “N=2 supersymmetric dynamics for pedestrians,” [arXiv:1312.2684 \[hep-th\]](#).
- [28] P. C. Argyres and N. Seiberg, “S-duality in N=2 supersymmetric gauge theories,” *JHEP* **0712** (2007) 088, [arXiv:0711.0054 \[hep-th\]](#).
- [29] K. A. Intriligator and B. Wecht, “The Exact superconformal R symmetry maximizes a,” *Nucl.Phys.* **B667** (2003) 183–200, [arXiv:hep-th/0304128 \[hep-th\]](#).
- [30] D. Kutasov, A. Schwimmer, and N. Seiberg, “Chiral rings, singularity theory and electric - magnetic duality,” *Nucl.Phys.* **B459** (1996) 455–496, [arXiv:hep-th/9510222 \[hep-th\]](#).
- [31] W. Lerche, “On Goldstone Fields in Supersymmetric Theories,” *Nucl.Phys.* **B238** (1984) 582.
- [32] Y. Tachikawa, “Six-dimensional D(N) theory and four-dimensional SO-USp quivers,” *JHEP* **0907** (2009) 067, [arXiv:0905.4074 \[hep-th\]](#).
- [33] O. Chacaltana and J. Distler, “Tinkertoys for Gaiotto Duality,” *JHEP* **1011** (2010) 099, [arXiv:1008.5203 \[hep-th\]](#).
- [34] O. Chacaltana and J. Distler, “Tinkertoys for the D_N series,” *JHEP* **1302** (2013) 110, [arXiv:1106.5410 \[hep-th\]](#).
- [35] O. Chacaltana, J. Distler, and Y. Tachikawa, “Gaiotto Duality for the Twisted A_{2N-1} Series,” [arXiv:1212.3952 \[hep-th\]](#).
- [36] O. Chacaltana, J. Distler, and A. Trimm, “Tinkertoys for the Twisted D-Series,” [arXiv:1309.2299 \[hep-th\]](#).

- [37] O. Chacaltana, J. Distler, and A. Trimm, “Tinkertoys for the E_6 Theory,” [arXiv:1403.4604 \[hep-th\]](#).
- [38] O. Chacaltana, J. Distler, and Y. Tachikawa, “Nilpotent orbits and codimension-two defects of 6d $N=(2,0)$ theories,” *Int.J.Mod.Phys.* **A28** (2013) 1340006, [arXiv:1203.2930 \[hep-th\]](#).
- [39] D. Gaiotto, A. Neitzke, and Y. Tachikawa, “Argyres-Seiberg duality and the Higgs branch,” *Commun.Math.Phys.* **294** (2010) 389–410, [arXiv:0810.4541 \[hep-th\]](#).
- [40] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, “Gauge Theories and Macdonald Polynomials,” *Commun.Math.Phys.* **319** (2013) 147–193, [arXiv:1110.3740 \[hep-th\]](#).
- [41] P. C. Argyres, M. R. Plesser, and N. Seiberg, “The Moduli space of vacua of $N=2$ SUSY QCD and duality in $N=1$ SUSY QCD,” *Nucl.Phys.* **B471** (1996) 159–194, [arXiv:hep-th/9603042 \[hep-th\]](#).
- [42] P. C. Argyres, K. A. Intriligator, R. G. Leigh, and M. J. Strassler, “On inherited duality in $N=1$ $d = 4$ supersymmetric gauge theories,” *JHEP* **0004** (2000) 029, [arXiv:hep-th/9910250 \[hep-th\]](#).
- [43] Y. Tachikawa and B. Wecht, “Explanation of the Central Charge Ratio $27/32$ in Four-Dimensional Renormalization Group Flows between Superconformal Theories,” *Phys.Rev.Lett.* **103** (2009) 061601, [arXiv:0906.0965 \[hep-th\]](#).
- [44] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, “The 4d Superconformal Index from q -deformed 2d Yang-Mills,” *Phys.Rev.Lett.* **106** (2011) 241602, [arXiv:1104.3850 \[hep-th\]](#).
- [45] D. Gaiotto, L. Rastelli, and S. S. Razamat, “Bootstrapping the superconformal index with surface defects,” *JHEP* **1301** (2013) 022, [arXiv:1207.3577 \[hep-th\]](#).
- [46] N. Mekareeya, J. Song, and Y. Tachikawa, “2d TQFT structure of the superconformal indices with outer-automorphism twists,” *JHEP* **1303** (2013) 171, [arXiv:1212.0545 \[hep-th\]](#).