Equivalence of Decoupling Schemes and Orthogonal Arrays

Martin Rötteler, *Member, IEEE*, and Pawel Wocjan

Abstract*—Decoupling schemes are used in quantum information processing to selectively switch off unwanted interactions in a multipartite Hamiltonian. A decoupling scheme consists of a sequence of local unitary operations which are applied to the system's qudits and alternate with the natural time evolution of the Hamiltonian. Several constructions of decoupling schemes have been given in the literature. Here we focus on two such schemes. The first is based on certain triples of submatrices of Hadamard matrices that are closed under pointwise multiplication (see Leung, "Simulation and reversal of n-qubit Hamiltonians using Hadamard matrices,"** ***J. Mod. Opt.**, vol. 49, pp. 1199–1217, 2002), the second uses orthogonal arrays (see Stollsteimer and Mahler, "Suppression of arbitrary internal couplings in a quantum register,"** ***Phys. Rev. A.*****, vol. 64, p. 052301, 2001). We show that both methods lead to the same class of decoupling schemes. We extend the first method to 2-local qudit Hamiltonians, where $d \geq 2$. Furthermore, we extend the second method to t-local qudit Hamiltonians, where $t \geq 2$ and $d \geq 2$, by using orthogonal arrays of strength t. We also establish a characterization of orthogonal arrays of strength t by showing that they are equivalent to decoupling schemes for t-local Hamiltonians which have the property that they can be refined to have time-slots of equal length. The methods used to derive efficient decoupling schemes are based on classical error-correcting codes.**

***Index Terms*—Decoupling schemes, Hadamard matrices, orthogonal arrays, t-local Hamiltonians.**

I. Introduction

An important problem in the control of quantum systems is how to manipulate a given Hamiltonian by applying external control operations in such a way that, in effect, the time-evolution of some other desired target Hamiltonian is simulated. Here we assume that the Hamiltonian is t-local, where $t \geq 2$, and that the individual subsystems of the quantum system are *qudits*, i.e., d-dimensional subsystems where $d \geq 2$. Typically, the available control operations are restricted to be local unitary operations only. In the theory of simulation of 2-local n-qubit Hamiltonians, a repertoire of techniques has been developed to use any entangling Hamiltonian for universal simulation of arbitrary couplings [1]–[14]. A cornerstone of this theory is the development of *decoupling schemes* and *selective coupling schemes*. Both are pulse sequences that switch off unwanted interactions in a given Hamiltonian. In the case of a decoupling scheme, all interactions have to be switched off. In contrast, the requirement for a selective coupling scheme is that all interactions, except for the interaction between two fixed subsystems, have to be switched off. Two methods have been proposed to achieve decoupling and selective coupling of a general 2-local n-qubit Hamiltonian:

Construction I Leung [7] gave a construction of decoupling schemes which uses certain triples S_x, S_y, S_z of submatrices of Hadamard matrices.

Construction II The decoupling schemes put forward by Stollsteimer and Mahler [3] are constructed by using orthogonal arrays.

The main goal of this paper is to show that these two constructions of decoupling schemes are equivalent. Furthermore, we generalize from decoupling schemes for 2-local n-qubit Hamiltonians to decoupling schemes for t-local n-qudit Hamiltonians. We show that for constant locality $t = 2$ and constant subsystem dimension $d \geq 2$ decoupling can be achieved using a number of control operations linear in n.

The paper is organized as follows. In Section II we introduce the necessary background from Average Hamiltonian Theory. This is the framework for decoupling schemes and (more generally) simulating Hamiltonians. In Section III we present the combinatorial objects of Hadamard matrices, sign matrices, and phase matrices. Sign matrices are used in Construction I to obtain decoupling schemes for 2-local qubit Hamiltonians. Phase matrices are our generalization of sign matrices to the qudit case. This generalization is based on nice error bases with abelian index groups. In Section IV, we present constructions of decoupling schemes from sign and phase matrices and from orthogonal arrays over nice error bases. While it was previously known [3] that orthogonal arrays of strength 2 can be used to construct decoupling schemes for 2-local qudit Hamiltonians, we extend this result by showing that orthogonal arrays of strength t yield decoupling schemes for t-local qudit Hamiltonians. In Section V we prove that the presented decoupling schemes based on phase matrices and orthogonal arrays of strength $t = 2$ are equivalent. The equivalence is interesting for two reasons: First, it gives a constructive method of turning phase matrices into orthogonal arrays and vice versa. Second, by using this correspondence we connect the problem of finding decoupling schemes to a problem studied extensively in combinatorics, namely the problem of finding orthogonal arrays. In Section VI, we show that equal length decoupling schemes for

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M. Rötteler was with the Institute for Quantum Computing, University of Waterloo, Canada. He is now with NEC Laboratories America, Inc., Princeton, NJ 08540 USA (e-mail: mroetteler@nec-labs.com).

P. Wocjan is with the Computer Science Department and the Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: wocjan@cs.caltech.edu).

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t-local qudit Hamiltonians correspond one-to-one to orthogonal arrays of strength t, whenever the number of levels is a square integer. Finally, we present conclusions in Section VII.

II. The Framework: Average Hamiltonian Theory

Decoupling schemes are used to switch off unwanted interactions in a system of interacting qudits which are governed by a fixed Hamiltonian. The ability to switch off unwanted interactions is important for the task of using a given Hamiltonian to simulate other Hamiltonians [1]–[3], [7]–[11], [14]. Here simulation is usually understood in a narrow sense in which the desired target Hamiltonian is approximated up to terms of quadratic and higher orders. In the following, we briefly introduce the features of Average Hamiltonian Theory [15], [16] that will be needed to develop the theory of decoupling schemes.

Assume that the Hamiltonian acts on an n-fold tensor product Hilbert space $\mathcal{H} := \mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$, where each $\mathbb{C}^d$ denotes the Hilbert space of a qudit. Let $B := \{\sigma_\alpha\}_{\alpha=1}^{d^2-1}$ be an arbitrary basis of the vector space of traceless matrices in $\mathbb{C}^{d\times d}$. The Pauli matrices σ_x, σ_y, and σ_z given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form such a basis B for $d = 2$.

Definition 1 (t-Local n-Qudit Hamiltonian): Let $t \geq 2$ and $d \geq 2$. Then the Hamiltonian H of a system of n coupled d-dimensional subsystems is said to be a t-local n-qudit Hamiltonian if it can be written in the form

$$H = \sum_{s=1}^{t} \sum_{(k_1,\ldots,k_s)} \sum_{\alpha_1,\ldots,\alpha_s=1}^{d^2-1} J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \sigma_{\alpha_1}^{(k_1)}, \ldots, \sigma_{\alpha_s}^{(k_s)} \tag{1}$$

where $J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \in \mathbb{C}$ and where the second sum runs over all s-tuples with (different) entries from $\{1,\ldots,n\}$. Here and in the following we use $\sigma_\alpha^{(k)}$ to denote the operator that acts as σ_α on the kth qudit, i.e., $\sigma_\alpha^{(k)} := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_\alpha \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, where $\sigma_\alpha \in B$.

In the setting discussed in this paper the only possibilities of external control are given by local unitaries on each qudit. We assume that it is possible to implement them independently, i.e., that they can be applied simultaneously. This is also referred to as the ability to perform selective pulses [3]. We assume that all control operations can be implemented arbitrarily fast ("fast control limit"). This allows us to assume that all control operations are elements of some finite subset of the group $\mathcal{U}(d)^{\otimes n}$, where $\mathcal{U}(d)$ denotes the group of unitary matrices acting on $\mathbb{C}^d$. The simulation of Hamiltonians is based on the following Average Hamiltonian Theory [15] approach. Letting the system evolve for some time t (in the following referred to as "wait") has the effect of applying the unitary $\exp(-iHt)$. Let $t_1, t_2, \ldots, t_N$ be real positive numbers and $V_1, V_2, \ldots, V_N \in \mathcal{U}(d)^{\otimes n}$ be control operations. Then the sequence

$$\text{apply } V_1, \text{wait } t_1, \text{apply } V_2, \text{wait } t_2, \ldots \text{apply } V_N, \text{wait } t_N$$

implements the evolution $\prod_{j=1}^{N} \exp(-iU_j^\dagger H U_j t_j)$, where $U_j = \prod_{i=1}^{j} V_i$. Without loss of generality we assume that the V_i are chosen such that $U_N = \prod_{i=1}^{N} V_i = \mathbf{1}$. We say that the sequence consists of N *time-slots* and use the shorthand notation $(t_1, U_1; t_2, U_2; \ldots; t_N, U_N)$, where we tacitly assume that the underlying Hamiltonian H is fixed. If the times t_j are small compared to the time scale of the natural evolution according to H, then the total time evolution of the sequence can be approximated by $e^{-i\bar{H}t/\tau}$ in which

$$\bar{H} := \sum_{j=1}^{N} t_j U_j^\dagger H U_j \tag{2}$$

is the average Hamiltonian. The quantity $\tau := \sum_j t_j$ is the slow down factor, i.e., the relative running time of the evolution, see, e.g., [5], [8].

We next introduce sequences for which $\bar{H}$ is the zero Hamiltonian. Since simulating the zero Hamiltonian means that the time evolution of the system is effectively stopped, these schemes are also used in dynamical suppression of decoherence in open quantum systems ("bang-bang" control), see [17]–[19].

Definition 2 (Decoupling Schemes): A decoupling scheme for t-local n-qudit Hamiltonians is a sequence $D = (t_1, U_1; \ldots; t_N, U_N)$ of control operations $U_j \in \mathcal{U}(d)^{\otimes n}$ and delays $t_j \in \mathbb{R}$ such that

$$\bar{H} = \sum_{j=1}^{N} t_j U_j^\dagger H U_j = 0 \tag{3}$$

holds for any t-local n-qudit Hamiltonian H. We call a decoupling scheme D *regular* if the lengths of the time-slots are the same, i.e., if $t_1 = t_2 = \ldots = t_N$.

Note that if the system consists of only one qudit, a unitary error basis [20], [21] in dimension d defines a regular decoupling scheme. Definition 2 includes decoupling schemes consisting of time slots of different length. Indeed, practical decoupling schemes with unequal time intervals exist and are used, most notably the famous WaHuHa sequence [15], [24], [25]. However, there is a preference for equal length schemes ($t_j \equiv 1/N$) in the literature [1]–[3], [7], [9].

III. Hadamard, Sign, and Phase Matrices

A. Hadamard Matrices

We denote the transpose of a matrix A by A^T. A Hadamard matrix of size N is a ± 1 matrix H_N of size $N \times N$ with the property that $H_N H_N^T = N\mathbf{1}_N$. Hadamard matrices have been studied extensively in combinatorics and several constructions have been found. We refer to [26]–[28] for background on and constructions of Hadamard matrices. We give some examples of Hadamard matrices of small sizes (here and in the following the entries ± 1 have been abbreviated to $+/-$):

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix}, \quad \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}, \quad \begin{pmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{pmatrix}. \tag{4}$$

TABLE I
EXAMPLE OF THREE ORTHOGONAL SIGN MATRICES S_x, S_y, S_z OF SIZE 7×8 WHICH ARE CLOSED UNDER SCHUR PRODUCT

$$S_x := \begin{pmatrix} + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{pmatrix}, \quad S_y := \begin{pmatrix} + & + & - & - & - & - & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \end{pmatrix}, \quad S_z := \begin{pmatrix} + & - & - & + & - & + & + & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & + & + & - & - \end{pmatrix}$$

TABLE II
EXAMPLE OF A PAIR-WISE ORTHOGONAL SCHUR-CLOSED SET OF SIGN MATRICES S_x, S_y, S_z OF SIZE 5×16

$$S_x := \begin{pmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \end{pmatrix},$$

$$S_y := \begin{pmatrix} + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \end{pmatrix},$$

$$S_z := \begin{pmatrix} + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \end{pmatrix}.$$

It is known that a necessary condition for the existence of a Hadamard matrix is that either $N = 2$ or $N \equiv 0 \bmod 4$. A long-standing conjecture is whether for any $N \equiv 0 \bmod 4$ a Hadamard matrix of size N exists [26]. Let H_2 be the leftmost Hadamard matrix in (4). To obtain a Hadamard matrix of size $N = 2^n$, construct $H_{2^n} := H_2 \otimes \ldots \otimes H_2$ (n tensor factors).

B. Sign Matrices

A sign matrix is a matrix with entries ± 1. An *orthogonal* sign matrix $S_{n,N}$ of size $n \times N$ is a sign matrix which satisfies $S_{n,N} S_{n,N}^T = N \mathbf{1}_n$. Examples of orthogonal sign matrices are obtained by selecting n rows of a Hadamard matrix of size N. Recall that the Schur product of two $n \times N$ matrices A and B, denoted by $C := A \circ B$, is defined as the entry-wise product: $C_{i,j} := A_{i,j} B_{i,j}$. If a set of $n \times N$ sign matrices has the property that all possible pair-wise Schur products are already contained in the set, then we say that the set is closed under taking Schur products, or *Schur-closed* for short.

Example 3: In Table I an example of a Schur-closed set consisting of three orthogonal sign matrices S_x, S_y, S_z of size 7×8 is given.

In the following S_{id} denotes the all-ones matrix of appropriate size. We call a set $\mathcal{S} = \{S_{id}\} \cup \{S_i : i = 1, \ldots, k-1\}$ of k sign matrices of size $n \times N$ (one of which is the all-ones matrix) *pair-wise orthogonal* if $S_\alpha S_\alpha^T = N \mathbf{1}_n$ for all $S_\alpha \in \mathcal{S}$ with $S_\alpha \neq S_{id}$ and $S_\alpha S_\beta^T = \mathbf{0}_n$ for all $S_\alpha, S_\beta \in \mathcal{S}$ with $S_\alpha \neq S_\beta$. Here $\mathbf{0}_n$ denotes the zero matrix of size $n \times n$.

Example 4: In Table II an example of a set of sign matrices which is pair-wise orthogonal and Schur-closed is given. The set consists of S_{id} and the three sign matrices S_x, S_y and S_z of size 5×16. Note that the orthogonal set of Example 3, albeit being Schur-closed and consisting only of orthogonal sign matrices, does not satisfy the stronger property of being pair-wise orthogonal. Indeed, every row of any of the three matrices in Table I also occurs as a row of the other two sign matrices.

C. Phase Matrices

The restriction of sign matrices to have only ± 1 as entries can be relaxed by allowing the entries to be more general complex phases. This leads to the concept of phase matrices which are defined as follows. Let $k \in$ and let $\omega = \exp(2\pi i/k) \in \mathbb{C}$ be a primitive kth root of unity. Then a phase matrix $P_{n,N}$ of order k is an $n \times N$ matrix with entries in $\{1, \omega, \ldots, \omega^{k-1}\}$. A phase matrix of size $n \times N$ is called orthogonal if $P_{n,N} P_{n,N}^\dagger = N \mathbf{1}_n$ holds. As in the case of sign matrices, we are interested in collections of phase matrices which are Schur-closed. This condition can be conveniently stated in terms of characters of some finite abelian group G. For the necessary background on characters of abelian groups we refer to the Appendix. In the following we assume that the elements of G are given in a fixed order $g_1, \ldots, g_{|G|}$ and that the irreducible characters $\{\chi_g : g \in G\}$ of G are in one-to-one correspondence with the elements of G with respect to the isomorphism $g \mapsto \chi_g$, so that $gh \mapsto \chi_g \chi_h$, cf. Theorem 22 in the Appendix. Recall that the exponent $e(G)$ of G is the smallest positive integer such that $g^{e(G)} = 1$ for all $g \in G$. Let $P_{g_1}, \ldots, P_{g_{|G|}}$ be phase matrices of order $e(G)$ which are labeled by the elements of G. Denoting the neutral element of G by id, we say that a set $\{P_g : g \in G \setminus \{id\}\}$ of phase matrices is Schur-closed if

$$P_g \circ P_h = P_{gh} \tag{5}$$

holds for all pairs $g, h \in G$, where we define P_{id} to be the all-ones matrix for the neutral element of G. In other words, under taking Schur products the matrices P_g form a group isomorphic to G. As in the case of sign matrices we call a set $\mathcal{P} = \{P_{id}\} \cup \{P_i : i = 1, \ldots, k-1\}$ of k phase matrices of size $n \times N$ (where P_{id} is the all-ones matrix) *pair-wise orthogonal* if $P_\alpha P_\alpha^\dagger = N \mathbf{1}_n$ for all $P_\alpha \in \mathcal{P}$ with $P_\alpha \neq P_{id}$ and $P_\alpha P_\beta^T = \mathbf{0}_n$ for all $P_\alpha, P_\beta \in \mathcal{P}$ with $P_\alpha \neq P_\beta$.

Finally, we observe that a Schur-closed set of phase matrices for the special case of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Schur-closed set of four sign matrices.

IV. CONSTRUCTING DECOUPLING SCHEMES

A. Decoupling Schemes From Sign and Phase Matrices

1) The Qubit Case: Decoupling Schemes From Sign Matrices: A general 2-local n-qubit Hamiltonian can be written in the form

$$H := \sum_{k<\ell} \sum_{\alpha\beta} J_{k\ell;\alpha\beta} \sigma_\alpha^{(k)} \sigma_\beta^{(\ell)} + \sum_k \sum_\alpha J_{k;\alpha} \sigma_\alpha^{(k)} \tag{6}$$

where $J_{k\ell;\alpha\beta} \in \mathbb{R}$, $J_{k;\alpha} \in \mathbb{R}$ and where σ_α are the Pauli matrices, i.e., $\alpha \in \{x, y, z\}$. We construct a regular decoupling scheme using tensor products of $\mathbf{1}_2, \sigma_x, \sigma_y$, and σ_z as control operations. Since both the terms of H and the control operations are tensor products of Pauli matrices, the conjugation of a term in (6) by a control operation results in multiplying the term by either $+1$ or -1 in each time-slot. The entries in the following table are the resulting signs when conjugating the column's Pauli operators by the row's Pauli operators:

	$\mathbf{1}_2$	σ_x	σ_y	σ_z
$\mathbf{1}_2$	$+$	$+$	$+$	$+$
σ_x	$+$	$+$	$-$	$-$
σ_y	$+$	$-$	$+$	$-$
σ_z	$+$	$-$	$-$	$+$

(7)

To achieve decoupling we use an idea from spin echo experiments on n qubits [1]: we design the scheme in such a way that any fixed term in (6) picks up either a $+$ or a $-$ sign in each time-slot and such that the sum of these signs is zero, that is, the term is canceled. Sufficient conditions for decoupling of general 2-local n-qubit Hamiltonians are given by the following theorem.

Theorem 5: A set $\mathcal{S} = \{S_{id}, S_x, S_y, S_z\}$ of sign matrices of size $n \times N$ defines a decoupling scheme for arbitrary 2-local n-qubit Hamiltonians if it is Schur-closed and pair-wise orthogonal.

Proof: We first show how to read off N local unitary operations from the sign matrices and then show the decoupling property. Denote the (k,j)'th entry of S_α for $\alpha \in \{x,y,z\}$ by $S_{\alpha;kj}$. Because of Schur-closedness the only possibilities for the tuple $(S_{\alpha;kj})_{\alpha\in\{id,x,y,z\}}$, where (k,j) is fixed, are $(++++), (++--), (+-+-)$, and $(+--+)$. Each of these possibilities uniquely corresponds to one row of the table in (7) and therefore defines a unique Pauli matrix $\sigma_{(k,j)}$. The signs acquired when conjugating the Pauli matrices $\mathbf{1}_2, \sigma_x, \sigma_y, \sigma_z$ by $\sigma_{(k,j)}$ are given by the (k,j)'th entries of S_{id}, S_x, S_y, S_z. Using this correspondence we define local unitary operations $U_j := \sigma_{(1,j)} \otimes \cdots \otimes \sigma_{(n,j)}$ for $j = 1, \ldots, N$ and define $D := (1/N, U_1; \ldots; 1/N, U_N)$. What remains to be shown is that D is a regular decoupling scheme, i.e., that in an arbitrary 2-local Hamiltonian all terms are decoupled.

Note that the sign acquired by the local term $\sigma_\alpha^{(k)}$ in (7) when conjugated by U_j is given by $S_{\alpha;kj}$ and that the sign acquired by the 2-local term $\sigma_\alpha^{(k)}\sigma_\beta^{(l)}$ in (1) when conjugated by U_j is given by the product of the signs $S_{\alpha;kj}$ and $S_{\beta;\ell j}$. The effect of applying the whole sequence D to a local or 2-local term is to multiply it with the sum of the resulting signs. The property of $\mathcal{S}$ to be pair-wise orthogonal is equivalent to the two conditions

$$\sum_{j=1}^{N} S_{\alpha;kj} = 0 \quad \text{and} \quad \sum_{j=1}^{N} S_{\alpha;kj}S_{\beta;\ell j} = 0 \tag{8}$$

for all $\alpha, \beta \neq id$ and all $k < \ell$.

The first condition in (8) ensures that all local terms are removed and the second ensures that all 2-local terms are removed. □

Example 6: Using Theorem 5 we obtain that the set of pair-wise orthogonal and Schur-closed sign matrices S_{id}, S_x, S_y, and S_z presented in Example 4 yields a regular 16 time-slot decoupling scheme for arbitrary 2-local 5-qubit Hamiltonians.

Example 7: The set consisting of S_{id}, S_x, S_y and S_z presented in Example 3 cannot be used to decouple arbitrary 2-local 7-qubit Hamiltonians since $S_\alpha S_\beta^T = \mathbf{0}_n$ is not satisfied for $\alpha, \beta \in \{id, x, y, z\}$ with $\alpha \neq \beta$. However, if the Hamiltonian of a system of seven qubits is of the particular form where only $\sigma_x^{(k)}\sigma_x^{(\ell)}, \sigma_y^{(k)}\sigma_y^{(\ell)}$, and $\sigma_z^{(k)}\sigma_z^{(\ell)}$ interaction terms occur, these matrices can be used for decoupling and selective coupling. More generally, a construction based on difference matrices for decoupling schemes for Hamiltonians of this special form is known [3].

2) Generalization to the Qudit Case: Phase Matrices: In the following we generalize the approach described in [7] to 2-local interactions between higher-dimensional systems, i.e., qudits. First, recall that a unitary error basis [20] is a collection of d^2 unitaries U_i that are orthogonal with respect to the inner product $\langle A|B\rangle := 1/d\mathrm{tr}(A^\dagger B)$. Bases of unitaries with this property have been studied by Schwinger [21]. Several explicit constructions are given in [20], [22], [23]. We express a 2-local qudit Hamiltonian using a so-called "nice" error basis and use the matrices from such a basis as control operations. Nice error bases are unitary error bases with a group structure [20], [29], [30].

Definition 8 (Nice Error basis): Let G be a group of order d^2 with identity element e. A *nice error basis* in dimension d is a set $\mathcal{E} = \{U_g \in \mathbb{C}^{d\times d} \mid g \in G\}$ of unitary matrices, which are labeled by the elements of G, such that 1) U_e is the identity matrix, 2) $\mathrm{tr}U_g = d\delta_{g,e}$ for all $g \in G$, and 3) $U_gU_h = \alpha(g,h)U_{gh}$ for all $g, h \in G$. The *factor system* $\alpha(g,h)$ is a function from $G \times G$ to the set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Condition (ii) of this definition shows that the matrices U_g are pair-wise orthogonal with respect to the trace inner product. The group G is called the *index group*.

Example 9: Let $d \in$ and let $\omega = \exp(2\pi i/d)$ denote a primitive dth root of unity. Let $S := \sum_{k=0}^{d-1} |k\rangle\langle k+1|$, where addition is performed modulo d, and let $T := \sum_{k=0}^{d-1} \omega^k |k\rangle\langle k|$. Then the set $\mathcal{E}_d := \{S^iT^j : i = 0, \ldots, d-1, j = 0, \ldots, d-1\}$ is a nice error basis in dimension d (see, e.g., [31]). This shows the existence of nice error bases for any dimension $d \in$. In this case the index group is the abelian group $G = \mathbb{Z}_d \times \mathbb{Z}_d$. The identity $ST = \omega TS$ is readily verified. The corresponding factor system α is given by $\alpha((i,j),(k,\ell)) = \omega^{-jk}$, for all $(i,j),(k,\ell) \in G$.

Next, we describe a way of representing a general 2-local n-qudit Hamiltonian using nice error bases with *abelian* index groups. Let $\mathcal{E} = \{U_g : g \in G\}$ be a nice error basis with abelian index group. Since the matrices U_g form a basis of $\mathbb{C}^{d\times d}$, a general 2-local Hamiltonian H may be written as

$$H := \sum_{k<\ell}\sum_{h,h'\neq e} J_{k\ell;hh'}U_h^{(k)}U_{h'}^{(\ell)} + \sum_k\sum_{h\neq e} J_{k;h}U_h^{(k)} \tag{9}$$

with $J_{k\ell;hh'} \in \mathbb{C}$ and $J_{k;h} \in \mathbb{C}$. The advantage of writing H in the form (9) becomes apparent from the construction of regular decoupling schemes using elements of $\mathcal{E}$ as control operations.

Since the terms of H and the control operations are tensor products of elements of the nice error basis $\mathcal{E}$, the conjugations of the terms by the control operations from each time-slot result in multiplying the terms by a complex phase factor. To characterize these phase factors we derive a generalization of the table in (7). To this end, we define $\chi(g,h)$ to be the phase factor that U_h acquires when it is conjugated by U_g, i.e., $\chi(g,h)$ is defined via the relation

$$U_g^\dagger U_h U_g = \chi(g,h) U_h. \tag{10}$$

Hence, the 2-local term $U_h \otimes U_{h'}$ acquires the phase factor $\chi(g,h)\chi(g',h')$ if it is conjugated by $U_g \otimes U_{g'}$. Let $\mathcal{X}$ be the corresponding $d^2 \times d^2$ matrix

$$\mathcal{X} := (\chi(g,h))_{g,h\in G}. \tag{11}$$

Lemma 10: Let $\mathcal{E} := \{U_g \mid g \in G\}$ be a nice error basis with an abelian index group G. Then the matrix $\mathcal{X} = (\chi(g,h))_{g,h\in G}$ is the character table of the group G.

We give a proof in the Appendix.

3) Criteria for Decoupling in Term of Phase Matrices: The following theorem gives a set of necessary and sufficient conditions for a set of phase matrices to form a decoupling scheme. It is a generalization of Theorem 5 to phase matrices.

Theorem 11: Let $\mathcal{E} = \{U_g : g \in G\}$ be a nice error basis in dimension $d \geq 2$ with abelian index group G. Then a set $\mathcal{P} = \{P_h : h \in G\}$ of phase matrices of order $e(G)$ and of size $n \times N$ defines a decoupling scheme for arbitrary 2-local n-qudit Hamiltonians if it is Schur-closed and pair-wise orthogonal.

Proof: As in the proof of Theorem 5, we first define N local unitary operations from the phase matrices and then show that they define a decoupling scheme for arbitrary n-qudit Hamiltonians. Denote the (k,j)'th entry of P_h by $P_{h;kj}$. Due to Schur-closedness of $\mathcal{P}$ the vector $(P_{h;kj})_{h\in G}$ for fixed (k,j) is equal to the list of values of an irreducible character of G and therefore is equal to a unique row of $\mathcal{X} = (\chi(g,h))_{g,h\in G}$. Hence, for each entry (k,j) we obtain a unique element $g_{kj} \in G$. The phase acquired when conjugating an element $U_h \in \mathcal{E}$ by $U_{g_{kj}}$ is then given by the entry (k,j) of P_h. Using this correspondence we define local unitary operations $\tilde{U}_j := U_{g_{1j}} \otimes \cdots \otimes U_{g_{nj}}$ for $j = 1,\ldots,N$ and define $D := (1/N, \tilde{U}_1; \ldots; 1/N, \tilde{U}_N)$.

The phase matrices in $\mathcal{P}$ describe the effects of the control operations on the terms of the Hamiltonian. The (k,j)'th entry $P_{h;kj}$ is equal to $\chi(g_{kj},h)$ and describes the phase factor that is acquired when $U_h^{(k)}$ is conjugated by $\tilde{U}_j$. The 2-local term $U_h^{(k)} U_{h'}^{(\ell)}$ acquires the phase factor $P_{h,kj} \cdot P_{h',\ell j}$ in the jth time-slot. The effect of applying the whole sequence D to a local or 2-local term is to multiply it by the sum of the resulting phases. Pair-wise orthogonality of $\mathcal{P}$ is equivalent to the two conditions

$$\sum_{j=1}^{N} P_{h;kj} = 0 \quad \text{and} \quad \sum_{j=1}^{N} P_{h;kj} P_{h';\ell j} = 0 \tag{12}$$

for $h, h' \neq id$ and all $k \leq \ell$. The first condition in (12) ensures that all local terms are removed. The second condition implies that all 2-local terms are removed. $\square$

The question of how to construct collections of phase matrices which satisfy (12) is addressed in Section V-A and a solution is presented for the case where all qudits have a dimension which is a power of a prime.

B. Decoupling Schemes From Orthogonal Arrays

Orthogonal arrays have been applied to the design of experiments to plan statistical data collections systematically. The books [26], [28], and [32] provide good introductions to the topic. In this section, we define orthogonal arrays (or OA's for short) and show how they give rise to efficient decoupling schemes for t-local qudit Hamiltonians with $t \geq 2$.

Definition 12 (Orthogonal Array of Strength t): Let $\mathcal{A}$ be a finite set of cardinality s and let $n, N \in \mathbb{N}$. An $n \times N$ array M with entries from $\mathcal{A}$ is an orthogonal array with $s = |\mathcal{A}|$ levels, strength t, and index λ if and only if every $t \times N$ subarray of M contains each possible t-tuple of elements in $\mathcal{A}$ precisely λ times as a column. In design theory the notation $OA_\lambda(N,n,s,t)$ is used to denote the parameters of the corresponding orthogonal array. Since λ can be computed as $\lambda = N/s^t$, the shorthand notation $OA(N,n,s,t)$ is also commonly used.

The terminology used for parameters of OAs in decoupling and Hamiltonian simulation differs from that used in statistics. We provide a dictionary of the different languages in Table III. Note that as a convention we write OA's as $n \times N$ matrices, whereas most authors in design theory would prefer to write the same OA's as $N \times n$ matrices. Besides typographic reasons we found the presentation using $n \times N$ matrices more useful since it establishes a correspondence with pulse sequences in NMR which are typically read from left to right like a musical score [15].

Example 13: As an example of small size we give an orthogonal array with parameters $OA(16,5,4,2)$. This means that we have 16 runs/time-slots, five factors/qubits, four different levels/pulses, and 2-locality/strength 2. The array is given by the matrix

$$\begin{pmatrix} 1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4 \\ 1&2&3&4&4&3&2&1&1&2&3&4&4&3&2&1 \\ 1&3&4&2&1&3&4&2&3&1&2&4&3&1&2&4 \\ 1&3&4&2&4&2&1&3&4&2&1&3&1&3&4&2 \\ 1&2&3&4&2&1&4&3&4&3&2&1&3&4&1&2 \end{pmatrix}.$$

over the alphabet $\mathcal{A} = \{1,2,3,4\}$. It is straightforward to check that indeed every pair of rows contains all the 16 possible pairs of symbols precisely once. This array was obtained from a Hamming code over $\mathbb{F}_4$. We explore this construction in more detail in Theorem 15.

The basic idea for using an OA M with parameters $OA(N,n,d^2,t)$ over an alphabet $\mathcal{A}$ of size d^2 for decoupling t-local qudit Hamiltonians is as follows. The elements of $\mathcal{A}$ are identified with the operators $U_1,\ldots,U_{d^2}$ of a unitary error basis. The entries of the $n \times N$ matrix $M = (m_{kj})$ determine a regular N time-slot scheme $D_M = (1/N, V_1, \ldots, 1/N, V_N)$ where the control operations V_j are given by $V_j = U_{m_{1j}} \otimes U_{m_{2j}} \otimes \cdots \otimes U_{m_{nj}}$ for $j = 1,\ldots,N$.

Theorem 14 (Decoupling schemes from OA's): We use the notation introduced in preceding paragraph. Let M be an $OA(N,n,d^2,t)$. Then D_M is a regular N time-slot decoupling scheme for arbitrary t-local n-qudit Hamiltonians.

TABLE III
DICTIONARY BETWEEN TERMINOLOGY USED IN THE THEORY OF DESIGN OF EXPERIMENTS TO DESCRIBE THE PARAMETERS OF AN ORTHOGONAL ARRAY $OA_\lambda(N, n, s, t)$ OVER AN ALPHABET $\mathcal{A}$ AND TERMINOLOGY USED IN THE THEORY OF DECOUPLING SCHEMES FOR t-LOCAL n-QUDIT HAMILTONIANS

Parameter	Design of Experiments	Decoupling Schemes for qudit systems
n	factors	subsystems (qudits)
N	runs	time-slots
$\mathcal{A}$	levels	elements of an operator basis
s	number of levels	(dimension of the subsystems)^2
t	strength	locality
λ	index	—

Proof: First, assume that the number of qudits is equal to the strength t. In this case, the resulting average Hamiltonian has the form

$$\frac{1}{d^{2t}} \sum_{i_1,\ldots,i_t=1}^{d^2} (U_{i_1}^\dagger \otimes \ldots \otimes U_{i_t}^\dagger) H (U_{i_1} \otimes \ldots \otimes U_{i_t}). \tag{13}$$

Clearly, the tensor products of all possible t-tuples of $U_1, \ldots, U_{d^2}$ form a vector space basis of the linear maps acting on $(\mathbb{C}^d)^{\otimes t}$. Therefore, it follows from [9], [22] that the average Hamiltonian in (13) is zero.

Let H be an arbitrary t-local n-qudit Hamiltonian as given in (1). Pick any subset $S = \{k_1, \ldots, k_s\} \subseteq \{1, \ldots, n\}$ of the qudits, where $s = |S|$, and denote by

$$H_{k_1,\ldots,k_s} = \sum_{\alpha_1,\ldots,\alpha_s=1}^{d^2-1} J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \sigma_{\alpha_1}^{(k_1)} \ldots \sigma_{\alpha_s}^{(k_s)}$$

the s-qudit Hamiltonian which describes the s-local couplings between the qudits in S. Note that $H_{k_1,\ldots,k_s}$ acts on s qudits. We define $\hat{H}_{k_1,\ldots,k_s}$ to be the operator acting on $(\mathbb{C}^d)^{\otimes n}$ obtained by embedding $(\mathbb{C}^d)^{\otimes s}$ into $(\mathbb{C}^d)^{\otimes n}$ according to the tuple $(k_1, \ldots, k_s)$. Hence, $\hat{H}_{k_1,\ldots,k_s}$ is equal to the sum of all s-local couplings between the qubits in S and we have that H can be written as $H = \sum_{S \subseteq \{1,\ldots,n\}} \hat{H}_S$.

Since $M = (m_{kj})$, where $k = 1, \ldots, n$ and $j = 1, \ldots, N$, is an $OA(N, n, d^2, t)$ all elements of $\{1, 2, \ldots, d^2\}^s$ for $s \leq t$ appear equally often in the list $(m_{k_1,j}, \ldots, m_{k_s,j})$ where $j = 1, \ldots, N$. Therefore, the average Hamiltonian corresponding to the s-local couplings in S is evaluated as follows:

$$\begin{aligned}
&\frac{1}{N} \sum_{j=1}^{N} (U_{m_{1j}}^\dagger \otimes \cdots \otimes U_{m_{nj}}^\dagger) \\
&\quad \times \hat{H}_{k_1,\ldots,k_s} (U_{m_{1j}} \otimes \cdots \otimes U_{m_{nj}}) \\
&\quad = \Bigg[\frac{1}{N} \sum_{j=1}^{N} (U_{m_{k_1,j}}^\dagger \otimes \cdots \otimes U_{m_{k_s,j}}^\dagger) \\
&\qquad \times \ H_{k_1,\ldots,k_s} (U_{m_{k_1,j}} \otimes \cdots \otimes U_{m_{k_s,j}}) \Bigg] \\
&\quad = \Bigg[\frac{1}{d^{2t}} \sum_{i_1,\ldots,i_s=1}^{d^2} (U_{i_1}^\dagger \otimes \ldots \otimes U_{i_s}^\dagger) \\
&\qquad \times \ H_{k_1,\ldots k_s} (U_{i_1} \otimes \ldots \otimes U_{i_s}) \Bigg] = 0.
\end{aligned}$$

The last equality is due to (13) and the fact that $N = \lambda d^{2t}$. By applying this argument to any s-subset of subsystems, where $s \leq t$, we obtain that indeed any t-local n-qudit Hamiltonian is decoupled. □

C. OAs From Error-Correcting Codes

One of the most widely used constructions of OAs is based on error-correcting codes. We briefly describe this connection. First, we recall some basic facts about error-correcting codes [33], [34]. A linear code over the finite field $\mathbb{F}_q$ is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$. The metric on the space $\mathbb{F}_q^n$ is called the Hamming weight. For $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ we have that $\mathrm{wt}(x) := |\{i \in \{1, \ldots, n\} : x_i \neq 0\}|$. The minimum distance of a linear code C is defined by $d = d_{\min} := \min\{\mathrm{wt}(c) : c \in C, c \neq 0\}$, where 0 denotes the zero vector. We often abbreviate this situation by saying that C is an $[n, k, d]_q$ code. We need one more definition which is the dual code $C^\perp$ of C defined by $C^\perp := \{x \in \mathbb{F}_q^n : x \cdot y = 0 \text{ for all } y \in C\}$.

The following theorem [32, Th. 4.6] establishes a connection between orthogonal arrays and error-correcting codes.

Theorem 15 (OAs From Linear Codes): Let C be a linear $[n, k, d]_q$ code over $\mathbb{F}_q$. Let $d^\perp$ be the minimum distance of the dual code $C^\perp$. Arrange the code words of C into the columns of a matrix $A \in \mathbb{F}_q^{n \times q^k}$. Then A is an $OA(q^k, n, q, d^\perp - 1)$.

For the case of by a 2-local n-qubit Hamiltonian we can construct decoupling schemes using N pulses from any $OA(N, n, 4, 2)$. Hence, in order to apply Theorem 15 we have to find linear codes over $\mathbb{F}_4$ with parameters $[n, k, d]$ for which the minimum distance $d^\perp$ of the dual code is at least 3.

Let q be a prime power and let $m \in \mathbb{N}$. Then the Hamming code $H_{q,m}$ of length $n = (q^m - 1)/(q - 1)$ is a single-error correcting code over $\mathbb{F}_q$ with parameters $[n, n-m, 3]_q$. The dual code $H_{q,m}^\perp$ of the Hamming code $[n, n-m, 3]_q$ has parameters $[n, m, q^{m-1}]$. By specializing $q = 4$ and by using Theorem 15 for $C = H_{4,m}^\perp$ we therefore obtain orthogonal arrays with parameters $OA(N, n, 4, 2)$, where $n = (4^m - 1)/3$ and $N = 4^m$ for any choice of $m \in \mathbb{N}$. The alphabet set is in this case the finite field $\mathbb{F}_4$ of four elements.

The procedure to obtain a decoupling scheme for a system of n qubits, where n is an arbitrary natural number, not necessarily of the form $n = (4^m - 1)/3$ is as follows: first let $m \in \mathbb{N}$ be the unique integer such that $n \leq \frac{4^m-1}{3} \leq 4n$. Then construct the orthogonal array with parameters $OA(4^m, (4^m-1)/3, 4, 2)$. The columns of this OA are code words of $H_{4,m}^\perp \subseteq \mathbb{F}_4^{(4^m-1)/3}$. Now, we simply use an arbitrary subset of n of the $(4^m - 1)/3$ rows of this OA to obtain a decoupling scheme for n qubits. In case the dimension is a prime power $d = p^r$, where $p \geq 2$, we use the dual of the Hamming code $[n, n-m, 3]_{d^2}$ to obtain an

$OA(N,n,d^2,2)$, where $n=(q^m-1)/(q-1)$ and $N=q^m$. Observing that $N=(q-1)n+1$, we have shown the following result.

Corollary 16: Let H be a 2-local n-qudit Hamiltonian, where each subsystem is of prime power dimension $d=p^r$. Then there exists an efficient decoupling scheme which uses $N=cn$ time slots, where c is a constant depending on d only.

V. Equivalence of Constructions I and II

We show that the methods based on phase matrices and orthogonal arrays of strength two lead to the same class of regular decoupling schemes if we use elements of a nice error basis with an abelian index group as control operations.

A. Phase Matrices From Orthogonal Arrays

Theorem 17: Let G be a finite Abelian group of exponent $e(G)$ and let $\mathcal{E}=\{U_g \,|\, g\in G\}$ be a nice error basis for $\mathbb{C}^d$ with index group G. Let M be an $OA(N,n,d^2,2)$. Then M gives rise to an orthogonal and Schur-closed collection of phase matrices $[P_h]_{h\in G}$ of size $n\times N$ and order $e(G)$.

Proof: Denote the entries of M by m_{kj}, where $k=1,\dots,n$ and $j=1,\dots,N$. Fix an ordering $g_1,\dots,g_{d^2}$ of the elements of G and assume that $g_1=id$ is the identity. To simplify notation, we identify the operators of $\mathcal{E}$ with the elements of $\mathcal{A}$ according to $1\mapsto g_1, 2\mapsto g_2,\dots,d^2\mapsto g_{d^2}$.

Starting from the given orthogonal array we construct d^2 phase matrices $P_{g_1},\dots,P_{g_{d^2}}$ as follows. The (k,j)'th entry of the phase matrix P_h, where $h\in\{g_1,\dots,g_{d^2}\}$, is defined as $P_{h;kj} := \chi(m_{k,j},h)$, where $(\chi(g,h))$ is the character table. Note that the matrix P_{g_1} is the all-ones matrix of size $n\times N$.

While the condition $P_g\circ P_h=P_{gh}$ is automatically guaranteed since the characters form a group, we have to show that the resulting vectors are pair-wise orthogonal. In order to do so we pick two rows k and ℓ of the original orthogonal array. We may assume that the two rows have the following form (or else we apply a suitable permutation of the columns) shown in (14) at the bottom of the page, since all pairs appear with the multiplicity $\lambda=N/d^4$.

For $\nu\in\mathbb{N}$ denote by $\vec{\mathbf{1}}_\nu$ the vector $(++\cdots+)$ of length ν. We define the vector $w_h\in\mathbb{C}^{d^2} := [\chi(g_1,h),\dots,\chi(g_{d^2},h)]$ for $h\in\{g_1,\dots,g_{d^2}\}$. The kth rows of $P_{g_1},\dots,P_{g_{d^2}}$ are the vectors

$$\vec{\mathbf{1}}_\lambda\otimes w_{g_1}\otimes\vec{\mathbf{1}}_{d^2}, \vec{\mathbf{1}}_\lambda\otimes w_{g_2}\otimes\vec{\mathbf{1}}_{d^2},\dots,\vec{\mathbf{1}}_\lambda\otimes w_{g_{d^2}}\otimes\vec{\mathbf{1}}_{d^2}$$

and the ℓth rows of $P_{g_1},\dots,P_{g_{d^2}}$ are the following vectors:

$$\vec{\mathbf{1}}_\lambda\otimes\vec{\mathbf{1}}_{d^2}\otimes w_{g_1}, \vec{\mathbf{1}}_\lambda\otimes\vec{\mathbf{1}}_{d^2}\otimes w_{g_2},\dots,\vec{\mathbf{1}}_\lambda\otimes\vec{\mathbf{1}}_{d^2}\otimes w_{g_{d^2}}.$$

Whenever g_i,g_j are not both equal to the identity g_1 all these vectors are orthogonal to each other since the columns of the character table are orthogonal. This shows that all rows of the matrices $P_{g_2},\dots,P_{g_{d^2}}$ are orthogonal. □

Example 18: As an application of Theorem 17 we obtain that the sign matrices constructed by Leung [7] can alternatively be obtained from well known families of orthogonal arrays based on Hamming codes. Indeed, in case of dimension $d=2$, we use Hamming codes to obtain orthogonal arrays with parameters $OA(4^m,(4^m-1)/3,4,2)$. We can now obtain a triple of sign matrices S_x, S_y, and S_z by using the substitution rules in Theorem 17. This leads to the same sign matrices as the ones constructed in [7] by a direct construction and in [9] using spreads in the geometry $\mathbb{F}_2^{2m}$.

In the case where the dimension is an arbitrary power of a prime $d=p^r$, we use the Hamming code $[n,n-m,3]_{d^2}$ to obtain an $OA(N,n,d^2,2)$, where $n=(q^m-1)/(q-1)$ and $N=q^m$. We can use Theorem 17 to construct a collection of phase matrices from this orthogonal array.

B. Orthogonal Arrays From Phase Matrices

In this section we provide the converse to the previous section by showing that orthogonal arrays of strength two can be constructed from phase matrices. To check whether a matrix is an orthogonal array we need the following lemma which gives a criterion in terms of group characters for deciding whether an element of the group ring is an equally weighted sum of all group elements.

Lemma 19: Let G be an abelian group of order $|G|$. Denote by $\chi_1,\chi_2,\dots,\chi_{|G|}$ all irreducible characters of G, where χ_1 is the trivial character (i.e., $\chi_1(h)=1$ for all $h\in G$). Let v be an arbitrary element of the group ring $\mathbb{C}[G]$, i.e., v is a formal sum of (weighted) group elements

$$v := \sum_{g\in G}\mu_g g, \quad \mu_g\in\mathbb{C}. \tag{15}$$

If $\chi_i(v)=0$ for all $i=2,\dots,|G|$ then we have $v=\frac{\mu}{|G|}\sum_{g\in G} g$, where $\mu:=\chi_1(v)=\sum_{g\in G}\mu_g$.

A proof of this lemma is given in the Appendix. We are now ready to state the main result of this section.

Theorem 20: Let G be a finite abelian group and let $[P_h]_{h\in G}$ be a Schur-closed collection of pair-wise orthogonal phase matrices of size $n\times N$. Then these phase matrices define an orthogonal array $OA(N,n,|G|,2)$.

Proof: For each $k\in\{1,\dots,n\}$ and $j\in\{1,\dots,N\}$ each vector $v_{k,j} := [P_h;k,j]_{h\in G}$ is a row of the character table of G. Therefore, it determines uniquely $g\in G$ such that the entries of $v_{k,j}$ are $\chi_g(h)$, i.e., the values of the irreducible character corresponding to g applied to $h\in G$. For each entry (k,j), we denote the so defined group element by g_{kj} and define the $n\times N$ matrix $M=(g_{kj})$. We claim that M is an orthogonal array $OA(N,n,|G|,2)$.

$$\left(\underbrace{\begin{array}{cccc|cccc|c|cccc} 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 & \dots & d^2 & d^2 & \dots & d^2 \\ 1 & 2 & \dots & d^2 & 1 & 2 & \dots & d^2 & \dots & 1 & 2 & \dots & d^2 \end{array}}_{\lambda \text{ times}}\right) \tag{14}$$

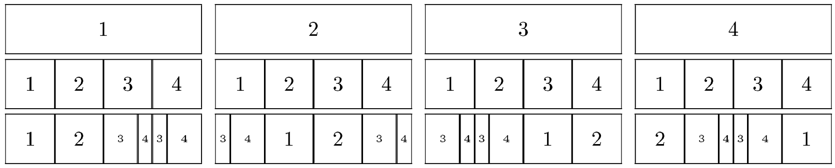

Fig. 1. A decoupling scheme for a system of three qubits which is not regular, i.e., the time slots cannot be rearranged into a form where all time slots have the same length. In this picture the time progresses from left to right and each row corresponds to one of the qubits. The transformations applied to the individual qubits correspond to the Pauli matrices as follows: $1 = \mathbf{1}_2, 2 = \sigma_x, 3 = \sigma_y$, and $4 = \sigma_z$. The time-slots indicated in the figure have four different basic lengths $t_1, t_2 = 1/4t_1, t_3 = \frac{1}{\sqrt{2}}t_2$, and $t_4 = \frac{\sqrt{2}-1}{\sqrt{2}}t_2$. For instance, the first intervals applied to the first qubit all have length t_1, whereas the interval lengths for qubit three are given by $t_2, t_2, t_3, t_4, t_4, t_3$, etc. The reason why this is a decoupling scheme is that for each pair of rows all possible combinations in $\{1, \dots, 4\}^2$ are applied for the same amount of time, which indicated by the lengths of the boxes.

Pick any two rows $(g_{kj})_j$ and $(g_{\ell j})_j$ of M. We define an element of the group ring $\mathbb{C}[G \times G]$ as the formal sum

$$r_{k\ell} := \sum_{j=1}^{N} (g_{kj}, g_{\ell j}).$$

To abbreviate the notation we denote by $\chi_{g,g'}$ the irreducible character of $G \times G$ corresponding to the element (g, g'). The decoupling conditions given in (12) are equivalent to $\chi_{g,g'}(r_{k\ell}) = 0$ for all $(g, g') \neq (e, e)$. By Lemma 19 this implies that all elements of $G \times G$ appear equally often in the sum $r_{k\ell}$. This shows that M is an orthogonal array $OA(N, n, |G|, 2)$ of strength $t = 2$ over G. $\square$

VI. Equivalence of Orthogonal Arrays and Regular Decoupling Schemes

A. Not All Decoupling Schemes are Regular

In the following, we show that not all decoupling schemes are regular, and that this is the case even if we are allowed to modify the scheme. This shows that the requirement of regularity is necessary for the results to follow in Section VI-B.

Let $D = (p_1, U_1, p_2, U_2, \dots, p_N, U_N)$ be a decoupling scheme for arbitrary n-qudit Hamiltonians. We say that $R_j = (q_{j,1}, U_{j,1}, \dots, q_{j,r_j}, U_{j,r_j})$ is a refinement of the jth time-slot (p_j, U_j) of D if the lengths $q_{j,1}, \dots, q_{j,r_j}$ of the r_j time slots sum up to p_j. Similarly, a decoupling scheme D' is a *refinement* of D if after a suitable reordering of the time-slots D' has the form $D' = (R_1, \dots, R_N)$, where the R_j's are refinements of the N time slots of D.

In general, we cannot refine a decoupling scheme in order to obtain a regular decoupling scheme. Indeed, the pulse sequence given in Fig. 1 provides an example of such a decoupling scheme. The control operations used in the scheme are the Pauli matrices $\mathbf{1}_2, \sigma_x, \sigma_y$, and σ_z. In the given example, the system consists of three qubits and in each time slot precisely one of the four Pauli matrices is applied.

It is easy to verify that the pulse sequence defined in Fig. 1 defines a decoupling scheme for any 2-local Hamiltonian: note that the sum of the times for each Pauli operator applied to the individual qubits is constant, i.e., the local terms are removed. Moreover, by considering pairs of rows we verify directly that also any pair of Pauli operators associated to the pair of symbols (a, b) with $a, b \in \{1, 2, 3, 4\}$ is applied for the same time t_2. For example in case of rows two and three we obtain for the pair $(1, 1)$ the total time t_2 and for $(3, 4)$ the total time $t_3 + t_4 = t_2$. However, the sequence cannot be subdivided into a finite number of intervals of equal lengths. Indeed, such a refinement would contradict the fact that $\sqrt{2}$ is not a rational number.

B. Equivalence of OAs and Regular Decoupling Schemes

In Section IV-B we have seen that orthogonal arrays of strength t can be used to construct decoupling schemes for t-local Hamiltonians. In order to establish a converse result we need some additional conditions on the class of schemes considered: 1) the schemes have to be regular (see Definition 2) and moreover we will assume that 2) the pulses are actually taken from a fixed set of unitaries which in addition will be assumed to form a unitary error basis. We begin by stating some standard concepts from quantum information theory which will be used in the proof. The *Shannon entropy* of a distribution $p_1, \dots, p_M$ is given by $H(p_1, \dots, p_M) = -\sum_{j=1}^{M} p_j \log_2 p_j$. The notion of entropy extends to density operators, and is usually called *von Neumann entropy*. Let ρ be an arbitrary density operator on $\mathbb{C}^d$. Then the spectral decomposition $\rho = \sum_{j=1}^{M} \lambda_j |\Psi_j\rangle\langle\Psi_j|$ is such that the eigenvalues $\lambda_1, \dots, \lambda_M$ form a probability distribution and the eigenvectors $|\Psi_1\rangle, \dots, |\Psi_M\rangle$ form an orthogonal basis of $\mathbb{C}^M$. The *von Neumann entropy* $S(\rho)$ of ρ is defined by $S(\rho) = -\sum_{j=1}^{M} \lambda_j \log_2 \lambda_j$. The von Neumann entropy takes its minimal value 0 on pure states $\rho = |\Psi\rangle\langle\Psi|$, and its maximal value $\log_2 M$ for the maximally mixed state $\rho = \mathbf{1}/M$. Let $U_1, \dots, U_N \in \mathbb{C}^{M \times M}$ be arbitrary unitary matrices, $p_1, \dots, p_N$ a probability distribution, and $|\Psi\rangle$ a state of $\mathbb{C}^M$. We have the following inequality (see [35, p. 518])

$$S\left(\sum_{j=1}^{N} p_j U_j^\dagger |\Psi\rangle\langle\Psi| U_j\right) \leq H(p_1, \dots, p_N) \leq \log_2 N. \tag{16}$$

Let D be an n-qudit regular N time slot scheme using the elements of the unitary error basis $\mathcal{E} := \{U_1, \dots, U_{d^2}\}$ as control operations. Denote by $U_{m_{1j}} \otimes \cdots \otimes U_{m_{nj}}$ the local operations that are performed on the qudits in the time-slots $j = 1, \dots, N$. The indices $m_{kj} \in \{1, \dots, d^2\}$ determine which elements of $\mathcal{E}$ are applied to the qudits in the time-slots. Denote the alphabet underlying the orthogonal array M by $\mathcal{A} = \{1, \dots, d^2\}$. Define the weight $w_{i_1,\dots,i_t}$ of each tuple $(i_1, \dots, i_t) \in \mathcal{A}^t$ to be the sum of all p_j's with $(m_{k_1,j}, \dots, m_{k_s,j}) = (i_1, \dots, i_t)$.

Theorem 21 (Equivalence): The scheme D defines a decoupling scheme for an arbitrary t-local n-qudit Hamiltonian if and only if the matrix $M = (m_{kj})$, where $k = 1, \dots, n$ and $j = 1, \dots, N$, is an orthogonal array $OA(N, n, d^2, t)$ of strength t.

Proof: We have already shown in Theorem 14 that an $OA(N, n, d^2, t)$ gives rise to a regular decoupling scheme with the stated properties. It remains to show that the converse direction is also true, i.e., whenever we have a regular decoupling scheme as stated in the theorem, then this scheme actually forms an orthogonal array. The main technical difficulty for this direction is to show the balancedness condition for any t-tuple of the rows of the scheme. The decoupling scheme implements the operator T_D given by

$$T_D(H) = \frac{1}{N}\sum_{i=1}^{N}(U_{m_{1j}} \otimes \cdots \otimes U_{m_{nj}})^\dagger H (U_{m_{1j}} \otimes \cdots \otimes U_{m_{nj}}).$$

Now suppose that $T_D(H) = 0$ for any t-local Hamiltonian H. Consequently, we have that $T_D(\hat{H}_{k_1,\dots,k_t}) = 0$ for all restrictions to t-tuples $(k_1, \dots, k_t)$ with different entries from $\{1, \dots, n\}$, where we use notation $\hat{H}$ introduced in the proof of Theorem 14 for the embeddings. Hence T_D defines a unitary depolarizer [22] for $(\mathbb{C}^d)^{\otimes t}$, i.e.

$$\sum_{i_1,\dots,i_t=1}^{d^2} w_{i_1\dots i_t}(U_{i_1} \otimes \cdots \otimes U_{i_t})^\dagger X (U_{i_1} \otimes \cdots \otimes U_{i_t}) = \frac{\mathrm{tr}(X)}{d^t}\mathbf{1}$$

for all operators X acting on $(\mathbb{C}^d)^{\otimes t}$. Let $|\Psi_1\rangle, \dots, |\Psi_{d^t}\rangle$ be an orthonormal basis of $(\mathbb{C}^d)^{\otimes t}$. We define the maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{d^t}}\sum_{r=1}^{d^t} |\Psi_r\rangle \otimes |\Psi_r\rangle$$

in the bipartite system $(\mathbb{C}^d)^{\otimes t} \otimes (\mathbb{C}^d)^{\otimes t}$ together with its corresponding density operator $\rho = |\Psi\rangle\langle\Psi|$, i.e.

$$\rho = \frac{1}{d^t}\sum_{r,s=1}^{d^t} |\Psi_r\rangle\langle\Psi_s| \otimes |\Psi_r\rangle\langle\Psi_s|.$$

We obtain that

$$\begin{aligned}
(\mathbf{1} \otimes T_D)(\rho) &= \sum_{i_1\dots i_t \in \mathcal{A}^t} w_{i_1\dots i_t} \\
&\quad \times (\mathbf{1}_{d^t} \otimes U_{i_1} \otimes \cdots \otimes U_{i_t})^\dagger \rho (\mathbf{1}_{d^t} \otimes U_{i_1} \otimes \cdots \otimes U_{i_t}) \\
&= \frac{1}{d^t}\sum_{r,s=1}^{d^t} |\Psi_r\rangle\langle\Psi_s| \otimes \sum_{i_1\dots i_t \in \mathcal{A}^t} w_{i_1\dots i_t} \\
&\quad \times (U_{i_1} \otimes \cdots \otimes U_{i_t})^\dagger |\Psi_r\rangle\langle\Psi_s| (U_{i_1} \otimes \cdots \otimes U_{i_t}) \\
&= \frac{1}{d}\sum_{r=1}^{d} |\Psi_r\rangle\langle\Psi_r| \otimes \mathbf{1}_{d^t}/d^t \\
&= \mathbf{1}_{d^t}/d^t \otimes \mathbf{1}_{d^t}/d^t = \mathbf{1}_{d^{2t}}/d^{2t}.
\end{aligned}$$

It follows from the inequalities (16) that we need at least d^{2t} different unitaries since the rank of each pure state $(\mathbf{1} \otimes U^\dagger)|\Psi\rangle\langle\Psi|(\mathbf{1} \otimes U)$ is one and since they have to sum up to d^{2t} (the rank of the maximally mixed state).

Since we use exactly d^{2t} different unitaries (tensor products of elements of the unitary error basis $\mathcal{E}$) as control operations, all weights $w_{i_1\dots i_t}$ must be equal due to the inequality (16). Now together with the fact that for regular schemes all time-slots have equal length, we conclude that $U_{i_1} \otimes \cdots \otimes U_{i_t}$ must appear with the same multiplicity. Therefore, by considering all t-tuples $(k_1, \dots, k_t)$ of t qudits we see that the decoupling D scheme must correspond to an orthogonal array $OA(N, n, d^2, t)$ with d^2 levels and strength t. □

VII. Conclusion

We have shown the equivalence between two important constructions of decoupling schemes for Hamiltonians acting on systems of qubits and, more generally, qudits. The first construction is based on triples of pair-wise orthogonal sign matrices which are closed under taking entry-wise products. The second construction is based on orthogonal arrays of strength two. We have generalized the construction based on orthogonal arrays in two directions. One is the generalization to the case where the subsystems have higher dimensions. The other shows that Hamiltonians with higher couplings can be dealt with by using orthogonal arrays of strength $t \geq 2$: the order of the couplings directly translates into the strength of the orthogonal array. Finally, we have shown the equivalence of regular decoupling schemes for t-local Hamiltonians and orthogonal arrays of strength t.

Appendix

In this appendix we present some basic facts from the character theory of finite abelian groups which are needed in this paper. First, we cite the following theorem on characters of abelian groups (see, e.g., [37, Ch. V, Sec. 6]):

Theorem 22 (Characters of Abelian Groups): Let G be a finite abelian group of order $|G|$. Then every irreducible representation ρ of G has degree 1, i.e., we have that $\rho : G \to \mathbb{C}^\times$ is a homomorphism which maps G to scalars. Furthermore, the number of different irreducible representations (irreducible characters) of G is given by $|G|$ and the characters form a group $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ under pointwise multiplication. Hence, we have that

$$\chi\tilde{\chi}(h) = \chi(h)\tilde{\chi}(h)$$

for all irreducible characters $\chi, \tilde{\chi}$ and $h \in G$. Moreover, the character group $\hat{G}$ is isomorphic to G. Thus, we can label the characters of G by the elements of G using an isomorphism which maps $h \mapsto \chi_h$ for all $h \in G$.

We next give the proofs of Lemmas 10 and 19 in this text.

Lemma 10: Let $\mathcal{E} := \{U_g \,|\, g \in G\}$ be a nice error basis with an Abelian index group G. Then the matrix $\mathcal{X} = (\chi(g,h))_{g,h \in G}$ is the character table of the group G.

Proof: Let α be the factor system corresponding to the nice error basis $\mathcal{E}$ with abelian index group G. We prove that $\mathcal{X}$ is a character table by showing that the rows of $\mathcal{X}$ form a group under pointwise multiplication that is isomorphic to G (see Theorem 22). We first show that

$$\chi(g,h)=\frac{\alpha(h,g)}{\alpha(g,h)}.$$

We have that

$$U_gU_h=\alpha(g,h)U_{gh} \tag{17}$$
$$U_hU_g=\alpha(h,g)U_{hg}=\alpha(h,g)U_{gh}. \tag{18}$$

By multiplying (18) by $U_g^\dagger$ from the left and using (17), we obtain

$$\begin{aligned} U_g^\dagger U_hU_g &= \alpha(h,g)U_g^\dagger U_{gh} \\ &= \frac{\alpha(h,g)}{\alpha(g,h)}U_g^\dagger U_gU_h \\ &= \frac{\alpha(h,g)}{\alpha(g,h)}U_h. \end{aligned}$$

Let $g,\tilde g$ be arbitrary elements of G. Note that we have $\overline{\alpha(\tilde g^{-1},g)}\alpha(\tilde g^{-1},g)=1$ (otherwise the matrix $U_{\tilde g^{-1}}U_g=\alpha(\tilde g^{-1},g)U_{\tilde g^{-1}g}$ would not be unitary). The group property follows:

$$\begin{aligned} &\chi(g,h)\chi(\tilde g^{-1},h)U_h \\ &\quad=U_g^\dagger U_{\tilde g^{-1}}^\dagger U_hU_{\tilde g^{-1}}U_g \\ &\quad=\overline{\alpha(\tilde g^{-1},g)}\alpha(\tilde g^{-1},g)U_{g\tilde g^{-1}}^\dagger U_hU_{g\tilde g^{-1}} \\ &\quad=U_{g\tilde g^{-1}}^\dagger U_hU_{g\tilde g^{-1}} \\ &\quad=\chi(g\tilde g^{-1},h)U_h \end{aligned}$$

which holds for all $g,\tilde g,h\in G$. The rows of $\mathcal{X}$ form a group that is isomorphic to G (as opposed to a proper subgroup of G) since there is a bijection between the rows of $\mathcal{X}$ and the elements of G. This is seen as follows. Assume that there are $g\neq\tilde g$ such that $\chi(g,h)=\chi(\tilde g,h)$ for all $h\in G$. This is equivalent to $U_g^\dagger U_hU_g=U_{\tilde g}^\dagger U_hU_{\tilde g}$. Set $U=U_{\tilde g}U_g^\dagger$. Then we have $UM=MU$ for all $M\in\mathbb{C}^{d\times d}$ since the matrices U_h form a basis of $\mathbb{C}^{d\times d}$. Therefore U must be a multiple of the identity matrix. Due to the properties of a nice error basis this is only possible for $g=\tilde g$. This proves that there is a bijection between the group elements of G and the rows of $\mathcal{X}$. □

Lemma 19: Let G be an abelian group of order $|G|$. Denote by $\chi_1,\chi_2,\ldots,\chi_{|G|}$ all irreducible characters of G, where χ_1 is the trivial character (i.e., $\chi_1(h)=1$ for all $h\in G$). Let v be an arbitrary element of the group ring $\mathbb{C}[G]$, i.e., v is a formal sum of (weighted) group elements

$$v:=\sum_{g\in G}\mu_g g,\quad \mu_g\in\mathbb{C}.$$

If $\chi_i(v)=0$ for all $i=2,\ldots,|G|$ then we have $v=\frac{\mu}{|G|}\sum_{g\in G}g$, where $\mu:=\chi_1(v)=\sum_{g\in G}\mu_g$.

Proof: Let $G:=\{g_1,\ldots,g_{|G|}\}$ be an arbitrary ordering of the group elements, where g_1 is the identity element of G. Denote by $\mathcal{X}$ the (normalized) character table of G, i.e.,

$$\mathcal{X}_{ij}:=|G|^{-1/2}\chi_i(g_j)$$

for $i,j=1,\ldots,|G|$. Recall that the (normalized) character table $\mathcal{X}$ is a unitary matrix and has the following form [36], [37]:

$$\mathcal{X}=\frac{1}{|G|^{1/2}}\begin{pmatrix}1 & \cdots & 1\\ \vdots & * & \\ 1 & & \end{pmatrix}. \tag{19}$$

The conditions given in the lemma can now be expressed as

$$|G|^{1/2}\mathcal{X}(\mu_1,\mu_2\ldots,\mu_{|G|})^T=(\mu,0,\cdots,0)^T.$$

Multiplying by the inverse $\mathcal{X}^{-1}$ we obtain

$$(\mu_1,\mu_2,\ldots,\mu_{|G|})^T=\frac{\mu}{|G|}(1,1,\ldots,1)^T$$

due to the special form in (19). This show that all coefficients μ_g in (15) are equal to $\mu/|G|$. □

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