# SPECTROSCOPIC SIGNATURES OF CRYSTAL MOMENTUM FRACTIONALIZATION: SUPPLEMENTAL MATERIAL

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## A. Relations for square lattice space group and time reversal symmetry

Following Ref. [8] of the main text [Phys. Rev. B 87, 104406 (2013), arXiv:1212.0593], we list the 10 relations that define the symmetry group consisting of square lattice space group and time reversal operations, in terms of the generators $T_x$, $P_x$, $P_{xy}$ and $\mathcal{T}$ defined in the main text (recall that $T_y = P_{xy} T_x P_{xy}$.) The relations are:

$$T_x T_y T_x^{-1} T_y^{-1} = 1 \tag{A.1}$$
$$T_x P_x T_x P_x^{-1} = 1 \tag{A.2}$$
$$T_y P_x T_y^{-1} P_x^{-1} = 1 \tag{A.3}$$
$$\mathcal{T} T_x \mathcal{T}^{-1} T_x^{-1} = 1 \tag{A.4}$$
$$P_x^2 = P_{xy}^2 = \mathcal{T}^2 = 1 \tag{A.5}$$
$$(P_x P_{xy})^4 = 1 \tag{A.6}$$
$$\mathcal{T} P_x \mathcal{T}^{-1} P_x^{-1} = 1 \tag{A.7}$$
$$\mathcal{T} P_{xy} \mathcal{T}^{-1} P_{xy}^{-1} = 1. \tag{A.8}$$

## B. No subtypes of Type D spectral periodicity

Here, we consider the four choices of $\sigma_i$ satisfying $\sigma_{txty} = 1$, $\sigma_{tS} = -1$, with Type D spectral periodicity. We show that each of these four cases has the same spectral periodicity, and thus there are no subtypes of Type D.

Since $\sigma_{tS} = -1$, single-spinon eigenstates can be grouped into degenerate doublets with crystal momenta $\mathbf{k}$ and $\mathbf{k} + (\pi, \pi)$. Therefore, each two-spinon state with single-spinon crystal momenta $\mathbf{k}$, $\mathbf{k}'$ is part of a degenerate quadruplet, in which the crystal momenta $\mathbf{q}$ and $\mathbf{q} + (\pi, \pi)$ each appear twice. Making a shift $\mathbf{k} \to \mathbf{k} + (\pi, \pi)$ has no effect on the quadruplet.

The four choices of the $\sigma_i$ under consideration can be specified by the ordered triple $(\sigma_{Ttx}, \sigma_{txpx}, \sigma_{typx})$, and are

$$\begin{aligned} \mathrm{D}_A &= (-1, -1, -1) \\ \mathrm{D}_B &= (1, 1, -1) \\ \mathrm{D}_C &= (-1, 1, 1) \\ \mathrm{D}_D &= (1, -1, 1). \end{aligned} \tag{B.1}$$

First, we note that $\mathrm{D}_A$ and $\mathrm{D}_B$ can be mapped into one another by shifting the origin of $\mathbf{k}$ by $(\pi/2, \pi/2)$, as discussed in the main text, and thus have the same spectral periodicity. The same holds for $\mathrm{D}_C$ and $\mathrm{D}_D$. It is thus sufficient to focus on $\mathrm{D}_A$ and $\mathrm{D}_C$.

We now consider the effect of acting with $P_x^s$, $P_{xy}^s$ and $\mathcal{T}^s$ on one spinon in the quadruplet described above. This action is given in Eq. (17) of the main text, and we see that for both $\mathrm{D}_A$ and $\mathrm{D}_C$, the only effect of the nontrivial $\sigma_i$'s is to augment the ordinary symmetry transformation of $\mathbf{k}$ with a shift of $(\pi, \pi)$ in some cases. For $\mathrm{D}_A$ this shift is present for both $P_x^s$ and $\mathcal{T}^s$, while for $\mathrm{D}_C$ it is only present for $\mathcal{T}^s$. Because such shifts have no effect on the quadruplet, and because the presence/absence of the $(\pi, \pi)$ shift for $P_x^s$ is the only difference between $\mathrm{D}_A$ and $\mathrm{D}_C$, the spectral periodicity is the same in both cases.

## C. Parton mean-field $\mathbb{Z}_2$ spin liquid states

Here, we use fermionic parton theory to exhibit mean-field, gapped $\mathbb{Z}_2$ spin liquid states with Type D and D1$d$ spectral periodicity. We dub these State-D and State-D1$d$, respectively. The PSG is specified by the action of symmetry generators on the two-component fermionic spinon field $\psi = (f_\uparrow, f_\downarrow^\dagger)^T$ [X.-G. Wen, Phys. Rev. B 65, 165113 (2002), arXiv:cond-mat/0107071], which can be chosen to be

$$\begin{aligned} T_x^s\, \psi(x,y) &= \sigma_{txty}^y \psi(x+1, y) \\ \mathcal{T}^s\, \psi(x,y) &= \sigma_{Ttx}^{x+y} (i\tau^y) g_T \psi(x,y) \\ P_x^s\, \psi(x,y) &= \sigma_{txpx}^x \sigma_{typx}^y g_{P_x} \psi(-x, y) \\ P_{xy}^s\, \psi(x,y) &= \sigma_{txty}^{xy} g_{P_{xy}} \psi(y, x). \end{aligned} \tag{C.1}$$

Here $g_T$, $g_{P_x}$, and $g_{P_{xy}}$ are $2 \times 2$ matrices and $\tau^{x,y,z}$ are the $2 \times 2$ Pauli matrices. It is straightforward to construct Hamiltonians quadratic in $\psi$ that are invariant under these transformations.

To construct State-D, we set $\sigma_{txty} = 1$, $\sigma_{txpx} = \sigma_{Ttx} = \sigma_{typx} = -1$, and construct a Bogoliubov–de Gennes Hamiltonian $\psi^\dagger \mathcal{H}_D \psi$ that involves hopping and pairing terms to next-nearest-neighbor, with

$$\begin{aligned} \mathcal{H}_D(\mathbf{k}) = &\left(u_0 + u_2 \cos k_x \cos k_y\right) \tau^x \\ &+ u_1 (\cos k_x + \cos k_y) \tau^z + u_2' \sin k_x \sin k_y \tau^y. \end{aligned} \tag{C.2}$$

In this PSG, $g_{P_{xy}} = \tau^0$ (the $2 \times 2$ identity matrix), $g_{P_x} = i\tau^x$, and $g_T = i\tau^z$. The physical (two-spinon) spectrum is shown in Fig. 2(a) of the main text for parameters $u_1 = u_0/4$, $u_2 = -u_0/2$, and $u_2' = u_0/4$. As expected, it has Type D spectral periodicity, with equivalent minima at $(0, 0)$ and $(\pi, \pi)$. As discussed in the main text, when we include fluctuations about this mean-field state we arrive at a $\mathbb{Z}_2$ spin liquid which we term State-D.

To construct State-D1$d$, we modify the above PSG by setting $\sigma_{Ttx} = 1$. Only pairing terms appear now, and a generic Hamiltonian with the same range is

$$\begin{aligned} \mathcal{H}_{D1d}(\mathbf{k}) = &\left(u_0 + u_2 \cos k_x \cos k_y\right) \tau^x \\ &+ \left[u_1 (\cos k_x + \cos k_y) + u_2' \sin k_x \sin k_y\right] \tau^y. \end{aligned} \tag{C.3}$$

An example spectrum is shown in Figs. 2(b) and 2(c) of the main text, with the same parameters as for State-D. As expected, the global minima at $(0, 0)$ and $(\pi, \pi)$ are degenerate but inequivalent, and the projection of the density of states to $q_x$ is periodic under $q_x \to q_x + \pi$.