It is intended to make a systematic search for a possible drift in other directions when the apparatus has been slightly modified to increase its sensitiveness and facilitate the numerous observations that will be necessary.

The writer is under great obligation to Dr. R. A. Millikan, whose continued interest has made this investigation possible.

ON THE EVALUATION OF CERTAIN INTEGRALS IMPORTANT IN THE THEORY OF QUANTA

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1. It is known[1] that the matrix of the hydrogen atom, determining the intensities of the hydrogen series lines and of their fine structure components, essentially depends on the integral

$$I = \int_0^\infty r^3 \chi\chi' dr, \tag{1}$$

where the functions χ and χ' are solutions of the Schroedinger equation[2]

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left(\frac{2\mu E}{K^2} + \frac{2\mu e^2}{K^2 r} - \frac{(k-1)k}{r^2}\right)\chi = 0. \tag{2}$$

The symbols μ, e stand for the mass and the charge of the electron; $K = h/2\pi$ (h Planck's constant). The energy E and the integer k have different values in χ and χ'. These functions are supposed to be finite for $r = 0$ and to vanish for $r = \infty$.

Since in the case of elliptic orbits the functions χ, χ' turn out to be polynomials multiplied by an exponential, the direct evaluation term by term is possible on principle. The numerical computations involved are, however, so lengthy as to make this method almost prohibitive in practice. We give, therefore, in this note a reduction of the integral (1) to a simple and convenient expression. Such a reduction is quite indispensable in the case of hyperbolic orbits. Moreover, the procedure applied has an interest beyond the special case of the Kepler motions, since quite analogous expressions occur in other problems of the quantum theory. In fact, the same method has been used by the author for reducing the intensity expressions of the components in the Stark effect.[3] Only the simple closed Kepler motion, neglecting relativity effect, and spin of the electron, will be considered in the following sections.

2. Our procedure is based on the existence of certain recurrent relations between functions χ with different parameters. We use the substitution

$$\chi = r^{k-1} \exp. (\alpha r).M, \qquad \alpha^2 = -2\mu E/K^2. \tag{3}$$

According to Schroedinger, only values of α are permissible for which

$$\mu e^2/\alpha K^2 = -(s + k) = -l, \tag{4}$$

where s and l are two other integers. Equation (2) is transformed into

$$\frac{d^2M}{dr^2} + 2\left(\alpha + \frac{k}{r}\right)\frac{dM}{dr} - \frac{2\alpha s}{r} M = 0. \tag{5}$$

The solution of this equation, satisfying our requirements of finiteness obviously is

$$M = 1 + \frac{s}{1.2k} 2\alpha r + \frac{s(s-1)}{1.2.2k(2k+1)} (2\alpha r)^2 + \ldots\ldots \tag{6}$$

We can consider this expression as derived from a hypergeometric function by a process of transition to the limit

$$M(l, 2k) = \lim_{\substack{x=0,\ \beta=\infty \\ \beta x = r}} F(-s, \beta, 2k, -2\,\alpha x). \tag{7}$$

I.e., we let x decrease, β increase indefinitely, keeping, however, βx finite and equal to r. The advantage of this representation is that from the relations existing between hypergeometric functions[4] we can immediately derive the following recurrences for M:

$$(2k + s - 1)M(s, 2k) - sM(s - 1, 2k) = (2k - 1)M(s, 2k - 1), \tag{8}$$

$$2\alpha r M(s, 2k) = (2k - 1)[M(s + 1, 2k - 1) - M(s, 2k - 1)], \tag{9}$$

$$2\alpha r M(s, 2k) = (2k + s)M(s + 1, 2k) - (2k + 2s)M(s, 2k) + sM(s-1, 2k), \tag{10}$$

$$\frac{dM(s, 2k)}{dr} = \frac{s}{r}[M(s, 2k) - M(s - 1, 2k)]. \tag{11}$$

The parameter α is supposed to have the same value in all the functions M of these formulae.

3. We shall consider the case in which the function χ depends on the arguments s and α, χ' on s', α' while k is the same in both. Therefore, we shall omit in this section the parameter k in order to abbreviate our symbols. Moreover, for the present, we shall not make use of the connection (4) between α and s, k. If we denote for short

$$R(s, s') = \int_0^\infty r\chi(s, \alpha)\chi(s', \alpha')dr, \tag{12}$$

we can obtain a relation from equation (2). Multiplying that equation by $r^2\chi'$, subtracting the corresponding equation for χ' multiplied by $r^2\chi$, and integrating with respe ct to r from 0 to ∞, we have

$$(\alpha^2-\alpha'^2)\int_0^\infty r^2\chi(s, \alpha)\chi(s', \alpha')dr + 2[(s + k)\alpha-(s' + k)\alpha']R(s, s') = 0. \tag{13}$$

If we transform the integral by means of our recurrence (10) and use the substitutions

$$R(s, s') = u^s V(s, s')/(\alpha + \alpha')^{2k}, \qquad u = (\alpha - \alpha')/(\alpha + \alpha') = (l - l')/(l + l'), \tag{14}$$

this relation can be written so

$$(s + 2k)u^2V(s + 1) - [(s - s') + (s + s' + 2k)u^2]V(s) + sV(s - 1) = 0. \tag{15}$$

We do not write the argument s' because in this relation (as in the following ones) it is kept constant. The interest of relation (15) lies in the fact that by means of it we can compute successively all the $V(s, s')$ if only $V(0, 0)$ is known. According to (3) and (6)

$$R(0, 0) = \int_0^\infty r^{2k-1} \exp.[(\alpha + \alpha')r]dr = \frac{(2k-1)!}{(\alpha + \alpha')^{2k}}; \qquad V(0, 0) = (2k - 1)! \tag{16}$$

It follows that $V(s)$ depends on $u^2 = z$ only. Noticing this and bearing in mind that M is a function of the product αr, we get a second relation by differentiating (12) with respect to α:

$$2z(1 - z)\frac{dV(s)}{dz} = -(s + 2k)zV(s + 1)-[s-(s + 2k)z]V(s) + sV(s-1). \tag{17}$$

Combining (15) and (17) we obtain a differential equation determining $V(s, s')$

$$4z^2(1 - z)\frac{d^2V(s)}{dz^2} + 4z[(s + 1)-(s + 2k + 1)z]\frac{dV(s)}{dz} + s[(2s-s') + (s' + 2s + 4k)z]V(s) = 0. \tag{18}$$

4. This last differential equation is of the hypergeometric type and its solution obviously is

$$V(s, s') = Cu^{-s'}F(s + 2k, -s', s - s' + 1, u^2) \tag{19}$$

where C is a constant depending on s, s', k. Its dependence on s can be obtained from (15) and is given by the factor $s!/(s-s')!$. The rest can be found from the limiting value (16) and from the requirement of symmetry

with respect to s and s'. We obtain in this way $C = (2k-1)!\,(2k-1)!\cdot s!/(s-s')!(s'+2k-1)!$ giving

$$R(s, s') = \frac{(2k-1)!\,(2k-1)!\,s!}{(s-s')!\,(s'+2k-1)!} \cdot \frac{u^{s-s'}}{(\alpha+\alpha')^{2k}} \cdot F(s+2k, -s', s-s'+1, u^2). \quad (20)$$

The symmetry of this expression is brought out if we rearrange it in falling powers of u^2

$$R(s, s') = (-1)^{s'} \frac{(2k-1)!\,(2k-1)!\,(s+s'+2k-1)!}{(s+2k-1)!\,(s'+2k-1)!} \cdot \frac{u^{s+s'}}{(\alpha+\alpha')^{2k}} \cdot F\left(-s, -s', -s-s'-2k+1, \frac{1}{u^2}\right). \quad (21)$$

For finding the intensities we must normalize the functions χ. To do this we have to know the value of the integral $\int_0^\infty r^2\chi^2 dr$. We can obtain it by applying (10) to $r\chi$. Since in our case $\alpha = \alpha'$, $u = 0$, the hypergeometric function is reduced to unity. Moreover $u^{s-s'}$ is equal to 1 in the middle term and vanishes in the two other terms, so that

$$1/A(k, s) = \int_0^\infty r^2\chi^2(s, k)dr = (-2\alpha)^{-2k-1}\, 2(s+k)\,(2k-1)! \cdot (2k-1)!\,s!/(s+2k-1)! \quad (22)$$

5. The intensity expressions contain as their essential factor the integral

$$I = \sqrt{A(k, s)A(k-1, s')} \int_0^\infty r^3\chi(s, k)\chi(s', k-1)\, dr. \quad (23)$$

We apply (9) two times

$$I = \sqrt{A(k, s)A(k-1, s')}\, \frac{(2k-2)(2k-1)}{4\alpha^2} \cdot \int_0^\infty r^2[\chi(s, k-1) - 2\chi(s+1, k-1) + \chi(s+2, k-1)]\chi(s', k-1)dr. \quad (24)$$

Relation (13) is always valid, except when $\alpha = \alpha'$, $s = s'$. Since, in our case, $1/\alpha \backsim s+k$, $1/\alpha' \backsim s'+k-1$, we may apply (13) to all three terms of expression (24). Using abbreviations (14) and relation (15), we arrive at

$$I = -\frac{(1-u^2)^{k-1/2}\, u^{s-1}}{2(\alpha+\alpha')(2k-3)!(2k-3)!} \sqrt{\frac{(s+2k-1)!\,(s'+2k-3)}{(s+k)(s'+k-1)s!\,s'!}} \cdot [V(s, s') - u^2V(s+2, s')]. \quad (25)$$

If we substitute expressions (21)

$$I(l, k; l', k-1) = (-1)^{s'} \frac{(1-u^2)^{k-1/2}\, u^{s+s'-1}}{2(\alpha+\alpha')\sqrt{l.l'}} \cdot$$

$$\cdot \frac{(l+l'-2)!}{\sqrt{s!\,s'!(l+k-1)!\,(l'+k-2)!}} \cdot \{u^2(l+l'-1)(l+l')\psi(s+2', s', k) -$$

$$-(l'+k-2)(l'+k-1)\psi(s, s', k)\} \quad (26)$$

using the abbreviation

$$\psi(s, s', k) = F\left(-s, -s', -s-s'-2k+3, \frac{1}{u^2}\right). \quad (27)$$

By an analogous reasoning we obtain for the case $k' = k + 1$ the expression

$$I(l, k; l', k+1) = (-1)^{s'+1} \frac{(1-u^2)^{k+1/2}\, u^{s+s'-1}}{2(\alpha+\alpha')\sqrt{l.l'}} \cdot$$

$$\cdot \frac{(l+l'-2)!}{\sqrt{s!\,s'!(l+k-1)!\,(l'+k)!}} \cdot \{u^2(l+l'-1)(l+l')\psi(s, s'+2, k+1) -$$

$$-(l'+k-1)(l'+k)\psi(s, s', k+1)\} \cdot \quad (28)$$

6. In the case of the Balmer series we have $l' = 2$ with the two possible cases $k' = 2, s' = 0$ and $k' = 1, s' = 1$. We arrive at the following three possible values

$$\left.\begin{aligned} I(l, 3; 2, 2) &= -B \cdot \frac{2l}{l^2-4}\sqrt{\frac{(l+2)!}{3!(l-3)!}}, \\ I(l, 2; 2, 1) &= B \cdot 2\sqrt{\frac{(l+1)!}{2!(l-2)!}}, \\ I(l, 1; 2, 2) &= B.l\sqrt{\frac{l!}{3!(l-1)!}} \\ B &= (1-u^2)^{5/2}\, u^{l-3}/(\alpha+\alpha')\sqrt{2l}. \end{aligned}\right\} \quad (29)$$

These formulae give, in fact, numerical values identical with those found by Eckart.

[1] C. Eckart, *Physic. Rev.*, October, 1926; E. Schroedinger, *Ann. Physik*, **80**, p. 437, 1926.

[2] E. Schroedinger, *Ann. Physik.*, **79**, p. 361, 1926.

[3] P. S. Epstein, *Physic. Rev.*, October, 1926.

[4] Compare: W. Laska, *Sammlung von Formeln*, p. 45, Braunschweig, 1888–1894.