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J. D. COLE AND Y. M. LYNN

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SUMMARY

The simple wave solutions, which in ordinary gas dynamics correspond to expansion flows or Prandtl-Meyer flows are generalized here to ideal magnetohydrodynamic flows. The one-dimensional unsteady (x, t) case is considered. Due to magnetic effects more than one component of field and velocity must be considered. To carry out the simple wave formalism the equations of motion (continuity, momentum, induction) are written in terms of flow velocities (u_1, u_2), Alfvén velocities (b_1, b_2) and sound speed (a). These velocities are then functions only of the phase $\xi = x_1 - U(\xi)t$; each phase line can be thought of as an infinitesimal wave propagating with a speed $c = U - u_1$ related to the flow. By elimination of (u_1, u_2) the system of five first-order ordinary differential equations can be reduced to three (homogeneous) equations. The vanishing of the determinant of coefficients provides a famous relation for wave speed c and reduces the problem to integration of two first-order equations. The further introduction of dimensionless variables, ratios of wave speeds, reduces the problem to integration of a single first-order equation. By studying the trajectories of this differential equation an overall view of all possible solutions is obtained; numerical integration is also carried out in the case of slow waves. As applications of this theory various physical problems are studied, the receding piston and waves produced by a current sheet.

MAGNETOHYDRODYNAMIC SIMPLE WAVES*

by

J. D. Cole and Y. M. Lynn

I. INTRODUCTION

Magnetohydrodynamic simple waves are the analogue in a conducting compressible fluid of the gas dynamic simple waves studied by Riemann (for a complete discussion of gas dynamic simple waves see Courant and Friedrichs, Ref. 1). As such these flows only exist as solutions to the equations of motion of an ideal (dissipationless) fluid. In the gas dynamical case this approximation corresponds to neglecting viscosity and thermal conductivity while in the MHD** case it is assumed in addition that the electric conductivity σ is infinite; the currents that flow are finite so that there is no dissipation and always (since by Ohms Law $\vec{j} = \sigma(\vec{E} + \vec{u} \times \vec{B})$)

$$\vec{E} = -\vec{u} \times \vec{B} \tag{1-1}$$

A natural question to ask concerns the validity of the approximation $\sigma = \infty$ since for most fluids σ is finite and has a considerably smaller value than in good metallic conductors. The question can, of course, only be answered in terms of certain dimensionless variables. For one-dimensional unsteady flow (x_1, t) the flow is described by non-linear equations very similar in form to those of ordinary gas dynamics, with the addition of a diffusion term (higher order spatial derivative) if σ is finite. Thus, the

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**MHD = Magnetohydrodynamic.

situation is familiar. Non-linear effects dominate the underlying motion; diffusion resolves discontinuities after some time and the actual motion is close to that of the ideal fluid. The time units are measured in a scale of

$$T^* = \frac{1}{\mu\sigma V^2} \tag{1-2}$$

and the length scale in

$$L^* = \frac{1}{\mu\sigma V} \tag{1-3}$$

where V = a characteristic velocity, for example a typical speed of sound or flow velocity. For $t \gg T^*$ an actual flow close to the ideal flow should be achieved. For more detailed calculations showing the asymptotic approach to the ideal fluid solution see Ref. 2.

Obtaining solutions of non-linear equations always presents formidable problems. In the case of simple waves an attempt is made to define a simple flow by constructing a solution which depends on a single parameter ξ which denotes the "phase". The form of the solution is typically

$$u_1(x_1, t) = u_1(\xi), \qquad \xi = x_1 - U(\xi)t \tag{1-4}$$

Such a wave evolves from itself and at any time represents a distorted image of its previous self. If the distorted image attains a triple value, the existence of a shock wave is implied. It is a real triumph of mind over matter that such flows actually exist. Since lines ξ = const. can separate regions of uniform state from regions of variation, these lines must be characteristics.

In the following part of the paper the theory of simple wave flows is first presented, following for the most part the masterly treatment of Friedrichs (Ref. 3). Using a slightly different set of variables here enables group theoretic arguments to be applied which reduces the problem to the integration of a single first-order differential equation of wave speeds. Friedrichs pointed out that the solution of this equation can be obtained explicitly. The details of calculations based on this idea are presented by Bazer (Ref. 4), who also uses the simple waves as elements in solving certain boundary-value problems. A similar treatment is presented by Kemp and Petschek (Ref. 5). In this paper the trajectories of the first-order differential equation are studied which provides an overall view of all the possible solutions. These trajectories are also used as the basis for numerical calculations of the properties of the solutions. In the final part of the paper some examples are discussed which show how the simple wave solution enters as a basic element of the complete solution.

II. SIMPLE WAVE ANALYSIS

Consider the equations of one-dimensional unsteady (x_1, t) motion of the usual MHD fluid ($\sigma = \infty$, zero displacement current). For the case considered here let the magnetic and velocity fields have only two components

$$\vec{u} = (u_1, u_2), \qquad \vec{B} = (B_1, B_2) .$$

The introduction of the third component does not generalize the simple wave solutions but does permit the existence of general transverse waves. These transverse waves may in fact have to be included if certain boundary conditions are to be satisfied. The equations read:

continuity $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1) = 0 \qquad (2\text{-}1a)$$

x_1-momentum $$\rho\left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial P}{\partial x_1} + \frac{B_2}{\mu}\frac{\partial B_2}{\partial x_1} = 0 \qquad (2\text{-}1b)$$

x_2-momentum $\qquad \rho(\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1}) - \frac{B_1}{\mu} \frac{\partial B_2}{\partial x_1} = 0$ (2-1c)

x_2-induction $\qquad \frac{\partial B_2}{\partial t} - \frac{\partial}{\partial x_1} (u_2 B_1 - u_1 B_2) = 0$ (2-1d)

isentropy $\qquad (\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1})(\frac{P}{\rho^\gamma}) = 0$ (2-1e)

(1-1) has been used to eliminate the electric field $\vec{E} = (0, 0, E_3)$ in the x_2-induction equation. The x_1-induction and div $\vec{B} = 0$ imply

$$B_1 = \text{constant} \tag{4-4}$$

The waves run along field lines of constant strength. The extra terms in the momentum equation can either be thought of as the components of the divergence of the Maxwell stress tensor T

$$T = \frac{1}{\mu}(\vec{B} \circ \vec{B} - \frac{1}{2}B^2 I) = \frac{1}{\mu}\begin{pmatrix} \frac{B_1^2 - B_2^2}{2} & B_1B_2 \\ B_1B_2 & \frac{B_2^2 - B_1^2}{2} \end{pmatrix} \tag{2-3}$$

or the components of the Lorentz force/volume $\vec{f}$

$$\vec{f} = (\vec{j} \times \vec{B}) = \frac{1}{\mu}(\text{curl } \vec{B} \times \vec{B}) \tag{2-4}$$

Note $\vec{j} = \vec{i}_3 \frac{1}{\mu} \frac{\partial B_2}{\partial x_1}$ so that currents flow only in the x_3-direction. As a first step the equations are written only in terms of the flow velocity and certain wave speeds characteristic of the fluid. The flow is assumed initially isentropic and it follows from (2-1e) that the entropy is constant everywhere. The local speed of sound for a polytropic gas is thus given by

$$a^2 = \frac{dP}{d\rho} = k\rho^{\gamma-1}; \quad k = \text{const.} \tag{2-5a}$$

or

$$\frac{d\rho}{\rho} = \frac{2}{\gamma-1}\frac{da}{a} \tag{2-5b}$$

with the fields in the $(1, 2)$ directions

$$b_1 = \frac{B_1}{(\mu\rho)^{1/2}}, \quad b_2 = \frac{B_2}{(\mu\rho)^{1/2}} \tag{2-6}$$

Note that, since B_1 is constant, the speed b_1 is a measure of the density only and as such is simply related to a. Note also

$$\frac{dB_1}{B_1} = \frac{db_1}{b_1} + \frac{1}{2}\frac{d\rho}{\rho} = \frac{db_1}{b_1} + \frac{1}{\gamma-1}\frac{da}{a} = 0 \tag{2-7a}$$

$$\frac{dB_2}{B_2} = \frac{db_2}{b_2} + \frac{1}{2}\frac{d\rho}{\rho} = \frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a} \tag{2-7b}$$

Introducing new dependent variables (u_1, u_2, a, b_1, b_2) into (2-1) and (2-2), these equations may be rewritten

$$\frac{2}{\gamma-1}\frac{1}{a}\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right)a + \frac{\partial u_1}{\partial x_1} = 0 \tag{2-8a}$$

$$\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right)u_1 + \frac{2}{\gamma-1}\,a\frac{\partial a}{\partial x_1} + b_2^2\left(\frac{1}{b_2}\frac{\partial b_2}{\partial x_1} + \frac{1}{\gamma-1}\frac{1}{a}\frac{\partial a}{\partial x_1}\right) = 0 \tag{2-8b}$$

$$\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right)u_2 - b_1 b_2\left(\frac{1}{b_2}\frac{\partial b_2}{\partial x_1} + \frac{1}{\gamma-1}\frac{1}{a}\frac{\partial a}{\partial x_1}\right) = 0 \tag{2-8c}$$

$$\frac{1}{b_2}\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right)b_2 + \frac{1}{\gamma-1}\frac{1}{a}\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right)a + \frac{\partial u_1}{\partial x_1} - \frac{b_1}{b_2}\frac{\partial u_2}{\partial x_1} = 0 \tag{2-8d}$$

and if $B_1 \neq 0$

$$\frac{db_1}{b_1} + \frac{1}{\gamma-1}\frac{da}{a} = 0 \tag{2-8e}$$

Simple wave solutions of (2-) are now investigated where each dependent variable is taken to depend on a single independent variable ξ which is constant on a straight line in the (x_1, t) plane (see Fig. 2-1):

$$\xi = x_1 - U(\xi)t \tag{2-9}$$

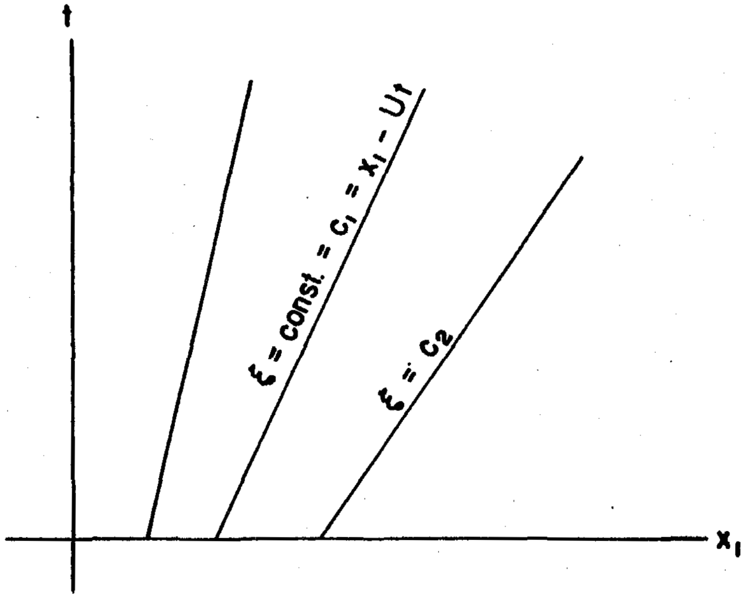


Fig. 2-1

$U(\xi)$ is the constant speed of a particular phase in the x_1-direction. The partial derivatives are thus expressed

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1}\frac{d}{d\xi} = \frac{1}{1 + U'(\xi)t}\frac{d}{d\xi} \tag{2-10a}$$

$$\frac{d}{dt} = \frac{\partial \xi}{\partial t}\frac{d}{d\xi} = -\frac{U(\xi)}{1 + U'(\xi)t}\frac{d}{d\xi} \tag{2-10b}$$

and the derivative following the fluid particles is

$$\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1}\right) = \frac{u_1 - U}{1 + U'(\xi)t}\frac{d}{d\xi}$$

$$= \frac{-c(\xi)}{1 + U'(\xi)t}\frac{d}{d\xi} \tag{2-11}$$

where $c = U - u_1$ = speed of the phase relative to fluid in the x_1-direction. For the special case of a centered wave the constant phase lines run through a fixed point, for example, the origin. When the partial derivatives in (2-8) are replaced by (2-10) and (2-11), $\frac{1}{1+U'(t)} \frac{d}{d\xi}$ appears in all derivatives. Further ξ may be eliminated in the parametric representation $u_1(\xi)$, $u_2(\xi)$, $a(\xi)$, $b_1(\xi)$, $b_2(\xi)$, $c(\xi)$ by regarding c as a function of the dependent variables. If the dependent variables are found, ξ may be determined from (2-9). Thus the system (2-8) becomes

$$\frac{2}{\gamma-1}\frac{c}{a}da - du_1 = 0 \tag{2-12a}$$

$$-c\,du_1 + \frac{2}{\gamma-1}a\,da + b_2^2\left(\frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a}\right) = 0 \tag{2-12b}$$

$$-c\,du_2 - b_1 b_2\left(\frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a}\right) = 0 \tag{2-12c}$$

$$-c\frac{db_2}{b_2} - \frac{c}{\gamma-1}\frac{da}{a} + du_1 - \frac{b_1}{b_2}du_2 = 0 \tag{2-12d}$$

$$\frac{db_1}{b_1} + \frac{1}{\gamma-1}\frac{da}{a} = 0 \tag{2-12e}$$

Since the system (2-12) is homogeneous in the unknowns (du_1, du_2, da, db_1, db_2), a non-trivial solution exists only if the determinant of the coefficients vanish. Since the coefficients depend only on (a, b_1, b_2), the calculations are simplified if first du_1 and du_2 are eliminated by using (2-12a) and combining (2-12c) and (2-12d). The result is

$$\{2(a^2 - c^2) + b_2^2\}\,d(a^2) + (\gamma-1)a^2\,d(b_2^2) = 0 \tag{2-13a}$$

$$(b_1^2 + c^2)b_2^2\,d(a^2) + (\gamma-1)(b_1^2 - c^2)a^2\,d(b_2^2) = 0 \tag{2-13b}$$

$$b_1^2\,d(a^2) + (\gamma-1)a^2\,d(b_1^2) = 0 \tag{2-13c}$$

The determinant of the coefficients is

$$\det = \{ c^4 - (a^2 + b_1^2 + b_2^2)c^2 + a^2 b_1^2 \} \tag{2-14}$$

The vanishing of the determinant determines two speeds, c_f, c_s, corresponding to fast and slow coupled waves. These wave speeds enter MHD theory in other contexts. The limiting speed of shock waves in the case of weak waves is given by (2-14) as well as the propagation speed for two families of characteristics. That is, each phase element of the simple wave propagates as a weak wave in the ambient environment.

Now using the condition (2-14) only two of the first-order equations (2-13) are independent. However this pair is invariant under any change of scale in the velocities, for example ($a \to Aa$, $b_1 \to Ab_1$; A = arbitrary constant). This implies the possibility of reducing the problem to the integration of a single first-order equation. Any new variables which are invariant as above can be used. An alternative way of saying the same thing is to say that dimensionless variables can be introduced to reduce the order of the system. Let

$$\alpha = \frac{a^2}{c^2}, \quad \beta_1 = \frac{b_1^2}{c^2}, \quad \beta_2 = \frac{b_2^2}{c^2} \tag{2-15}$$

The basic condition (2-14) on the wave speed is thus

$$(\alpha - 1)(\beta_1 - 1) = \beta_2 \tag{2-16}$$

Using

$$\frac{d\alpha}{\alpha} = \frac{d(a^2)}{a^2} - \frac{d(c^2)}{c^2} \tag{2-17a}$$

$$\frac{d\beta_1}{\beta_1} = \frac{d(b_1^2)}{b_1^2} - \frac{d(c^2)}{c^2} \tag{2-17b}$$

$$\frac{d\beta_2}{\beta_2} = \frac{d(b_2^2)}{b_2^2} - \frac{d(c^2)}{c^2} \tag{2-17c}$$

we can arrive at first order differential equations in any of the planes (α, β_1), (α, β_2) or (β_1, β_2):

$$\frac{d\beta_1}{d\alpha} = \frac{2(\alpha-1) + \gamma\alpha(\beta_1-1)}{(2-\gamma)(\alpha-1)} \frac{\beta_1}{\alpha} \tag{2-18a}$$

$$\frac{d\beta_2}{d\alpha} = \frac{2(\alpha-1)[(\alpha-1)^2 + \beta_2(2\alpha-1)] + \gamma\alpha\beta_2^2}{(2-\gamma)\alpha(\alpha-1)^2} \tag{2-18b}$$

$$\frac{d\beta_2}{d\beta_1} = \frac{(\beta_1-1)(\beta_1+\beta_2-1)[\gamma(\beta_1-1)+2] + 2\beta_1\beta_2}{(\beta_1-1)[\gamma(\beta_1-1)(\beta_1+\beta_2-1) + 2\beta_2]} \frac{\beta_2}{\beta_1} \tag{2-18c}$$

A study of the trajectories of any one of the equations (2-18) provides a complete description of all possible simple waves. Some details are presented in the next section.

III. SIMPLE WAVE SOLUTIONS

In principle, the MHD simple wave problem can be obtained by first solving any of the three first order differential equations (2-18a, b, c) to get the relations between various wave speeds. The physical quantities (i.e. the magnetic field strength, density and flow velocity) are then obtained from known differential equations and boundary conditions. It can be shown, however, that the exact solution of the flow problem cannot be expressed explicitly. A study of integral curves on the coordinates of wave speed has been made in this paper. The variation of physical quantities can be seen readily in these coordinates. The solution of the problem is obtained by graphical means.

The choice of any one of (2-18a, b, c) to work with is completely arbitrary. It is found, however, to be most convenient to use (2-18a) and study the integral curves in the (α, β_1) plane; hence most of our analysis will be based on this.

It should be noted that Friedrichs (Ref. 3) first obtained the solution of simple waves in terms of a first-order differential equation which is given by

$$2(q-1)ds = \gamma d(q-1)(s-q^{-1}) \tag{3-1}$$

where the variables q, s in our notation are $\frac{1}{\alpha}$, $\frac{\alpha}{\beta_1}$ respectively. One may easily verify that (3-1) is identical with (2-18a). Bazer (Ref. 4) has extended (3-1) for the solution of a shear flow discontinuity problem, the flow velocities were obtained in terms of exact integrals and some approximation has been used to get the solution in analytical form.

In order to study the integral curves (trajectories) of (2-18a) on the (α, β_1) plane, we start with the examination of the singular points which are located at (0, 0), (1, 0) and (1, 1). We will denote them by L, M and N respectively. The behavior of the trajectories near L, M, N are studied by the usual method of local linearization. For the value of γ in the range $1 < \gamma \leqslant \frac{5}{3}$ of ordinary gases, the results are given as follows:

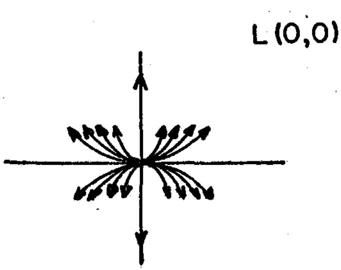


Fig. 3-1

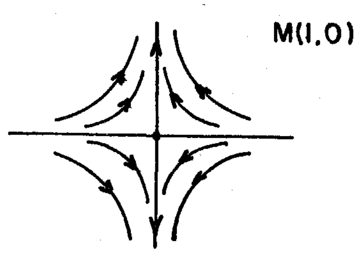


Fig. 3-2

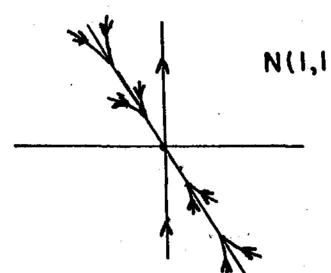


Fig. 3-3

(a) $L(0, 0)$ is a nodal point, all trajectories, except one lying on β_1-axis, are tangent to α-axis.

(b) $M(0, 1)$ is a saddle point with two exceptional trajectories lying on α-axis and the vertical line $\alpha = 1$.

(c) $N(1, 1)$ is a nodal point, all trajectories are tangent to the line $(\alpha-1) + (\gamma-1)(\beta_1-1) = 0$ with the exceptional one being on $\alpha = 1$.

From (2-15), (2-16)

$$\alpha = \frac{a^2}{c^2} \geqslant 0$$

$$\beta_1 = \frac{b_1^2}{c^2} \geqslant 0$$

$$\beta_2 = \frac{b_2^2}{c^2} = (\alpha-1)(\beta_1-1) \geqslant 0$$

hence on the (α, β_1) plane, the regions of physical significance are bounded by:

(a) $1 \geqslant \alpha \geqslant 0$, $1 \geqslant \beta_1 \geqslant 0$ corresponds to fast wave region.

(b) $\alpha \geqslant 1$, $\beta_1 \geqslant 1$ corresponds to slow wave region.

Knowing the general behavior about singular points, the entire family of integral curves can then be constructed either by isocline method or more precisely by the numerical integration method. Their qualitative nature is illustrated in Fig. 3-4. Similarly, the behavior of integral curves on (α, β_2) and (β_1, β_2) planes are given in Fig. 3-5 and 3-6 for reference, they may appear to be convenient in describing the physical quantities under certain circumstances.

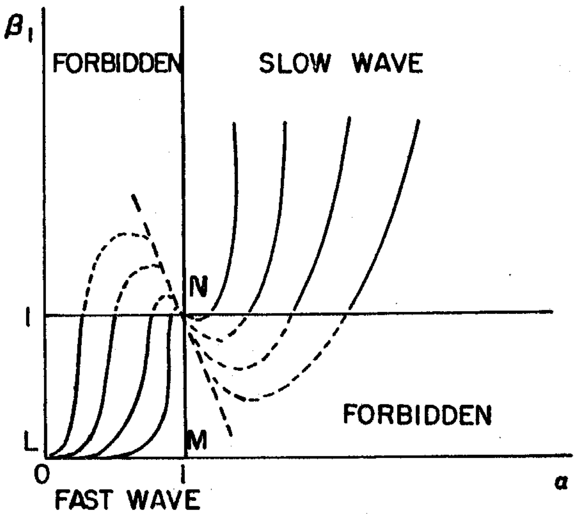


Fig. 3-4

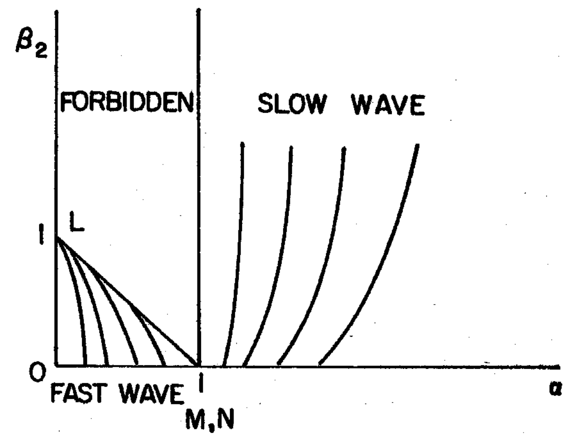


Fig. 3-5

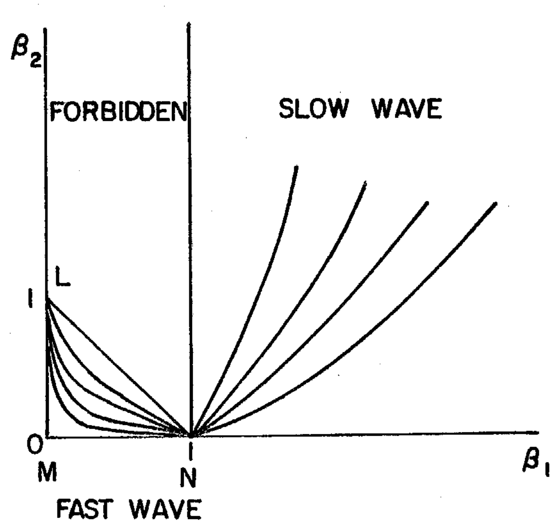


Fig. 3-6

We introduce the fundamental parameter of the initial condition of the problem, K_I, which is defined as

$$\frac{1}{a_I}\frac{B_{1I}}{(\mu\rho_I)^{1/2}} = \frac{b_{1I}}{a_I} \tag{3-2}$$

This provides a measure of the relative importance of the imposed longitudinal magnetic field and the mechanical state of the system at the initial moment. It may be seen later that this parameter governs a whole family of solutions and plays an important role in scaling the solution curves.

As in many other physical problems, it is most convenient to use dimensionless variables which, normalized with respect to the initial physical quantities, are defined as follows:

$$\begin{aligned}
&\overline{\rho} = \frac{\rho}{\rho_I}, && \overline{P} = \frac{P}{P_I}, && \overline{B_2} = \frac{B_2}{B_{1I}} \\
&\overline{u_1} = \frac{u_1}{a_I}, && \overline{u_2} = \frac{u_2}{a_I}, && \overline{a} = \frac{a}{a_I} \\
&\overline{b_1} = \frac{b_1}{a_I}, && \overline{b_2} = \frac{b_2}{a_I}, && \overline{c} = \frac{c}{a_I}
\end{aligned} \tag{3-3}$$

Now, we are in a position to obtain the expressions of the curves on (α, β_1) plane along which certain physical variables remain constant for each given value of K_I.

From

$$\frac{\beta_1}{\alpha} = \frac{b_1^2}{a^2} = \frac{\overline{b_1}^2}{\overline{a}^2} = \frac{K_1^2}{\overline{\rho}^{\gamma}}$$

we have

$$\overline{\rho} = K_1^{\frac{2}{\gamma}}\left(\frac{\alpha}{\beta_1}\right)^{\frac{1}{\gamma}} \tag{3-4a}$$

From (2-16)

$$\overline{B_2} = \left(\frac{\beta_2}{\beta_1}\right)^{1/2} = \left[\frac{(\alpha-1)(\beta_1-1)}{\beta_1}\right]^{1/2} \tag{3-4b}$$

Since $\overline{a}^2 = \overline{\rho}^{\gamma-1}$, we have

$$\overline{a} = K_1^{\frac{\gamma-1}{\gamma}} \left(\frac{\alpha}{\beta_1}\right)^{\frac{\gamma-1}{2\gamma}} \tag{3-4c}$$

From $\overline{b_1}^2 = \frac{K_1^2}{\overline{\rho}}$ and (3-4a)

$$\overline{b_1} = K_1^{\frac{\gamma-1}{\gamma}} \left(\frac{\beta_1}{\alpha}\right)^{\frac{1}{2\gamma}} \tag{3-4d}$$

From $\frac{\overline{b_2}^2}{\overline{b_1}^2} = \frac{\beta_2}{\beta_1}$ and (2-16)

$$\overline{b_2} = K_1^{\frac{\gamma-1}{\gamma}} \frac{[(\alpha-1)(\beta_1-1)]^{1/2}}{(\alpha\beta_1^{\gamma-1})^{\frac{1}{2\gamma}}} \tag{3-4e}$$

From $\beta_1 = \frac{\overline{b_1}^2}{\overline{c}^2}$ and (3-4d)

$$\overline{c} = \left(\frac{1}{\alpha\beta_1^{\gamma-1}}\right)^{\frac{1}{2\gamma}} \tag{3-4f}$$

One may see from above that all the variables except $\overline{B_2}$ are dependent on K_1. Furthermore, all velocities have the same multiplying factor $K_1^{(\gamma-1)/\gamma}$. For the convenience of labelling these constant lines, we have only to give the corresponding values at $K_1 = 1$ on each curve. They are denoted by the subscript "o" and expressed in terms of (α, β_1) coordinates as follows:

$$(\overline{\rho})_o = \left(\frac{\alpha}{\beta_1}\right)^{\frac{1}{\gamma}} \tag{3-5a}$$

$$(\overline{a})_o = (\frac{\alpha}{\beta_1})^{\frac{\gamma-1}{2\gamma}} \tag{3-5b}$$

$$(\overline{b_1})_o = (\frac{\beta_1}{\alpha})^{\frac{1}{2\gamma}} \tag{3-5c}$$

$$(\overline{b_2})_o = \frac{(\alpha-1)^{1/2}(\beta_1-1)^{1/2}}{(\alpha\beta_1^{\gamma-1})^{1/2\gamma}} \tag{3-5d}$$

$$(\overline{c})_o = (\frac{1}{\alpha\beta_1^{\gamma-1}})^{1/2\gamma} \tag{3-5e}$$

The values for K_1 different from one are then obtained by multiplying the appropriate scaling factors. They are:

$$\overline{\rho} = K_1^{\frac{2}{\gamma}}(\overline{\rho})_o \tag{3-6a}$$

$$\overline{a} = K_1^{\frac{\gamma-1}{\gamma}}(\overline{a})_o \tag{3-6b}$$

$$\overline{b_1} = K_1^{\frac{\gamma-1}{\gamma}}(\overline{b_1})_o \tag{3-6c}$$

$$\overline{b_2} = K_1^{\frac{\gamma-1}{\gamma}}(\overline{b_2})_o \tag{3-6d}$$

$$\overline{c} = K_1^{\frac{\gamma-1}{\gamma}}(\overline{c})_o \tag{3-6e}$$

There are two basic variables of state, $\overline{\rho}$, $\overline{B_2}$ appearing in this problem, all other physical quantities are related to them by known differential equations and boundary conditions.

It is obvious from (3-5) that the main advantage of choosing (α, β_1) plane for our analysis lies in the fact that the lines of $(\overline{\rho})$ = const. are rays from the origin. This simplifies greatly the graphical representation and gives a clear picture of the variation of $(\overline{\rho})_o$ on (α, β_1) plane. Since the values of $(\overline{a})_o$ and $(\overline{b_1})_o$ are directly proportional to $(\overline{\rho})_o$ only, their lines of constant value are centered rays also.

Several integral curves for $\gamma = \frac{5}{3}$ on the slow wave region of (α, β_1) plane have been obtained by applying Adams' method of numerical integration to (2-18a). The results are plotted on Fig. 3-7 for the region $0 \leq \alpha \leq 12$ and $0 \leq \beta_1 \leq 12$. Various other curves of constant β_2, $\overline{B_2}$, $(\overline{\rho})_o$ and $(\overline{c})_o$ can also be plotted on the same graph, but, in order to avoid the possible confusions among them, they have been prepared on separate figures. Those are constant β_2 lines and integral curves on Fig. 3-7, constant $(\overline{c})_o$ and constant $(\overline{\rho})_o$ lines on Fig. 3-8, and constant $\overline{B_2}$ and constant $(\overline{\rho})_o$ lines on Fig. 3-9. The relations between these constant lines can, however, be correlated by using transparent papers for the graphs. The corresponding integral curves in the fast wave region can be constructed in a similar way.

The change of $(\overline{u})_o$ along each integral curve is obtained as follows:

From (2-5b) and (2-12a)

$$du_1 = \frac{2}{\gamma-1}\, c \frac{da}{a} = c \frac{d\rho}{\rho} \tag{3-7a}$$

In terms of dimensionless variables, it is

$$d\overline{u_1} = \overline{c} \frac{d\overline{\rho}}{\overline{\rho}} \tag{3-7b}$$

For $K_1 = 1$, we have then

$$d(\overline{u_1})_o = (\overline{c})_o \frac{d(\overline{\rho})_o}{(\overline{\rho})_o} \tag{3-7c}$$

where $\overline{u_1} = K_1^{\frac{\gamma-1}{\gamma}} (\overline{u_1})_o$ can be directly seen from here and (3-6a, e).

We may approximate the differentials by finite intervals, i. e.

$$\Delta(\overline{u_1})_o = (\overline{c})_o \frac{\Delta(\overline{\rho})_o}{(\overline{\rho})_o} \tag{3-7d}$$

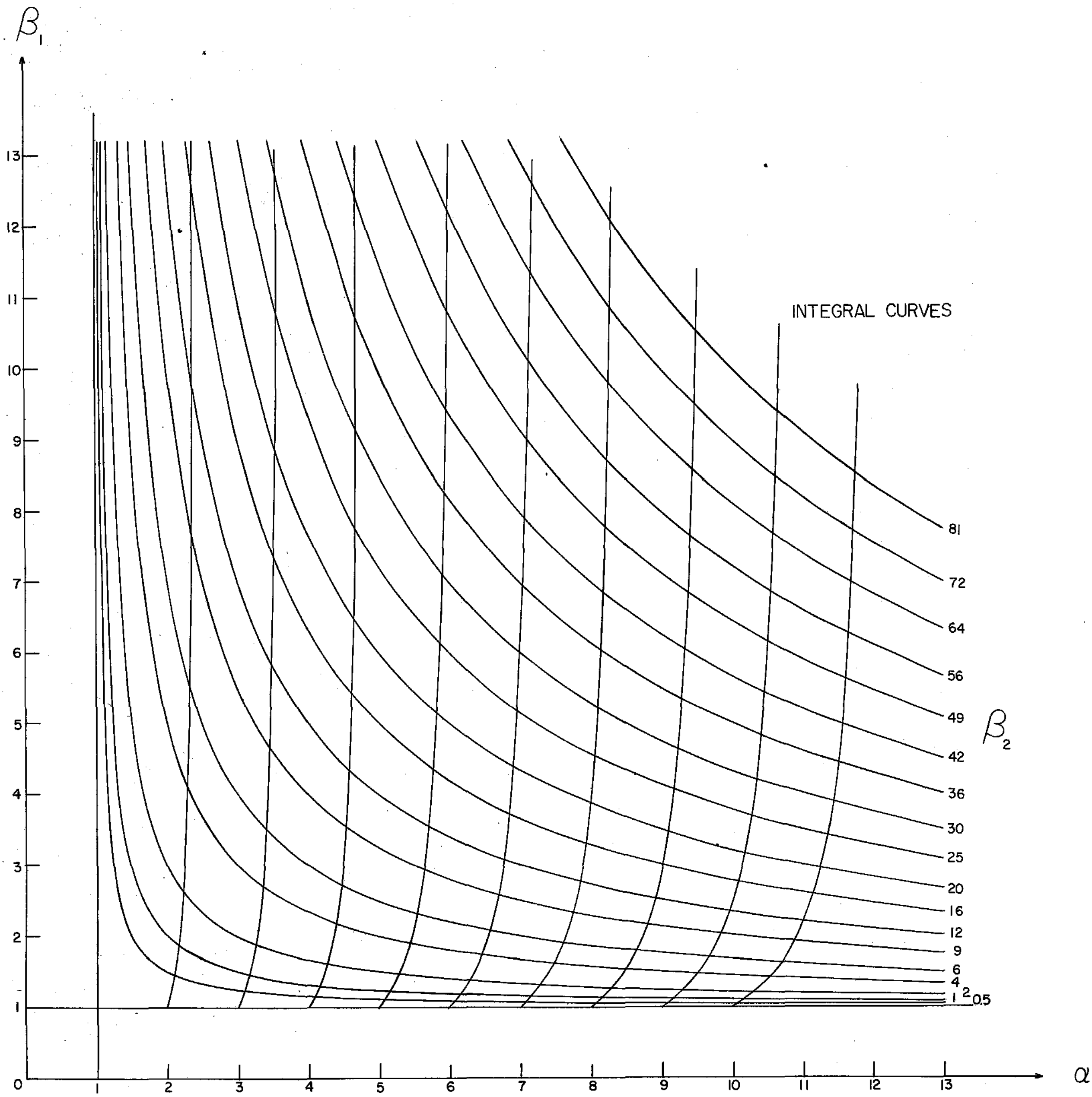

β_1
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
α
INTEGRAL CURVES
81
72
64
56
49
42
36
30
25
20
16
12
9
6
4
1
2
0.5
β_2

Fig. 3-7

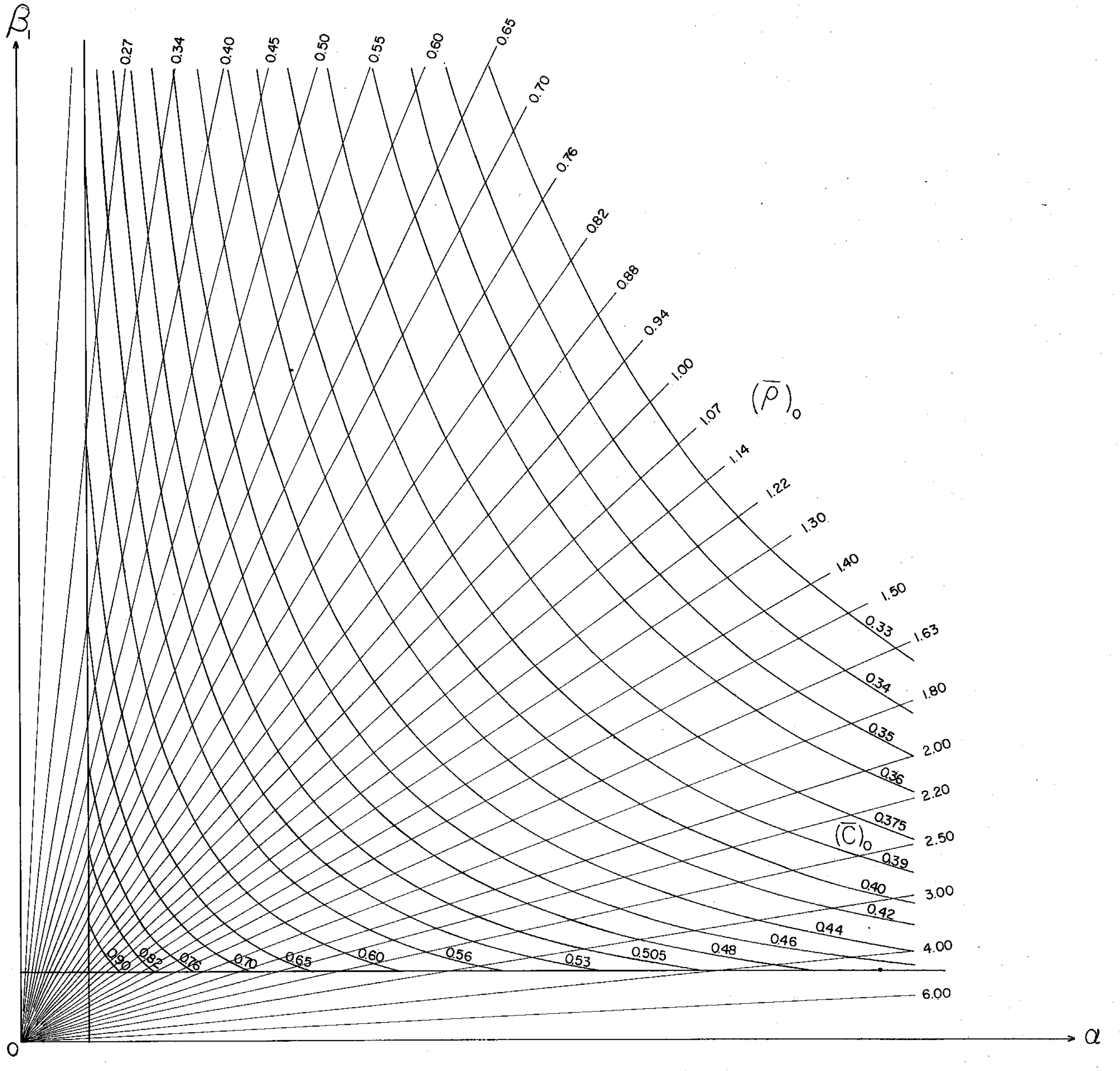

β_1
0.27
0.34
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.76
0.82
0.88
0.94
1.00
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
6.00
$(\bar{\rho})_0$
$(\bar{C})_0$
0.33
0.34
0.35
0.36
0.375
0.39
0.40
0.42
0.44
0.46
0.48
0.505
0.53
0.56
0.60
0.65
0.70
0.76
0.82
0.90
O
α

Fig. 3-8

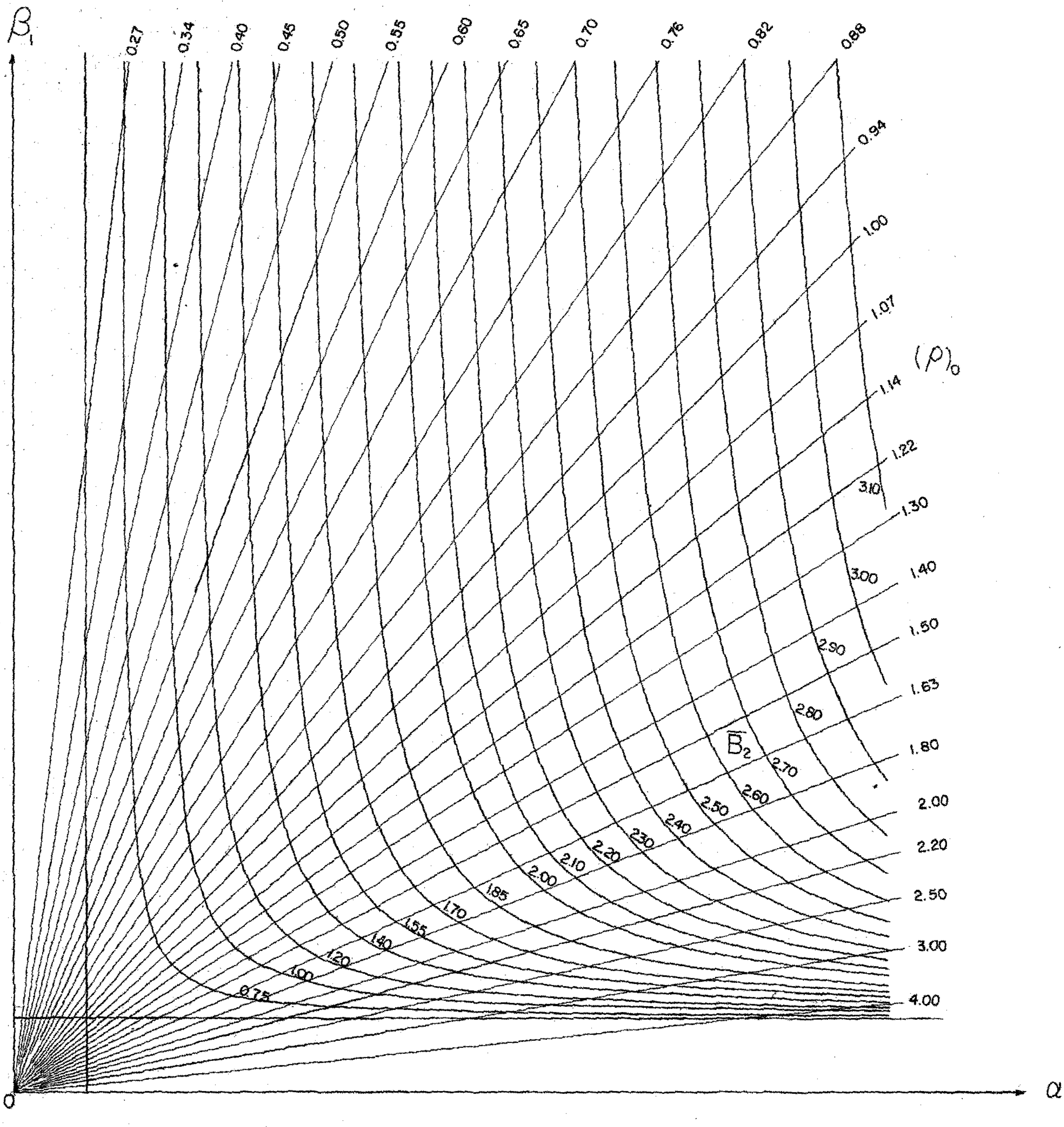

β_1
0.27
0.34
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.76
0.82
0.88
0.94
1.00
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
$(\rho)_0$
$\overline{B}_2$
3.10
3.00
2.90
2.80
2.70
2.60
2.50
2.40
2.30
2.20
2.10
2.00
1.85
1.70
1.55
1.40
1.20
1.00
0.75
O
α

Fig. 3-9

For each finite interval $\Delta(\overline{\rho})_o$ on a given curve, read off the mean values of $(\overline{\rho})_o$ and $(\overline{c})_o$ within it, and obtain the finite change $\Delta(\overline{u_1})_o$ along the interval from (3-7c). This same procedure can be applied successively along each integral curve. Note here $(\overline{u_1})_o$ on a given integral curve is a function of $(\overline{\rho})_o$ only. It is convenient for the graphical representation to introduce a notation w_1 which is defined by the following relations:

$$(\overline{u_1})_o = w_1 - w_1' \tag{3-8a}$$

and

$$w_1 = 0 \text{ at } \beta_1 = 0 \text{ on each integral curve} \tag{3-8b}$$

where w_1' is a constant. Hence from (3-8a)

$$\Delta(\overline{u_1})_o = \Delta w_1$$

and the value of w_1 on each integral curve as a function of $(\overline{\rho})_o$ is given on Fig. 3-10. The constant w_1' is determined by knowing $(\overline{u_1})_o = (\overline{u_1})_{oA}$ at a specific point A on a given integral curve where $(\overline{\rho})_o = (\overline{\rho})_{oA}$, then $w_1 = W_{1A}$ is read from Fig. 3-10. Thus

$$w_1' = w_{1A} - (\overline{u_1})_{oA} \tag{3-9}$$

Similarly, eliminating a from (2-12c) and (2-12d) we obtain the differential equation for u_2

$$du_2 = \frac{b_1 b_2}{b_1^2 - c^2} du_1 \tag{3-10a}$$

and express it in dimensionless variables

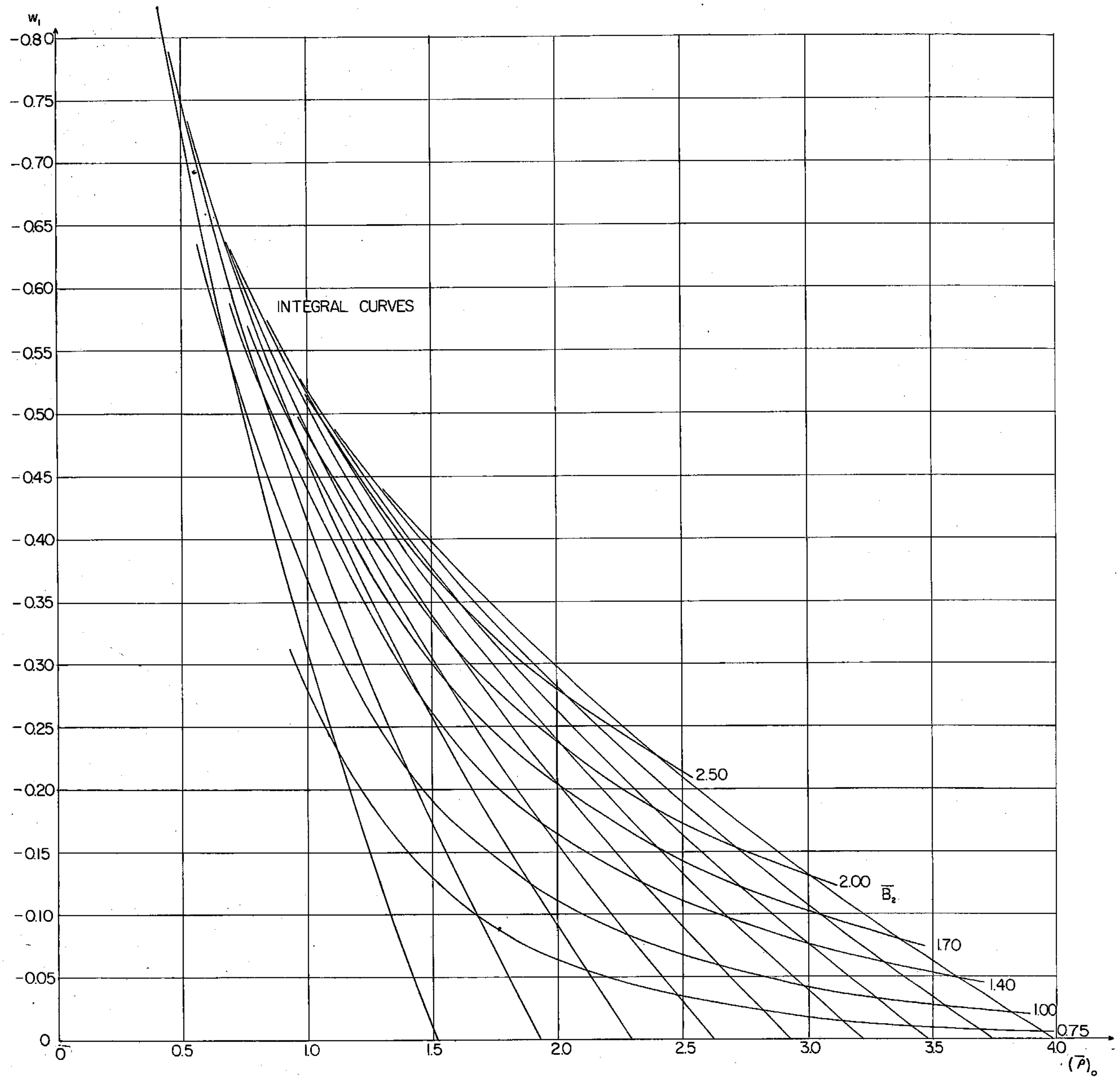

w_1
-0.80
-0.75
-0.70
-0.65
-0.60
-0.55
-0.50
-0.45
-0.40
-0.35
-0.30
-0.25
-0.20
-0.15
-0.10
-0.05
0
INTEGRAL CURVES
2.50
2.00 $\overline{B}_2$
1.70
1.40
1.00
0.75
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$(\overline{P})_o$

Fig. 3-10

$$d\overline{u_2} = \frac{\overline{b_1}\,\overline{b_2}}{\overline{b_1}^2 - \overline{c}^2}\, d\overline{u_1} = \frac{\overline{b_2}/\overline{b_1}}{1-(\overline{c}^2/\overline{b_1}^2)}\, d\overline{u_1}$$

$$= \operatorname{sgn}(B_1B_2)\,\frac{\beta_2^{1/2}}{(1-\frac{1}{\beta_1})\beta_1^{1/2}}\, d\overline{u_1}$$

$$= \operatorname{sgn}(B_1B_2)\left(\frac{\beta_1(\alpha-1)}{\beta_1-1}\right)^{1/2} d\overline{u_1} \qquad \text{(3-10b)}$$

where the value inside the square root is a function of coordinates α, β_1 only. They can be read off directly from (α,β_1) plane. For $K_1 = 1$, we get

$$d(\overline{u_2})_o = \operatorname{sgn}(B_1B_2)\left(\frac{\beta_1(\alpha-1)}{\beta_1-1}\right)^{1/2} d(\overline{u_1})_o \qquad \text{(3-10c)}$$

where $\overline{u_2} = K_1^{\frac{1}{\gamma-1}}(\overline{u_2})_o$ is obvious.

To approximate (3-10c) in terms of finite small intervals along the integral curves, use the values $\Delta(\overline{u_1})_o$ obtained from (3-7d); then

$$\Delta(\overline{u_2})_o = \operatorname{sgn}(B_1B_2)\left(\frac{\beta_1(\alpha-1)}{\beta_1-1}\right)^{1/2}\Delta(u_1)_o \qquad \text{(3-10d)}$$

As in the case of $(\overline{u_1})_o$, we define w_2 by the relations

$$(\overline{u_2})_o = w_2 - w_2' \qquad \text{(3-11a)}$$

and

$$w_2 = 0 \text{ at } \beta_1 = 1 \text{ on each integral curve} \qquad \text{(3-11b)}$$

where w_2' is a constant. Then from (3-11a)

$$\Delta(\overline{u_2})_o = \Delta w_2$$

always. w_2 on each integral curve as a function of $(\overline{\rho})_o$ is given in Fig. 3-11. The constant w_2' is determined by the condition

$$(\overline{u_2})_{oA} = w_{2A} - w_2'$$

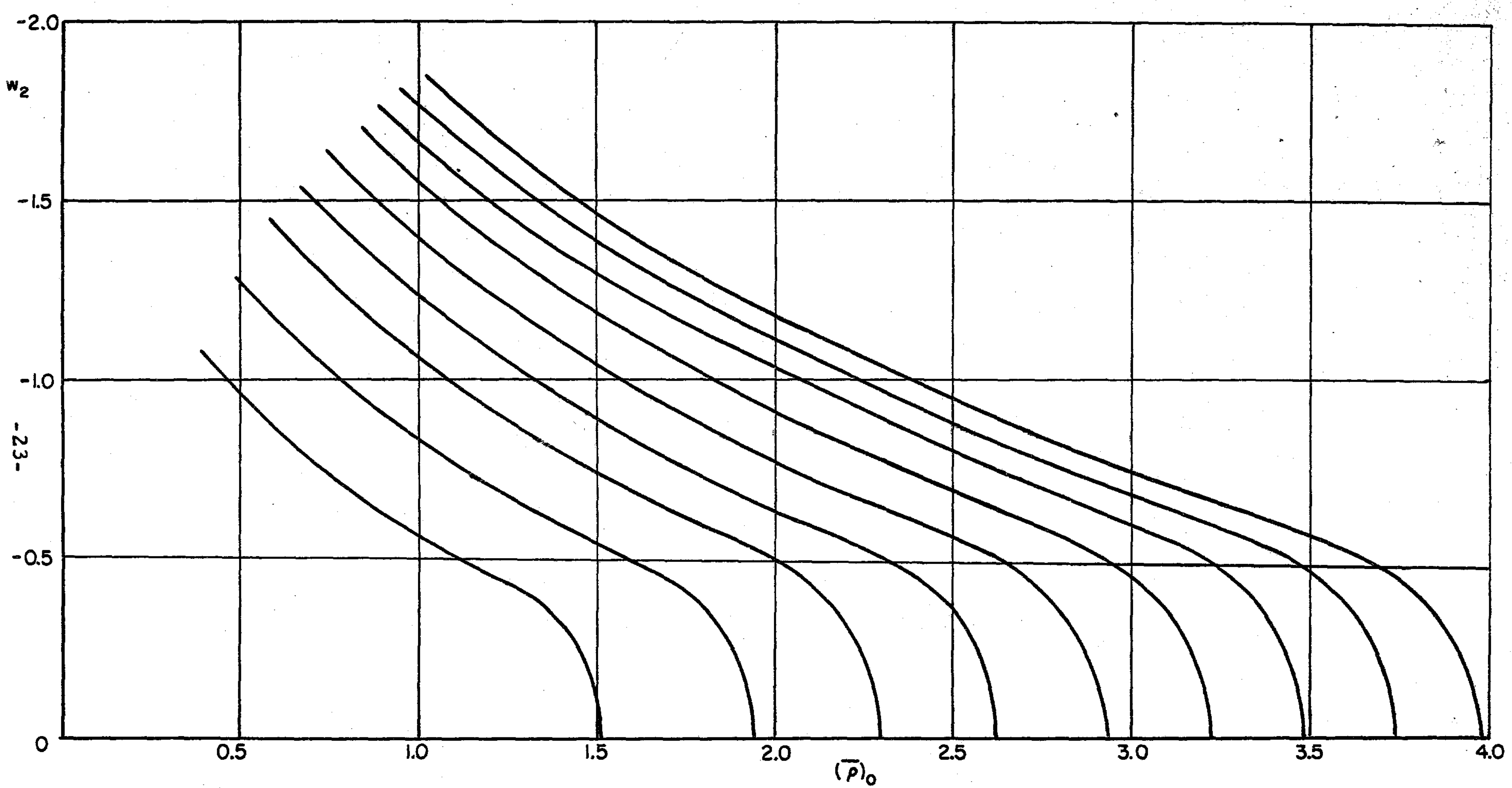


FIG. 3-11

Thus, (3-11a) becomes

$$(\overline{u_2})_o = w_2 - w_{2A} - (\overline{u_2})_{oA} \tag{3-12}$$

We may see from Fig. 3-7 that the integral curves extend vertically toward infinity as $\beta_1 \to \infty$ which corresponds to the condition that the gas expands near vacuum $(\overline{\rho_o} \to 0)$. The graphical solution is no longer possible but we may get, on the other hand, an approximate equation for the integral curves. The approximate solution for various state quantities can also be obtained in this limiting case.

The differential equation (2-18a) at $\beta_1 \to \infty$ is approximated by

$$\frac{d\beta_1}{d\alpha} \cong \frac{\gamma}{2-\gamma} \frac{\beta_1^2}{\alpha-1}$$

Its solution is

$$\alpha \cong 1 + (\text{const}) \left[\exp\left(-\frac{2-\gamma}{\gamma} \frac{1}{\beta_1}\right) \right]$$

and as $\beta_1 \to \infty$

$$\alpha \to \text{constant}$$

Therefore, we expand α in terms of orders of $\frac{1}{\beta_1}$

$$\alpha = D_o\left(1 + \frac{D_1}{\beta_1} + \frac{D_2}{\beta_1^2} + \dots\right) \tag{3-13}$$

where D_o is the value of α at $\beta_1 \to \infty$ and $D_1, D_2, \dots$ are constants determined by substituting (3-13) into (2-18a) and equating the coefficients of each order of $\frac{1}{\beta_1}$ on both sides of the equation. (3-13) becomes

$$\alpha = D_o\left(1 - \frac{2-\gamma}{\gamma} \frac{D_o-1}{D_o} \frac{1}{\beta_1} + \frac{2-\gamma}{\gamma^2} \frac{D_o-1}{D_o}\left(2-\gamma-\frac{1}{D_o}\right) \frac{1}{\beta_1^2} + \dots\right) \tag{3-14}$$

When $\gamma = \frac{5}{3}$, (3-14) reads

$$\alpha = D_o\left(1 - \frac{D_o - 1}{5D_o}\frac{1}{\beta_1} + \frac{(D_o-1)(D_o-3)}{25D_o^2}\frac{1}{\beta_1^2} + \ldots\right) \tag{3-15}$$

To include the first order term of $\frac{1}{\beta_1}$ on the right-hand side of (3-15) is generally sufficient for problems at large values of β_1. The approximate equations of state quantities expressed as functions of $\overline{\rho_o}$ are given below, they serve mainly as a continuation of the graphical solutions in the low density regions.

From (3-5a) for $\gamma = \frac{5}{3}$

$$(\overline{\rho})_o^{\frac{5}{3}} = \frac{\alpha}{\beta_1}$$

Substituting (3-15) into the above equation, it becomes

$$\frac{1}{\beta_1^2} - \frac{5D_o}{D_o - 1}\frac{1}{\beta_1} + \frac{5(\overline{\rho})_o^{5/3}}{D_o - 1} = 0 \tag{3-16}$$

This is a quadratic equation in $\frac{1}{\beta_1}$ which can be solved in terms of $(\overline{\rho})_o$. Neglecting the higher order terms of $(\overline{\rho})_o$, we get

$$\frac{1}{\beta_1} = \frac{(\overline{\rho})_o^{5/3}}{D_o}\left[1 + \frac{1}{5}\frac{D_o - 1}{D_o^2}(\overline{\rho})_o^{5/3} + \ldots\right] \tag{3-17}$$

From (3-4b) and substituting the value of α from (3-15),

$$\overline{B_2} = \left[(\alpha-1)\left(1 - \frac{1}{\beta_1}\right)\right]^{1/2} = (D_o-1)^{1/2}\left(1 - \frac{3}{5\beta_1}\right)$$

$$= (D_o-1)^{1/2}\left[1 - \frac{3}{5D_o}(\overline{\rho})_o^{5/3}\right] \tag{3-18}$$

From $\overline{b_1}^2 = \dfrac{K^2}{\overline{\rho}}$, $\beta_1 = \dfrac{\overline{b_1}^2}{\overline{c}^2}$, and (3-17), we have

$$(\overline{c})_o = \frac{1}{\left[\beta_1(\overline{\rho})_o\right]^{1/2}} = \frac{(\overline{\rho})_o^{1/3}}{D_o^{1/2}}\left[1 + \frac{D_o-1}{10\,D_o^2}(\overline{\rho})_o^{5/3}\right] \quad (3\text{-}19)$$

Then, from (3-7c)

$$d(\overline{u_1})_o = (\overline{c})_o \frac{d(\overline{\rho})_o}{(\overline{\rho})_o} = \frac{1}{D_o^{1/2}(\overline{\rho})_o^{2/3}}\left[1 + \frac{D_o-1}{10D_o^2}(\overline{\rho})_o^{5/3}\right] d(\overline{\rho})_o \quad (3\text{-}20)$$

Integrating we have

$$(\overline{u_1})_o - (\overline{u_1})_o' = \frac{(\overline{\rho})_o^{1/3}}{D_o^{1/3}}\left[3 + \frac{D_o-1}{20D_o^2}(\overline{\rho})_o^{5/3}\right] \quad (3\text{-}21)$$

where $(\overline{u_1})_o'$ is a constant to be determined by boundary conditions.

From (3-10c) and (3-15)

$$d(\overline{u_2})_o = \mathrm{sgn}(B_1B_2)\left(\frac{\alpha-1}{1-\frac{1}{\beta_1}}\right)^{1/2} d(\overline{u_1})_o$$

$$= \mathrm{sgn}(B_1B_2)(D_o-1)^{1/2}\left(1 + \frac{2}{5}\,\frac{1}{\beta_1}\right) d(\overline{u_1})_o$$

Substituting (3-20) and (3-17) into the above equation, then

$$d(\overline{u_2})_o = \mathrm{sgn}(B_1B_2)\left(\frac{D_o-1}{D_o}\right)^{1/2}\frac{1}{(\overline{\rho})_o^{2/3}}\left[1 + \frac{5D_o-1}{10D_o^2}(\overline{\rho})_o^{5/3}\right] d(\overline{\rho})_o$$

After the integration of the above equation, we get

$$(\overline{u_2}) - (\overline{u_2})_o' = \mathrm{sgn}(B_1B_2)\left(\frac{D_o-1}{D_o}\right)^{1/2}(\overline{\rho})_o^{1/3}\left[3 + \frac{5D_o-1}{20D_o^2}(\overline{\rho})_o^{5/3}\right] \quad (3\text{-}22)$$

where $(\overline{u_2})_o'$ is a constant to be determined by boundary conditions.

For the integral curves calculated on Fig. 3-7, the values of D_o are obtained by taking the value of α at $\beta_1 = 10$ on each integral curve and finding D_o from (3-15) retaining only the first order term in $\frac{1}{\beta_1}$. If we denote the state quantities at $\beta_1 = 10$ on each integral curve by the subscript "s", the values of α_s, $(\overline{\rho_o})_s$, $(\overline{B_2})_s$, $(\overline{w_1})_s$, $(\overline{w_2})_s$ together with D_o for each integral curve are given in the following table.

The value of α at $\beta_1 = 1$ to identify the trajectory	2	3	4	5	6	7	8	9	10
α_s	2.37	3.57	4.78	6.00	7.16	8.29	9.51	10.68	11.83
D_o	2.40	3.62	4.85	6.10	7.30	8.45	9.68	10.89	12.07
$(\overline{\rho_o})_s$	0.422	0.539	0.642	0.736	0.818	0.894	0.970	1.040	1.106
$(\overline{B_2})_s$	1.11	1.52	1.84	2.12	2.36	2.56	2.77	2.95	3.12
$(\overline{w_1})_s$	-0.806	-0.716	-0.655	-0.609	-0.581	-0.555	-0.528	-0.507	-0.490
$(\overline{w_2})_s$	-1.07	-1.26	-1.40	-1.48	-1.56	-1.65	-1.69	-1.74	-1.77

This completes the necessary information to continue the solution for the integral curves shown on Fig. 3-7 up to vacuum.

Each given physical state $(\overline{\rho}, \overline{B_2}; K_1)$ corresponds to one point on (α, β_1) plane.* The position is determined by the point of intersection of constant $\overline{B_2}$ lines and constant $(\overline{\rho})_o$ lines (where $(\overline{\rho})_o = \overline{\rho} K_1^{-2/\gamma}$).

Knowing the initial state and the changes of all physical quantities along integral curves, we are able to solve certain boundary value problems

*Note, there are always two points on (α, β_1) plane corresponding to a given physical state $\overline{B_2}$ and $\overline{\rho}$. One of them is in the fast wave region and the other in the slow wave region. However, if we limit our interest to slow waves only, the position can be said to be determined uniquely in the region of interest.

by satisfying the required conditions at the final state. This is equivalent to the choice of a particular integral curve that meets all requirements.

So far, the problem is solved in terms of the parameter ξ though it does not appear explicitly in the solutions. To transform the solutions back to (x_1, t) plane, we have to make use of (2-9)

$$\xi = x_1 - U(\xi)t$$

where $U(\xi) = u_1 + c$ = the constant speed of a particular phase. Since all physical variables remain unchanged on this phase line and the corresponding value of c for each u_1 from the solution are known, the slopes of the phase lines on $(x_1 - t)$ plane are determined. They should be attached to the given boundary B on (x_1, t) plane, identified by the same u_1. Other physical variables known to be at the same state of u_1 from the solution are then assigned to each phase line. At a given time t_1, the distribution of any physical quantity in the space is obtained from Fig. 3- 2 by crossing all phase lines with a horizontal line $t = t$, thus the values at each position of x_1 is obtained from the graph.

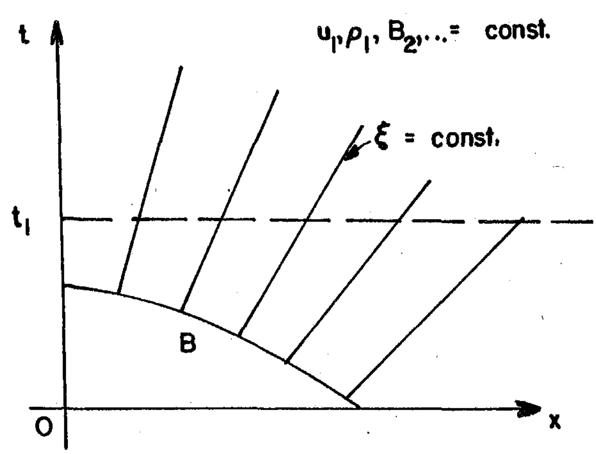


Fig. 3-12

Some examples will be given in Section 5 to illustrate this procedure.

IV. SOLUTION FOR THE CASE OF PURELY TRANSVERSE FIELD

When the longitudinal magnetic field B_1 is absent initially, it remains so at all later times. This corresponds to a limiting case of the problem. The analysis is greatly simplified and the solution can be expressed in an analytical form.

As to the governing differential equations, we see that (2-1c) becomes

$$\rho\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1}\right)u_2 = 0$$

if $\rho \neq 0$, u_2 = constant following each fluid particle path. If u_2 is constant throughout initially, it remains so afterwards.

Now, (2-1d) has exactly the same form as (2-1a), that is

$$\frac{\partial B_2}{\partial t} + \frac{\partial}{\partial x_1}(B_2 u_1) = 0$$

or it may be written

$$\frac{DB_2}{Dt} + B_2 \frac{\partial u}{\partial x} = 0$$

Likewise we may write (2-1a) in the similar form

$$\frac{D\rho}{Dt} + \rho \frac{\partial u}{\partial x} = 0$$

Eliminating $\frac{\partial u}{\partial x}$ between the above two equations, we have

$$\frac{D}{Dt}\left[\log\left(\frac{B}{\rho}\right)\right] = 0$$

Thus $\frac{B}{\rho}$ = constant along each streamline and this relation holds in a general sense even for the non-isentropic case. In our simple wave problem, $\frac{B}{\rho} = \lambda$ = constant in the entire flow region where λ is determined by the initial condition. From (2-5a)

$$a^2 = \frac{dP}{d\rho} = k\rho^{\gamma-1}$$

where k is a constant throughout the whole flow region.

Equations (2-1) are now reduced to the following two equations only

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1) = 0 \tag{4-1a}$$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1}\right) + a^2\frac{\partial\rho}{\partial x_1} + \frac{\lambda^2}{\mu}\rho\frac{\partial\rho}{\partial x_1} = 0 \tag{4-1b}$$

Writing these in terms of velocity variables u_1, a, we have

$$\frac{\partial a}{\partial t} + u_1\frac{\partial a}{\partial x_1} + \frac{\gamma-1}{2}a\frac{\partial u_1}{\partial x_1} = 0 \tag{4-2a}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + \frac{2a}{\gamma-1}\left[1 + \Lambda a^{\frac{2(2-\gamma)}{\gamma-1}}\right]\frac{\partial a}{\partial x_1} = 0 \tag{4-2b}$$

where $\Lambda = \lambda^2/\mu k^{1/(\gamma-1)}$.

As regards the simple wave solution, the entire family of integral curves on (α, β_1) plane as well as on (α, β_2) and (β_1, β_2) planes degenerates to a single straight line LM. Since $\beta_1 = 0$ now, the situation can be better visualized from the (α, β_2) plane where LM is described by the expression $\alpha + \beta_2 = 1$. The same result can also be obtained from (2-16) by putting $\beta_1 = 0$. Thus

$$c^2 = a^2 + b_2^2 = k\rho^{\gamma-1} + \frac{\lambda^2}{\mu}\rho \tag{4-3}$$

One can readily see from here or from Fig. 3.6 that c corresponds to the limiting case of a fast wave.

It is more convenient to use dimensionless variables normalized with respect to their initial values as follows,

$$\bar{\rho} = \frac{\rho}{\rho_I}, \qquad \bar{B_2} = \frac{B_2}{B_{1_I}}^{*}, \qquad \bar{u_1} = \frac{u_1}{a_I},$$

$$\bar{a} = \frac{a}{a_I}, \qquad \bar{b_2} = \frac{b_2}{a_I}, \qquad \bar{c} = \frac{c}{a_I}$$

Introduce the fundamental parameter of the initial condition

$$K_2 = \frac{B_{2_I}}{(\mu \rho_I)^{1/2}} \frac{1}{a_I} = \frac{b_{2_I}}{a_I}$$

(4-3) becomes

$$\bar{c}^2 = \bar{a}^2 + \bar{b_2}^2 = \bar{\rho}^{\gamma-1} + K_2^2 \bar{\rho}$$

hence

$$\bar{c} = \pm (\bar{\rho}^{\gamma-1} + K_2^2 \bar{\rho})^{1/2} \qquad (4\text{-}4)$$

where the positive sign refers to forward facing waves and the negative sign refers to backward facing waves.

Express (2-12a) in dimensionless form,

$$\frac{2}{\gamma-1} \frac{\bar{c}}{\bar{a}} d\bar{a} - d\bar{u_1} = 0$$

or

$$d\bar{u_1} = \bar{c} \frac{d\bar{\rho}}{\bar{\rho}} \qquad (4\text{-}5a)$$

Substitute the value of $\bar{c}$ in (4-4) into the above equation, then

$$d\bar{u_1} = \pm [\bar{\rho}^{\gamma-1} + K_2^2 \bar{\rho}]^{\frac{1}{2}} \frac{d\bar{\rho}}{\bar{\rho}} = \pm \bar{\rho}^{\frac{\gamma-3}{2}} [1 + K_2^2 \bar{\rho}^{2-\gamma}]^{\frac{1}{2}} d\bar{\rho}$$

---

*We have assumed that $B_{2_I} \neq 0$, otherwise the magnetic flux lines can never get into the fluid because of the ∞ electric conductivity, no magnetic effect can take place and the problem reduces to the ordinary gasdynamic one.

For any two states A and B on a certain trajectory

$$(\overline{u_t})_A - (\overline{u_t})_B = \pm\int_{\overline{\rho}_A}^{\overline{\rho}_B} \overline{\rho}^{\frac{\gamma-3}{2}} [1 + K_2^2 \overline{\rho}^{2-\gamma}]^{\frac{1}{2}} d\overline{\rho}$$

$$= \pm [r(\rho_B) - r(\rho_A)] \qquad (4\text{-}5b)$$

where

$$r(\overline{\rho_1}) \equiv \int_0^{\overline{\rho_1}} \overline{\rho}^{\frac{\gamma-3}{2}} [1 + K_2^2 \overline{\rho}^{2-\gamma}]^{\frac{1}{2}} d\overline{\rho} \qquad (4\text{-}5c)^*$$

We see here that

$$\overline{u_t} \mp r = \text{constant} \qquad (4\text{-}6)$$

has a formal resemblance to the Riemann invariant except the additional term $K_2^2 \overline{\rho}^{2-\gamma}$ taking into account the MHD effect due to the transverse magnetic field. This is sometimes called generalized Riemann invariant.

The definite integral (4-5c) can be expressed in terms of a hypergeometric function (see Appendix)

$$r(\overline{\rho_1}) = \frac{2}{\gamma-1} \overline{\rho_1}^{\frac{\gamma-1}{2}} F\left(-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2^2 \overline{\rho_1}^{2-\gamma}\right) \qquad (4\text{-}7)$$

or since the number of degrees of freedom of gas molecules is related to γ by

$$\gamma = \frac{n+2}{n}$$

(4-7) can be expressed as

$$r(\overline{\rho_1}) = n\overline{\rho_1}^{\frac{1}{n}} F\left(-\frac{1}{2}, \frac{1}{n-2}; \frac{n-1}{n-2}; -K_2^2 \overline{\rho_1}^{\frac{n-2}{n}}\right) \qquad (4\text{-}8)$$

---

* A similar integral has been obtained by Mitchner (Ref. 6) and Golitsyn (Ref. 7).

Therefore one gets for arbitrary value of γ

$$\overline{u_1} \mp \frac{2}{\gamma-1}\,\overline{\rho}^{\frac{\gamma-1}{2}} F(-\tfrac{1}{2}, \tfrac{\gamma-1}{2(2-\gamma)}; \tfrac{3-\gamma}{2(2-\gamma)}; -K_2^2\,\overline{\rho}^{\,2-\gamma}) = \text{constant}$$

or, in terms of initial conditions

$$\overline{u_1} = (\overline{u_1})_I \mp \frac{2}{\gamma-1}\Big[F(-\tfrac{1}{2}, \tfrac{\gamma-1}{2(2-\gamma)}; \tfrac{3-\gamma}{2(2-\gamma)}; -K_2^2) - F(-\tfrac{1}{2}, \tfrac{\gamma-1}{2(2-\gamma)}; \tfrac{3-\gamma}{2(2-\gamma)}; -K_2^2\,\overline{\rho}^{\,2-\gamma})\,\overline{\rho}^{\frac{\gamma-1}{2}}\Big] \tag{4-10}$$

For monatomic gas, $\gamma = \frac{5}{3}$ (or $n = 3$)

$$r(\overline{\rho_1}) = 2\,\overline{\rho_1}^{\frac{1}{3}} F(-\tfrac{1}{2}, 1; 2; -K_2^2\,\overline{\rho_1}^{\frac{1}{3}})$$

and it can be reduced to an elementary function from the following identity relation of hypergeometric functions (see e. g. ref. 8)

$$F(k, \ell; m+1; z) = \frac{m}{(k-\ell)z}\left[F(k-1, \ell; m; z) - F(k, \ell-1; m; z)\right]$$

hence

$$r(\overline{\rho_1}) = \frac{2}{K_2^2}\left[F(-\tfrac{3}{2}, 1; 1; -K_2^2\,\overline{\rho_1}^{\frac{1}{3}}) - 1\right]$$

$$= \frac{2}{K_2^2}\left[(1 + K_2^2\,\overline{\rho_1}^{\frac{1}{3}})^{\frac{3}{2}} - 1\right] \tag{4-11}$$

An interesting feature arises for this special case of simple waves. Consider a certain amount of ideal conducting gas in an infinitely long tube being separated from vacuum zones on both sides by two diaphragms. The whole arrangement is initially in a transverse magnetic field B_{2_I}. After the breaking of the diaphragm, the gas flows in both directions and the pressure balance across each of the contact surfaces is

$$\frac{B_2^2}{2\mu} + P = \frac{B_{2_I}^2}{2\mu} \tag{4-12}$$

Since $B_2 = \lambda\rho$, $P = \text{Const.} \times \rho^\gamma$ always and B_{2_I} is different from zero, ρ can never equal zero. Physically we see that the gas cannot expand completely to vacuum due to the presence of transverse magnetic field. It comes to rest finally and is confined by two contact surfaces which are supported by the magnetic pressure only. The gas density is decreased and can be calculated from (4-12). Owing to the jump of transverse magnetic field across each of the contact surfaces, a current sheet is flowing indefinitely with time on each of them; these currents have the same magnitude but are in opposite directions.

It is also of interest to note that the governing equations (4-2a, b) of this transverse magnetic field case belong to the reducible type of the non-linear partial differential equations. When the flow region ceases to be governed by simple waves as in the problem of interaction of waves, one can always perform the hodograph transformation to interchange the dependent and independent variables of (4-2a, b), since the Jacobian of the transformation is different from zero now. The transformation relations are

$$\frac{\partial u_1}{\partial t} = j\,\frac{\partial x_1}{\partial a}\,, \qquad \frac{\partial u_1}{\partial x_1} = -j\,\frac{\partial t}{\partial a}$$

$$\frac{\partial a}{\partial t} = -j\,\frac{\partial x_1}{\partial u_1}\,, \qquad \frac{\partial a}{\partial x_1} = j\,\frac{\partial t}{\partial u_1}$$

where $j = \frac{\partial u_1}{\partial t}\frac{\partial a}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial a}{\partial t}$ = the Jacobian of the transformation. After substituting these into (4-2a, b), we get

$$-\frac{\partial x_1}{\partial u_1} + u_1\frac{\partial t}{\partial u_1} - \frac{\gamma-1}{2}a\frac{\partial t}{\partial a} = 0 \tag{4-13a}$$

$$\frac{\partial x_1}{\partial a} - u_1\frac{\partial t}{\partial a} + \frac{2a}{\gamma-1}[1 + \Lambda a^{2(2-\gamma)/(\gamma-1)}]\frac{\partial t}{\partial u_1} = 0 \tag{4-13b}$$

Eliminating x_1 from above by cross differentiation, we arrive at a second order linear differential equation of $t(u_1, a)$

$$\frac{\partial^2 t}{\partial a^2} - \frac{4}{(\gamma-1)^2}[1 + \Lambda a^{\frac{2(2-\gamma)}{\gamma-1}}]\frac{\partial^2 t}{\partial u_1^2} + \frac{\gamma+1}{\gamma-1}\frac{1}{a}\frac{\partial t}{\partial a} = 0 \tag{4-14}$$

This is a hyperbolic equation due to the fact that Λ and a are always greater than or equal to zero. It can be solved by satisfying the proper boundary conditions of the given problem formulated on the hodograph plane.

V. EXAMPLES

We shall now apply the simple wave solution outlined above to several physical problems. As the flow region adjacent to a constant state is always a simple wave, this appears in many occasions of actual fluid motion. It plays an essential role in building up the flow pattern in many problems and can even be applied to plane or three-dimensional flow where the simple wave solutions may be approximated in certain local areas. For a conducting gas, the motion of the fluid may be set up either by the usual mechanical devices or by electromagnetic means. Some of these possibilities will be discussed below to illustrate their properties.

A. RECEDING PISTON PROBLEM

We consider here an infinitely conducting gas at rest initially with an imposed magnetic field oriented at an arbitrary direction. A piston is attached to one end of the gas and the gas extends to infinity at the other end. Since we are essentially dealing with a one-dimensional problem, the piston surface can be considered to be perpendicular to x_1 direction and be situated at $x_1 = 0$ initially. At $t > 0$, the piston moves with a velocity $u_p(t)$ (where $u_p(t) < 0$) in $-x_1$ direction, the gas then undergoes an expansion near the face of the piston. This is a typical example of a fluid motion started mechanically. The situation can be seen from x_1-t plane as shown in Fig. 5-1. A wave propagates into the gas at rest after the motion of the piston and the gas particles are disturbed after the arrival of the first wave front. Since the flow region is connected to a constant state at rest, it is a simple wave. In view of the relative simplicity of the case of having transverse magnetic field only, and its solution being able to be expressed in analytical form, we will discuss it first. Some essential features of the problem can also be visualized from it, and these will throw some light on treating the case of arbitrary

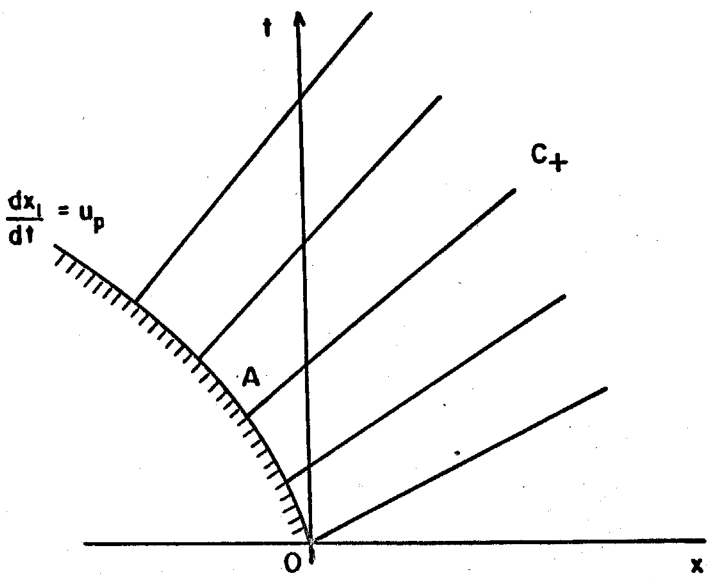


Fig. 5-1

trarily oriented magnetic field which can only be analyzed by graphical methods.

1. In a transverse magnetic field $(\vec{B}_I = B_{2_I}\vec{e_2})$

The solutions obtained in section 4 can be directly applied here. First of all, we consider the boundary conditions of the problem. The mechanical boundary conditions is specified by the requirement that before cavitation occurs the gas velocity has always the same value as the piston velocity $u_p(t)$ along the piston path. No magnetic boundary condition need be imposed here because of the infinite conductivity of the gas. The family of C_+ characteristics issuing from the piston path has the property that all physical quantities remain the same along each of them. On the other hand, we know from (4-6) that $\overline{u_1} - r$ = constant on each trajectory of the particle path. For $\gamma = \frac{5}{3}$, we have from (4-10),

$$u_1 - \frac{2a_I}{K_2^2}\left(1 + K_2^2\,\overline{\rho}^{\frac{1}{3}}\right)^{\frac{3}{2}} = \text{const.}$$

Now $u_1 = 0$ at $\overline{\rho} = 1$. Hence

$$u_1 = \frac{2a_I}{K_2^2}\left\{\left(1 + K_2^2\,\overline{\rho}^{\frac{1}{3}}\right)^{\frac{3}{2}} - \left(1 + K_2^2\right)^{\frac{3}{2}}\right\} \qquad (5\text{-}1)$$

Applying this equation to the piston path, the density variation along the piston path is obtained as

$$\overline{\rho}_p(t_p) = \frac{1}{K_2^6}\left\{\left[\left(1 + K_2^2\right)^{\frac{3}{2}} + \frac{K_2^2\,u_p(t_p)}{2a_I}\right]^{\frac{2}{3}} - 1\right\}^3 \qquad (5\text{-}2)$$

where $\overline{\rho}_p(t_p)$ is expressed in terms of t_p, the time elapsed from the start of the motion, and the position is fixed on the piston path. By using the relations of C_+ characteristics, the expression of $\overline{\rho}$ in terms of arbitrary space and time coordinates is possible. It is given as follows:

For any point A on (x_1, t) plane, the corresponding C_+ characteristics passing through A is given by

$$\frac{x_1 - x_{1p}}{t - t_p} = u_p + C_p$$

However

$$x_{1p} = \int_0^{t_p} u_p \, dt$$

$$u_{1p} = u_p(t_p)$$

from (4-4)

$$C_p = a_I \overline{C_p} = a_I(\overline{\rho_p}^{\frac{2}{3}} + K_2^2 \overline{\rho_p})^{\frac{1}{2}}$$

Then

$$x_1 - \int_0^{t_p} u_p \, dt = (t - t_p)\{u_p + a_I \overline{\rho_p}^{\frac{1}{3}}[1 + K_2^2 \overline{\rho_p}^{\frac{1}{3}}]^{\frac{1}{2}}\} \qquad (5\text{-}3)$$

Since $u_p = u_p(t_p)$ is known, $\overline{\rho_p} = \overline{\rho_p}(u_p) = \overline{\rho_p}(t_p)$ from (5-2), in principle we are able to solve for t_p in terms of x_1, t from (5-3), with initial constants K_2 and a_I. After obtaining $\overline{\rho} = \overline{\rho}(x_1, t)$, we know that $\overline{B_2}(x_1, t)$ has the identical form. All the other physical quantities $\overline{p}(x_1, t)$, $\overline{u_1}(x_1, t)$ can also be determined from known relations. The strength of the current sheet I, flowing on the piston surface is obtained as follows

$$\text{curl } \vec{B} = \mu \vec{J} = \mu I \delta(x_1) \vec{e_3}$$

$$\frac{\partial B_2}{\partial x} = \mu I \delta(x_1)$$

$$\therefore \; B_2(p^+) - B_2(p^-) = \mu I$$

$$I = \frac{B_{2I}}{\mu}[\overline{B_2}(p^+) - 1] \qquad (5\text{-}4)$$

The problem is now completely solved. However, as one may see from (5-2) and (5-3), the explicit solution for $t_p(x_1, t)$ is not easy to obtain.

2. In an arbitrarily oriented magnetic field $(\vec{B}_I)$

The problem is now more complicated than the previous one, we have to use the graphical method. The procedures are described as follows:

(a) Since the piston recedes from the gas, it causes an expansion process. Moreover, the wave motion is a slow simple wave because of the forward facing of the C_+ characteristics (phase lines) as shown in Fig. 5-1. At the initial state, the physical quantities expressed in dimensionless quantities are

$$\overline{\rho}_I = 1, \qquad \overline{B_2}_I = \frac{B_{2I}}{B_{1I}} = \frac{B_{2I}}{B_1}$$

$$\overline{u_1}_I = \frac{u_{1I}}{a_I} = 0, \quad \overline{u_2}_I = \frac{u_{2I}}{a_I} = 0$$

$$\overline{a}_I = 1, \qquad \overline{b_1}_I = \frac{b_{1I}}{a_I}, \qquad \overline{b_2}_I = \frac{b_{2I}}{a_I}$$

with the parameter

$$K_1 = \frac{B_{1I}}{(\mu\rho_I)^{1/2}} \frac{1}{a_I} = \frac{b_{1I}}{a_I} = \frac{\overline{b_{1I}}}{\overline{a_I}}$$

From (3-6) we have

$$(\overline{\rho})_{oI} = K_1^{\gamma/2}\,\overline{\rho}_I = K_1^{\gamma/2}$$

Knowing the values of $(\overline{\rho})_{oI}$ and $\overline{B_2}_I$ we can determine the point I of the initial state on the (α, β_1) plane by using Fig. 3-9.

(b) Read off the coordinates (α, β_1) of point I, the corresponding integral curve is determined from Fig. 3-7. This is the trajectory along which the phase of the physical variables changes. It is shown in Fig. 5-2.

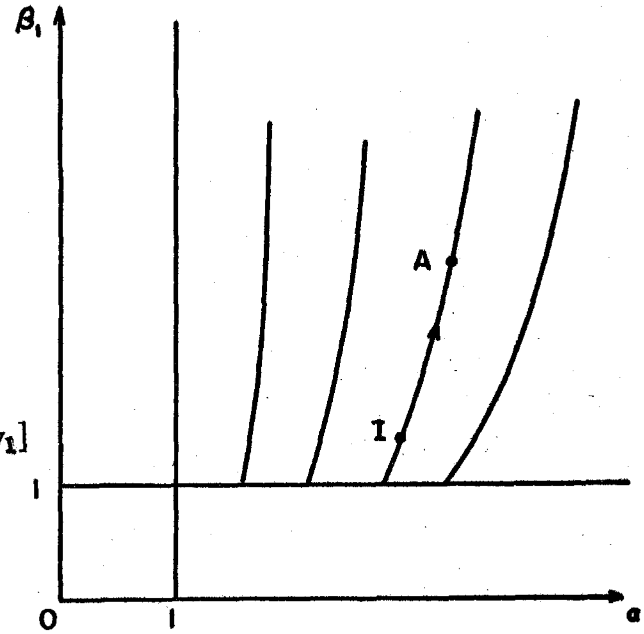


Fig. 5-2

(c) The point I on $[(\bar{\rho})_o, w_1]$ plane [see Fig. (5-3)] is also determined by $(\bar{\rho})_{oI}$ and $\overline{B_2}_I$. Then for each velocity of piston, say $(\bar{u}_p)_A$, we can determine the position of the state A on $[(\bar{\rho})_o, w_1]$ plane, hence the corresponding values $(\bar{\rho}_o)_A, (\overline{B_2})_A$ are known from this plane. This in turn determines the position of A on (α, β_1) plane also, and all the other physical quantities can be read off from the graph.

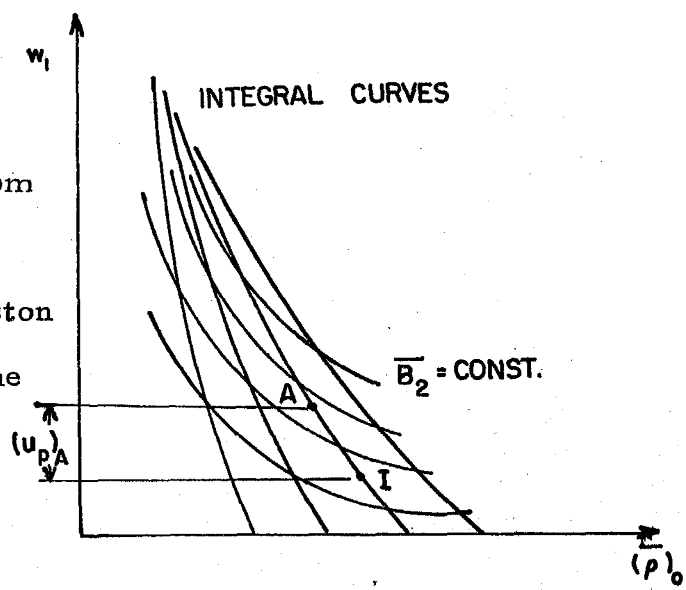


Fig. 5-3

(d) For each point on the piston path, say at A, where $u_p = (u_p)_A$, the straight characteristics C_+ leading from it has the slope

$$\left(\frac{dx}{dt}\right)_A = u_{pA} + C_A$$

where C_A is determined from c-curves on Fig. 3-8.

The characteristics C_+ can be drawn from the piston path now, this is actually the phase line. All the other physical variables associated with $(u_p)_A$, which are known from (c), can be attached to the phase line.

Carrying out the same process for several points on the piston path, we get a family of C_+ characteristics issuing from it with all known values of physical quantities on it. The change of $(u_2)_0$ is obtained from Fig. 3-11. The problem is thus solved.

To show explicitly the hydromagnetic effect on the gas flow for the receding piston problem, we consider the relatively simple case that u_p = constant. The corresponding solution of ordinary gas-dynamic case is well known that at any given time the velocity distribution is a linear function of space throughout the simple wave region. This serves as a standard of reference with which the solution of an ideal conducting gas in magnetic fields of different directions are compared.

I. The ordinary gasdynamic solution (cf. Ref. 1)

Because of the uniform motion of the piston starting impulsively at $t = 0$, the piston path is represented by a straight line through the origin on the (x_1, t) plane. The simple wave zone consists of the phase lines which are centered rays

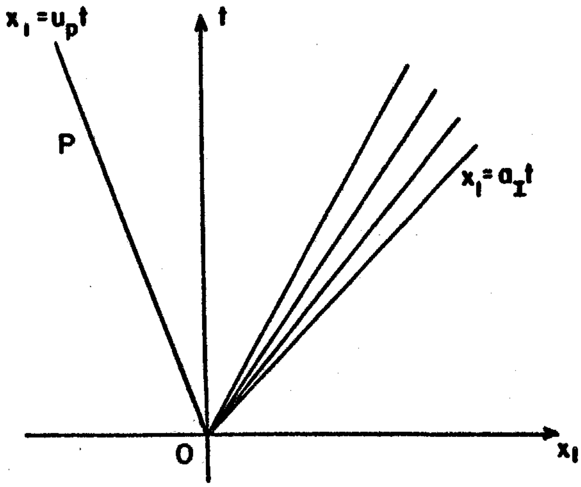


Fig. 5-4

now. The following equation holds along each ray.

$$\frac{dx_1}{dt} = \frac{x_1}{t} = u_1 + a \tag{5-5}$$

The Riemann invariant relation across the phase lines is

$$\frac{u_1}{2} - \frac{a}{\gamma-1} = - \frac{a_I}{\gamma-1} \tag{5-6}$$

since $u_I = 0$ here. Eliminating a from (5-5) and (5-6), we get

$$\frac{x_1}{t} = \frac{\gamma+1}{2} u_1 + a_I \tag{5-7a}$$

If we write (5-7a) in terms of dimensionless variables $\overline{u_1} = \frac{u_1}{a_I}$, $\overline{x_1} = \frac{x_1}{a_I t}$, it becomes

$$\overline{u_1} = \frac{2}{\gamma+1} (\overline{x_1} - 1) \tag{5-7b}$$

The gas at the surface of the piston has always the same longitudinal velocity as that of the piston until a critical value of the piston velocity is reached which can be obtained directly from (5-6) by putting $a=0$, i. e.

$$(u_p)_{cr} = - \frac{2a_I}{\gamma-1} \tag{5-8}$$

If $u_p > (u_p)_{cr}$, the simple waves are bounded by $\overline{u_p} < \overline{x_1} < 1$ and the gas is at rest in the region $\overline{x_1} > 1$. When $u_p < (u_p)_{cr}$, a caviation zone exists which separates the piston and the gas. The simple waves are bounded by $\overline{x_1} = 1$ and $\overline{x_1} = 1 + \frac{\gamma+1}{2} \overline{u_p}$. The gas is in a constant state between $\overline{x_1} = \overline{u_p}$ and $\overline{x_1} = 1 + \frac{\gamma+1}{2} \overline{u_p}$ and it is at rest in the region $\overline{x_1} > 1$.

For the sake of convenience in comparison, we have assumed $\gamma = \frac{5}{3}$ and $\overline{u_p} = - \frac{1}{5}$ for all examples calculated. Hence

$$\overline{u_1} = \frac{3}{4}(\overline{x_1} - 1) \tag{5-9}$$

and the simple waves are in the region $\frac{11}{15} < \overline{x} < 1$.

2. In a longitudinal field.

The motion must start from $\overline{B_2} = 0$ and the situation can be easily seen from Fig. 3-9 where $\overline{B_2} = 0$ is represented by two straight lines $\alpha = 1$ and $\beta_1 = 1$. Since the gas must undergo an expansion, we have to exclude the part of the line $\beta_1 = 1$ for $\alpha < 1$. The exact position representing the initial condition on $(\alpha - \beta_1)$ plane is then determined by $K_1^2 = b_{1_I}^2/a_I^2 = \beta_1/\alpha$. If $K_1 > 1$, the only solution is $c = a$ and the motion corresponds exactly to an ordinary gasdynamic one. If $K_1 < 1$, two different solutions are possible, either $c = a$ or $c = b_1$. The former one corresponds to the ordinary gasdynamic case and the latter one corresponds to a MHD case where a transverse magnetic field can be induced during the expansion process of the gas. The choice of one of these solutions depends usually on the boundary conditions, but it is undeterminate in the present problem.

3. In a transverse field.

An analytical solution of the problem is possible in this case. From (2-9) and the fact of centered simple wave, we have

$$\frac{dx_1}{dt} = \frac{x_1}{t} = u_1 + c \tag{5-10a}$$

or expressing it in terms of dimensionless variables,

$$\overline{x_1} = \overline{u_1} + \overline{c} \tag{5-10b}$$

From (4-6) we have

$$\overline{u_1} \mp r = \text{constant}$$

where we must choose the negative sign for the condition to be applied across the forward facing phase lines (c^+ characteristics). The value of r is obtained from (4-11) and the constant is determined by satisfying the initial condition that $\overline{u_1} = 0$ when $\overline{\rho} = 1$. Hence

$$\overline{u_1} = \frac{2}{K_2^2} \left[(1 + K_2^2 \, \overline{\rho}^{\frac{1}{3}})^{\frac{3}{2}} - (1 + K_2^2)^{\frac{3}{2}} \right] \qquad (5\text{-}11)$$

From (4-4), we have for $\gamma = \frac{5}{3}$

$$\overline{c} = \overline{\rho}^{\frac{1}{3}} (1 + K_2^2 \, \overline{\rho}^{\frac{1}{3}})^{\frac{1}{2}} \qquad (5\text{-}12a)$$

Eliminating $\overline{\rho}$ between (5-11) and (5-12a), we get

$$\overline{c} = \frac{\overline{u_1}}{2} + \frac{1}{K_2^2} \left\{ (1 + K_2^2)^{\frac{3}{2}} - \left[\frac{K_2^2 \, \overline{u_1}}{2} + (1 + K_2^2)^{\frac{3}{2}} \right]^{\frac{1}{3}} \right\} \qquad (5\text{-}12b)$$

Substituting this into (5-10b), we get the expression of $\overline{x}$ in terms of $\overline{u}$ and K_2

$$\overline{x_1} = \frac{3\overline{u_1}}{2} + \frac{1}{K_2^2} \left\{ (1 + K_2^2)^{\frac{3}{2}} - \left[\frac{K_2^2 \, \overline{u_1}}{2} + (1 + K_2^2)^{\frac{3}{2}} \right]^{\frac{1}{3}} \right\} \qquad (5\text{-}13)$$

This is an implicit solution of the velocity distribution as a function of $\overline{x_1}$. One may easily verify that it reduces to (5-9) in the limit $K_2 \to 0$. As $K_2 \gg 1$, we get

$$\overline{u_1} \cong \frac{2}{3}(\overline{x_1} - K_2) \qquad (5\text{-}14)$$

It is a linear function of space. This result is by no means accidental, it can also be obtained from a different approach. We see from (4-4) that at $K_2 \gg 1$,

$$\bar{c}^2 = K_2^2\, \bar{\rho} \tag{5-15a}$$

If we denote $\tilde{a} = \frac{\bar{c}}{K_2}$, (5-15a) becomes

$$\tilde{a} = \bar{\rho} \tag{5-15b}$$

It corresponds to the ordinary gasdynamic case for $\gamma = 2$.

Write $\tilde{u}_1 = \frac{\bar{u}_1}{K_2}$, from (4-5a) we get

$$d\tilde{u}_1 = \tilde{c}\, \frac{d\bar{\rho}}{\bar{\rho}}$$

If we denote $\tilde{x}_1 = \frac{\bar{x}_1}{K_2}$, (5-5) becomes

$$\tilde{x}_1 = \tilde{u}_1 + \tilde{a}$$

The equations have exactly the same form as those of the ordinary gasdynamic case, and the solution can be taken over from (5-7b)

$$\tilde{u}_1 = \frac{2}{3}(\tilde{x}_1 - 1) \tag{5-16}$$

This is identical to (5-14) if we express it in terms of original notations.

The escape velocity of the piston is obtained from (5-11) by putting $\bar{\rho} = 0$. It is

$$(\bar{u}_p)_{cr} = -\frac{2}{K_2^2}\left[(1 + K_2^2)^{\frac{3}{2}} - 1\right]$$

As $K_2 \to 0$,

$$(\bar{u}_p)_{cr} = -3$$

and as $K_2 \to \infty$,

$$(\bar{u}_p)_{cr} = -2K_2$$

which can also be obtained from (5-8) by putting $\gamma = 2$ and replacing

$\frac{(u_p)_{cr}}{a_I}$ by $\frac{(\overline{u_p})_{cr}}{K_2}$.

We have calculated the longitudinal velocity profiles for $K_2 =$ 0.75, 1.00 and 1.50 respectively which are compared with the ordinary gasdynamic one as shown in Fig. 5-5. They are very nearly straight lines and are displaced to the right of the ordinary gasdynamic profile since they are the limiting case of fast simple waves.

4. In an arbitrarily oriented magnetic field.

The solution is obtained by using the graphical method mentioned above. Some of the velocity profiles at $\overline{B_2} = 0.75$, 1.50 with $K_1 = 0.75$, 1.00, 1.50 are given in Fig. 5-6. They appear also very nearly as straight lines. The dominant effect due to magnetic field consists in displacing them to the left of the ordinary gasdynamic velocity profile, since only slow simple waves are possible in this case.

B. CURRENT SHEET PROBLEM

This is an example of the fluid motion initiated by electromagnetic means. A steady surface current sheet flowing in x_3-direction is suddenly discharged at $x_1 = 0$ into an ideal conducting gas at rest which is in an arbitrarily oriented uniform magnetic field. At the surface of the current sheet a tangential magnetic field is induced. It exerts a magnetic pressure on the gas next to it and which in turn initiates the fluid motion. There is no characteristic length and time in this problem and all physical variables must remain constant along the rays issuing from the origin of $(x_1$-$t)$ plane. As in the one-dimensional piston problem in ordinary gasdynamics, a shock discontinuity is formed but we have here a general hydromagnetic shock instead. Since there is no movement of

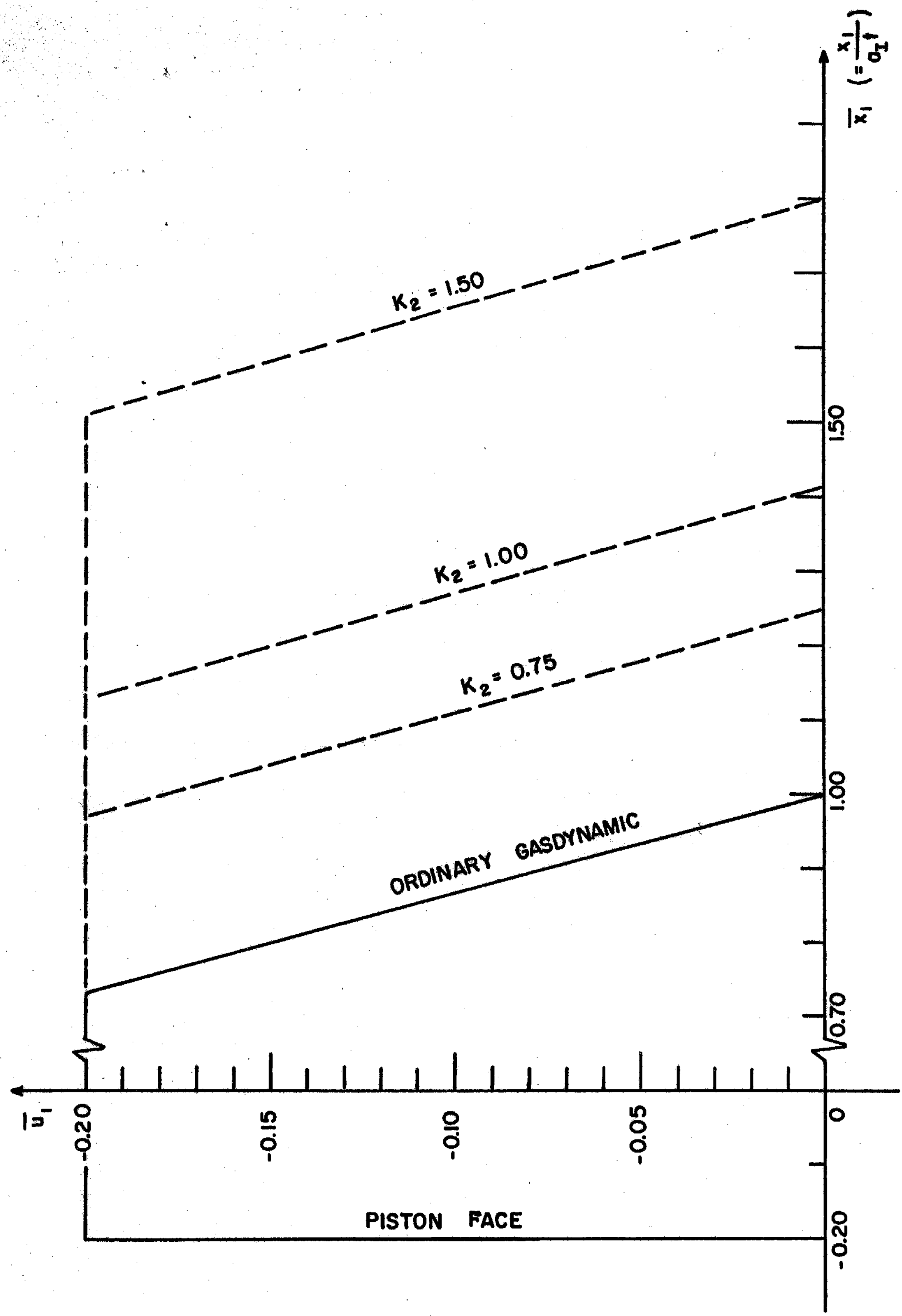
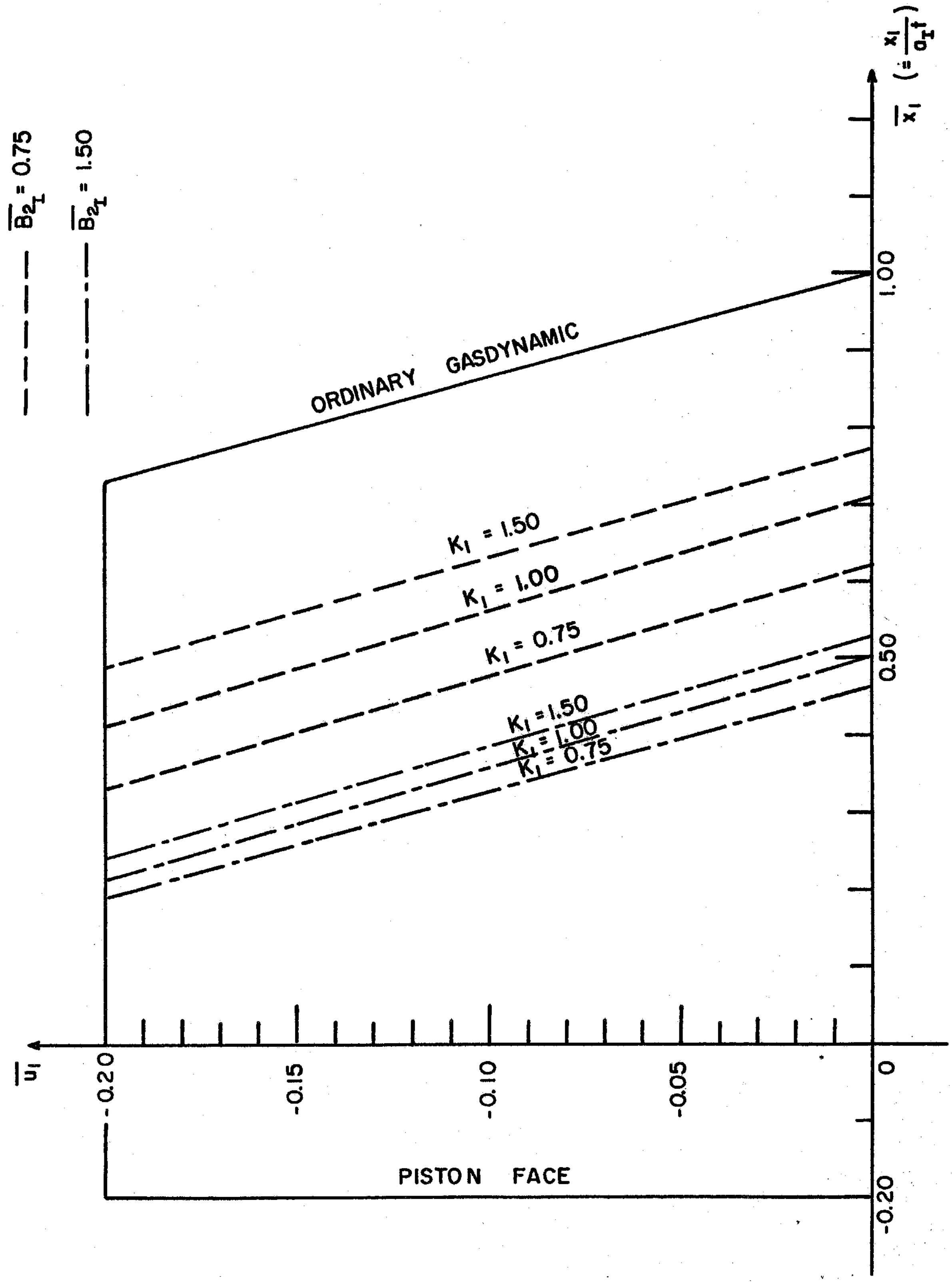

$K_2 = 1.50$
$K_2 = 1.00$
$K_2 = 0.75$
ORDINARY GASDYNAMIC
PISTON FACE
$\bar{x}_1 \ (= \frac{x_1}{a_I t})$
$\bar{u}_1$
0.70
1.00
1.50
-0.20
-0.15
-0.10
-0.05
0
-0.20

FIG. 5-5



$\overline{B_{2_I}} = 0.75$
$\overline{B_{2_I}} = 1.50$
ORDINARY GASDYNAMIC
$K_1 = 1.50$
$K_1 = 1.00$
$K_1 = 0.75$
$K_1 = 1.50$
$K_1 = 1.00$
$K_1 = 0.75$
$\overline{x_1}$ $(= \frac{x_1}{a_I t})$
1.00
0.50
0
-0.20
$\overline{u_1}$
-0.20
-0.15
-0.10
-0.05
PISTON FACE

FIG. 5-6

the mechanical boundary followed by the pressure, the gas behind the shock undergoes an expansion rather than remaining in a constant state. The expansion process is actually described by MHD simple waves. When the strength of the current sheet is not too large such that the density of the gas next to it is still finite, the boundary condition requirement at $x_1 = 0$ is simply $u_1 = 0$. For large values of the current intensity, the gas may expand to vacuum and be displaced from the current sheet by the magnetic pressure. Now, the boundary condition on the current sheet should be $\rho = 0$. It can be shown that a constant electric field always exists in the vacuum region.

This idea has been used to develop an extremely fast shock wave together with a very high temperature plasma in a magnetic annular shock tube. It was studied both experimentally by Patrick (Ref. 9) and theoretically by Kemp and Petschek (Ref. 5).

A somewhat simplified method of treating this problem has been worked out by assuming $\gamma = 2$. One may easily verify that the integral curves on Fig. 3-9 become vertical lines. The analysis is simplified a great deal and some qualitative behavior of the flow pattern can be easily visualized. Physically it corresponds to the somewhat unrealistic case that the gas has two degrees of freedom. (The gas is sometimes called "hydrodynamical gas"). From the isentropic condition, we have

$$P = \rho RT \sim \text{Const } \rho^2$$

Hence

$$T \sim \rho$$

We have

$$r(\overline{\rho_1}) = \int_0^{\overline{\rho_1}} \overline{\rho}^{\frac{\gamma-3}{2}} [1 + K_2^2 \, \overline{\rho}^{2-\gamma}]^{\frac{1}{2}} \, d\overline{\rho}$$

Let $t = \dfrac{\overline{\rho}}{\overline{\rho_1}}$, then

$$r(\overline{\rho_1}) = \overline{\rho_1}^{\frac{\gamma-1}{2}} \int_0^1 t^{\frac{\gamma-3}{2}} [1 + K_2^2 \, \overline{\rho_1}^{2-\gamma} \, t^{2-\gamma}]^{\frac{1}{2}} \, dt$$

Let $s = t^{2-\gamma}$ and $G = K_2^2 \, \overline{\rho_1}^{2-\gamma}$, we have then

$$r(\overline{\rho_1}) = \frac{1}{2-\gamma} \, \overline{\rho_1}^{\frac{\gamma-1}{2}} \int_0^1 S^{\frac{3\gamma-5}{2(2-\gamma)}} (1 + GS)^{\frac{1}{2}} \, dS$$

This conforms with the standard form of hypergeometric function (see Ref. 8) since

$$F(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(v)\Gamma(w-v)} \int_0^1 t^{v-1}(1-t)^{w-v-1}(1-tz)^{-u} \, dt$$

with $\mathrm{Re}(w) > \mathrm{Re}(v) > 0$. Here we have

$$z = -G$$

$$u = -\frac{1}{2}$$

$$v = 1 + \frac{3\gamma-5}{2(2-\gamma)} = \frac{\gamma-1}{2(2-\gamma)}$$

$$w = 1 + v = \frac{3-\gamma}{2(2-\gamma)}$$

Then

$$r(\overline{\rho_1}) = \frac{1}{2-\gamma} \, \overline{\rho_1}^{\frac{\gamma-1}{2}} \frac{\Gamma[\frac{\gamma-1}{2(2-\gamma)}]\Gamma(1)}{\Gamma[\frac{\gamma-1}{2(2-\gamma)} + 1]} F(-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -G)$$

$$= \frac{2}{\gamma-1} \, \overline{\rho_1}^{\frac{\gamma-1}{2}} F(-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2^2 \, \overline{\rho_1}^{2-\gamma})$$

REFERENCES

1. Courant, R. and K. O. Friedrichs, Supersonic Flow and Shock Waves Interscience Press, 1948.

2. Cole, J. D., Magnetohydrodynamic Waves. Air Force OSR-TN-59-13. January, 1959. Also appears in Proceedings of Lockheed Symposium on Magnetohydrodynamics, 1959.

3. Friedrichs, K. O. Magnetohydrodynamic Waves. Los Alamos Report 1954. See also, Friedrichs, K. O. and Kranzer, H. Notes on Magnetohydrodynamics. VIII. Nonlinear Wave Motion. Institute of Mathematical Sciences, New York University.

4. Bazer, J. Resolution of an Initial Shear-Flow Discontinuity in One Dimensional Hydromagnetic Flow. The Astrophysical Journal, Vol. 128, No. 3, November 1958.

5. Kemp, N. and Petschek, H. Theory of the Flow in the Magnetic Annular Shock Tube. Avco-Everett Research Report 60, July, 1959.

6. Mitchner, M. Magnetohydrodynamic Flow in a Shock Tube. The Physics of Fluids, Vol. 2, No. 1, Jan.-Feb. 1959.

7. Golitsyn, G. S. Unidimensional Motion in Magnetohydrodynamics. JETP 35(8), 1959.

8. Magnus, W. and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics. Chelsea Publishing Company, 1954.

9. Patrick, R. M., The Production and Study of High Speed Shock Waves in a Magnetic Annular Shock Tube. Arco-Everett Research Report 59, July, 1959.

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12. Abstract:

A study is made of simple wave flows of an infinitely conducting perfect gas (polytropic) in the presence of an arbitrarily oriented magnetic field. From group theoretic considerations the problem is reduced to the solution of a first-order differential equation which is integrated numerically. Properties of solutions and examples are discussed.