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Mei Zhuang, Toshi Kubota, and Paul E. Dimotakis, Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, CA

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On the Instability of Inviscid, Compressible Free Shear Layers

Mei Zhuang*, Toshi Kubota° and Paul E. Dimotakis†

Graduate Aeronautical Laboratories
California Institute of Technology, Pasadena, California

Abstract

The linear spatial instability of inviscid compressible laminar mixing of two parallel streams, comprised of the same gas, has been investigated with respect to two-dimensional wave disturbances. The effects of the velocity ratio, temperature ratio, and the temperature profile across the shear layer have been examined. A nearly universal dependence of the normalized maximum amplification rate on the convective Mach number is found, with the normalized maximum amplification rate decreasing significantly with increasing convective Mach number in the subsonic region. These results are in accord with those of recent growth rate experiments in compressible turbulent free shear layers and other similar recent calculations.

Introduction

The instability of inviscid, laminar, two-dimensional shear layers in both incompressible and compressible flow has been studied in the past.

For incompressible parallel flow, the linear spatial instability of the hyperbolic tangent and Blasius mixing layers was investigated for different values of the ratio between the difference and sum of the velocities of the two co-flowing streams by Monkewitz & Huerre[1]. They found that the maximum growth rate is approximately proportional to the velocity ratio.

For compressible flow, Lessen, Fox & Zien[2] found that increasing the Mach number of the flow tends to stabilize the flow. Gropengiesser[3] studied the instability characteristics of boundary layers at various free stream Mach numbers and temperature ratios. The linear stability of a shear layer of an inviscid fluid with two-dimensional temporally growing disturbances was considered by Blumen, Drazin & Billings[4]. They showed that the flow is unstable with respect to two-dimensional disturbances at all values of the Mach number. They also showed that there exists a second unstable mode which is supersonic and decays weakly with distance from the shear layer. For compressible flow, however, the effects of shear layer Mach number, temperature ratio, velocity ratio, and temperature profile on the stability characteristics are very complicated. These authors offer no prediction about what the combined influences of these flow parameters will be. Recently, Ragab & Wu[5] studied the influence of the velocity ratio on the stability characteristics of the compressible shear layer, and they also investigated the effect of the convective Mach number, as proposed by Papamoschou & Roshko[6]. Their results indicate the convective Mach number is a parameter which correlates the compressibility effects on the spreading rate of mixing layers.

Papamoschou & Roshko performed experiments on compressible shear layers and suggested the convective Mach number (M_c) as the appropriate parameter scaling the effects of compressibility. This is defined for each stream as:

$$M_{c1} = \frac{U_1 - U_c}{a_1}, \qquad M_{c2} = \frac{U_c - U_2}{a_2}, \tag{1}$$

where U_1, U_2 and a_1, a_2 are the free stream velocities and speeds of sound. The quantity U_c is the convective velocity of the large scale structures and was estimated as $\tilde{U}_c$ by Papamoschou & Roshko assuming that the dynamic pressure match at stagnation points in the flow (Coles[7], Dimotakis[8]). For compressible isentropic flow, i.e. (Papamoschou & Roshko)

$$\left(1+\frac{\gamma_1-1}{2}\tilde{M}_{c1}^2\right)^{\frac{\gamma_1}{\gamma_1-1}} = \left(1+\frac{\gamma_2-1}{2}\tilde{M}_{c2}^2\right)^{\frac{\gamma_2}{\gamma_2-1}}, \tag{2}$$

where γ_1, γ_2 are the ratios of the specific heats of the two streams, and

$$\tilde{M}_{c1} = \frac{U_1 - \tilde{U}_c}{a_1}, \qquad \tilde{M}_{c2} = \frac{\tilde{U}_c - U_2}{a_2}. \tag{3}$$

For γ_1 equals γ_2, $\tilde{U}_c$ can be obtained by

$$\tilde{U}_c = \frac{a_2 U_1 + a_1 U_2}{a_1 + a_2}, \tag{4}$$

which, for equal static free stream pressures and specific heats, reduces to the incompressible expression[8]. They suggested that the growth rate of a compressible shear layer, normalized by the growth rate for an incompressible shear layer, might be expressible as a universal function of the convective Mach number $\tilde{M}_{c1}$, which is valid over a wide range of velocity and temperature ratios of a shear layer. They also found that the normalized growth rate decreases significantly with increasing $\tilde{M}_{c1}$.

The numerical calculations described here were performed under the assumptions of linear instability theory. The convective velocity is estimated as $\hat{U}_c = C_r$ (Mack[9] 1975). Therefore, a convective Mach number ($\hat{M}_c$) for each stream can be

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* Graduate Student.
° Professor.
† Professor. Member AIAA

written as:

$$\hat{M}_{c1} = \frac{U_1 - C_r}{a_1}\ , \qquad \hat{M}_{c2} = \frac{C_r - U_2}{a_2}\ , \tag{5}$$

where C_r is the real phase velocity of the disturbances.

The purpose of the present studies is to investigate the combined influence of the convective Mach number ($\hat{M}_c$), which is different from the one used by Ragab & Wu ($\tilde{M}_c$), the velocity and temperature ratios, and the temperature profiles of the flow on the linear stability behavior of compressible shear layers. Studies are made of the case of inviscid flow under the assumptions that the gases in the two streams are the same, the main flow can be treated parallel, and that the disturbances in the flow are of small amplitude. The range of the unstable frequencies and wave numbers were numerically calculated for a two-dimensional, spatially growing disturbance.

Basic disturbance equations

We consider a two-dimensional flow of two parallel streams. With upper stream quantities as the reference and the local layer thickness δ as the length scale, the dimensionless quantities of the flow in Cartesian co-ordinates can be written as usual

$$u_x = \bar{U} + u', \quad u_y = v', \quad T = \bar{T} + T',$$

$$\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p',$$

or, for the general field quantity

$$Q(x,y,t) = \bar{Q}(y) + Q'(x,y,t),$$

where $\bar{Q}$ is a profile of the main flow, and Q' is the corresponding disturbance amplitude.

Consider now the disturbance to be a wave propagating in the x-direction. The disturbance quantities in dimensionless form can be expressed as[2]

$$\{ u', v', T', \rho', p' \} =$$

$$\Big\{ f(y), \alpha\phi(y), \theta(y), r(y), \pi(y) \Big\} \exp\,[i\alpha(x - ct)], \tag{6}$$

where α is a complex wave number, and c is a complex wave velocity. In the case of negligible viscous effects, the linearized disturbance equations for a 2-D compressible fluid with the same gas constants and specific heats are given by[2]:

$$\text{Continuity} : i\,(\bar{U} - c)\, r + \bar{\rho}\,(\phi' + i\,f) + \bar{\rho}'\,\phi = 0 \tag{7a}$$

$$\text{Momentum} : \gamma M_1^2\,\bar{\rho}\,[i(\bar{U} - c)\,f + \bar{U}'\,\phi] = -\,i\,\pi \tag{7b}$$

$$\gamma M_1^2\,\alpha^2\,\bar{\rho}\,[i(\bar{U} - c)\,\phi] = -\,\pi' \tag{7c}$$

$$\text{Energy} : \bar{\rho}\,[i(\bar{U} - c)\theta + \bar{T}'\phi] = -(\gamma - 1)(\phi' + if) \tag{7d}$$

$$\text{State} : \frac{\pi}{\bar{p}} = \frac{r}{\bar{\rho}} + \frac{\theta}{\bar{T}}\ , \tag{7e}$$

where M_1 is the upper stream Mach number and primes here correspond to d/dy. These equations can be reduced to a second order differential equation for the pressure disturbance[2], i.e.

$$\pi'' - \left(\frac{2\,\bar{U}'}{\bar{U} - c} - \frac{\bar{T}'}{\bar{T}}\right)\pi' - \alpha^2\left[1 - \frac{M_1^2}{\bar{T}}\,(\bar{U} - c)^2\right]\pi = 0. \tag{8}$$

Asymptotic Behavior of the Eigenfunctions

The asymptotic behavior of the eigenfunction $\pi(y)$ for $y \to \pm\infty$ is found from Equation (8). With $y \to \pm\infty$, $\bar{U}$ and $\bar{T}$ are constants, and $\bar{U}'$, $\bar{T}'$ are zeros. In that limit, Equation (8) becomes

$$\pi'' - \lambda_k^2\,\pi = 0\ , \tag{9}$$

with

$$\lambda_k^2 = \alpha^2\left[1 - \frac{M_1^2}{\bar{T}_k}\,(\bar{U}_k - c)^2\right] = \Lambda_k = \Lambda_{kr} + i\Lambda_{ki}\ , \tag{10}$$

and k = 1, 2. Therefore, from (10) we get

$$\lambda_k = \lambda_{kr} + i\,\lambda_{ki} = \pm\,\Lambda_k^{1/2}$$

and the solution for large $|y|$ can be written as

$$\pi = A_k \exp(-\lambda_k\,|y|)\ , \tag{11}$$

where A_k is a complex constant.

Since we have only considered the case of amplified disturbances ($\alpha_i < 0$), the boundary conditions for both supersonic and subsonic disturbances can be expressed by $\pi_r(y \to \pm\infty) \to 0$ and $\pi_i(y \to \pm\infty) \to 0$. In order to satisfy the boundary conditions, we set $\lambda_{kr} > 0$, and get

$$y = y_1 \to +\infty, \qquad \pi = A_1 \exp(-\lambda_1 y) \tag{12a}$$

$$y = y_2 \to -\infty, \qquad \pi = A_2 \exp(\lambda_2 y)\ , \tag{12b}$$

where

$$\lambda_k = \lambda_{kr} + i\,\lambda_{ki} =$$

$$\left[\frac{1}{2}\,(|\Lambda_k| + \Lambda_{kr})\right]^{1/2} + i\,\mathrm{sign}(\Lambda_{ki})\left[\frac{1}{2}\,(|\Lambda_k| - \Lambda_{kr})\right]^{1/2}.$$

Formulation of the Eigenvalue problem

The eigenvalue problem is defined as follows. For a given real disturbance frequency β ($\beta = \alpha c$), the eigenvalues α_r and α_i are to be determined in such a way that the eigenfuctions $\pi_r(y)$ and $\pi_i(y)$ satisfy the boundary conditions. Specifically, we used a Runge-Kutta method to solve the eigenvalue equation, with (12a) and (12b) as boundary conditions. The equation was integrated from one side of the boundary ($y = y_1$) to the other side ($y = y_2$). The correct α was obtained for a given β by matching to the boundary conditions.

Velocity and Temperature Distributions

Lock's[10] numerical calculation of the velocity distribution for a compressible laminar boundary layer, suggest that the velocity profile for compressible laminar shear layers is well approximated by a hyperbolic tangent profile. So we assume that the dimensionless mean velocity profile is described by a hyperbolic tangent profile represented by the form

$$\bar{U}(y) = \eta(y) + U_R\,[1 - \eta(y)]\ , \tag{13}$$

where $U_R = U_2/U_1$ is the velocity ratio across the shear layer, and $2\eta(y) - 1$ is approximated by a hyperbolic tangent. See mean velocity profiles $\bar{U}(y)$ in Fig. 1.

We note that the linearized flow equations do not prescribe the mean temperature profile. Accordingly, two different kinds of temperature profiles have been considered. One conforms to the Crocco-Busemann[11,12] relation, wherein the total temperature profile $T_t(y)$ for an equal ratio of the specific heats of the two free streams is represented by

$$T_t(y) = T_{t1}\,\eta(y) + T_{t2}\,[1 - \eta(y)] \; , \qquad (14)$$

where T_{t1}, T_{t2} are the free stream total temperatures. This yields the dimensionless mean static temperature profile,

$$\bar{T}(y) = c_1 + c_2\,\bar{U}(y) - \frac{(\gamma - 1)\,M_1^{\,2}}{2}\,\bar{U}^2(y) \; , \qquad (15)$$

where M_1 is the upper stream Mach number and c_1, c_2 are constants which satisfy the boundary conditions on the temperature profile. Such mean temperature profiles $\bar{T}(y)$ for $M_1 = 5$ are shown on Fig. 2. The other kind of dimensionless temperature profile is obtained by assuming that the dimensionless density distribution across the shear layer can also be approximated by a hyperbolic tangent profile, i.e.

$$\bar{\rho}(y) = \eta(y) + \rho_R\,[1 - \eta(y)] \; , \qquad (16)$$

where $\rho_R = \rho_2/\rho_1$ is the density ratio across the shear layer. Therefore, for a shear layer comprised of the same gas, the dimensionless temperature profile is $\bar{T}(y) = 1/\bar{\rho}(y)$. See Fig. 3.

Results

For a given combination of free stream Mach number M_1, temperature ratio T_R (T_2/T_1) and velocity ratio U_R, the linear instability characteristics were calculated, yielding the most unstable eigenvalue $(\alpha_m = \alpha_{mr} + i\,\alpha_{mi})$ and its corresponding real frequency β_m. The real phase velocity C_r of the disturbances was obtained as β_m/α_{mr}. This yields the convective Mach number $\hat{M}_{c1}$ and $\hat{M}_{c2}$ from Eq. (5).

Different combinations of velocity and temperature ratios using a velocity and temperature profile from Eqs. (13) and (15) were investigated for a convective Mach number $\hat{M}_{c1}$ from 0 to about 1.5. The velocity profiles for $U_R = 0.25, 0.5, 0.75$ appear in Fig. 1 and the temperature profiles for $T_R = 0.5, 1.0, 1.5$ in Fig. 2. Results shown in Figs. 4 - 9, which were obtained from nine different combinations of T_R and U_R, indicate that if the most unstable eigenvalue for a compressible shear layer is normalized by its value corresponding to an incompressible shear layer (at the same velocity and temperature ratio), the ratio is well approximated as a function of the convective Mach number only, i.e.

$$\frac{\delta_x(\hat{M}_{c1})}{\delta_x(0)} = \frac{\max\left\{ -\alpha_i\,(U_2/U_1, T_2/T_1, \hat{M}_{c1}) \right\}}{\max\left\{ -\alpha_i\,(U_2/U_1, T_2/T_1, \hat{M}_{c1} = 0) \right\}}$$

$$= F(\hat{M}_{c1}) \; , \qquad (17)$$

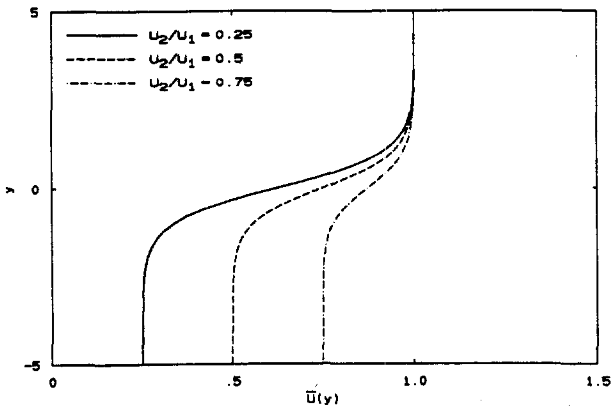


Fig. 1 Hyperbolic tangent mean velocity profiles for different values of the velocity ratio U_2/U_1.

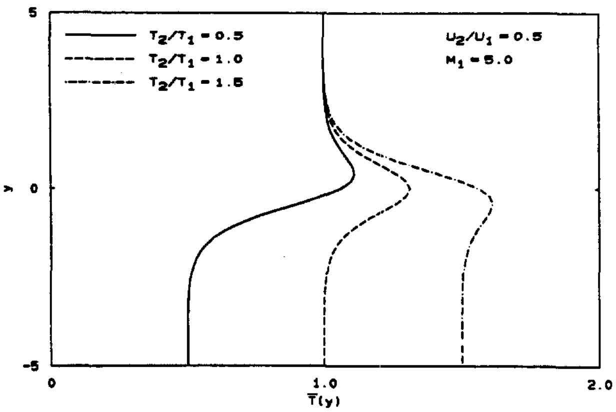


Fig. 2 Crocco-Busemann mean temperature profiles for different values of the temperature ratio T_2/T_1 for the case $U_2/U_1 = 0.5$ and $M_1 = 5.0$. —— $\hat{M}_{c1} = 1.54$, ---- $\hat{M}_{c1} = 1.13$, -·-· $\hat{M}_{c1} = 1.0$.

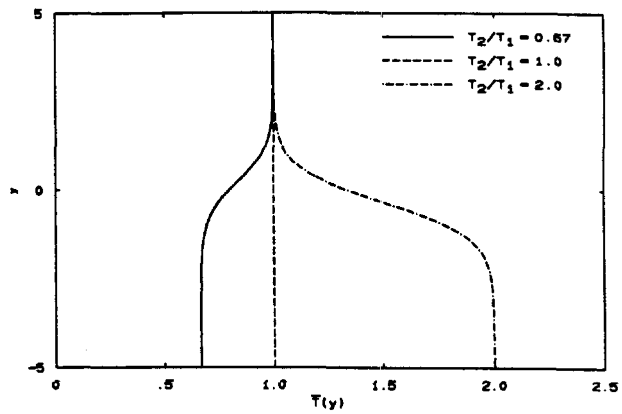


Fig. 3 Hyperbolic tangent $\bar{\rho}(y)$ mean temperature profiles for different values of the temperature ratio T_2/T_1.

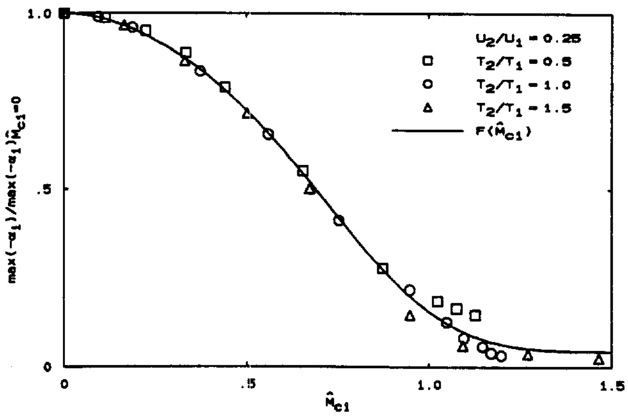
where $\delta_x = d\delta/dx$ for the shear layer of the particular free stream conditions. The solid line estimate of $\delta_x(\hat{M}_{c1})/\delta_x(0)$ in Figs. 4 - 9 was computed by using all the data of the nine different cases, and least squares fitting the normalized maximum amplification rate versus the convective Mach number $\hat{M}_{c1}$, for the range of $\hat{M}_{c1}$ from 0 to about 1.5 with a function of the form

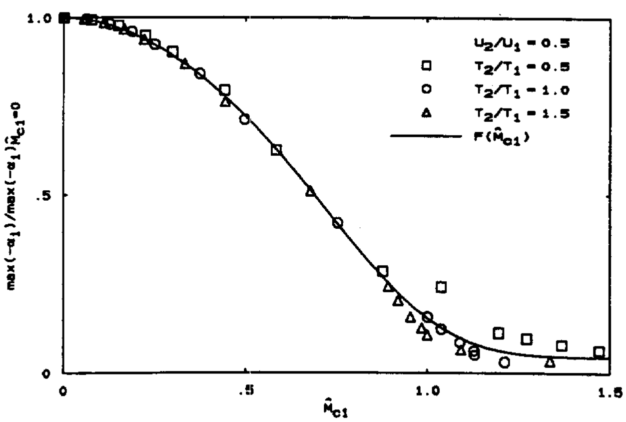
$$\frac{\delta_x(\hat{M}_{c1})}{\delta_x(0)} = 1 + p_0\left(e^{-(p_2\hat{M}_{c1}^2 + p_3\hat{M}_{c1}^3 + p_4\hat{M}_{c1}^4)} - 1\right), \quad (18)$$

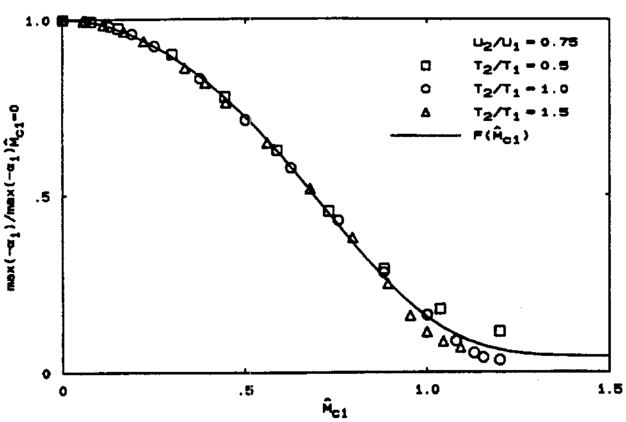
where

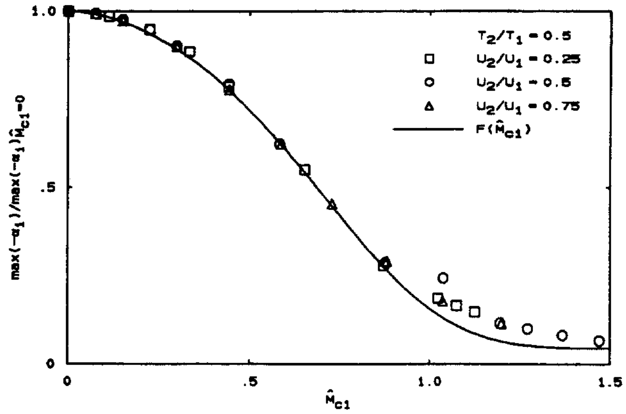
$$p_0 = 0.956174 \qquad p_2 = 1.53471$$
$$p_3 = -1.22389 \qquad p_4 = 1.83827\ .$$

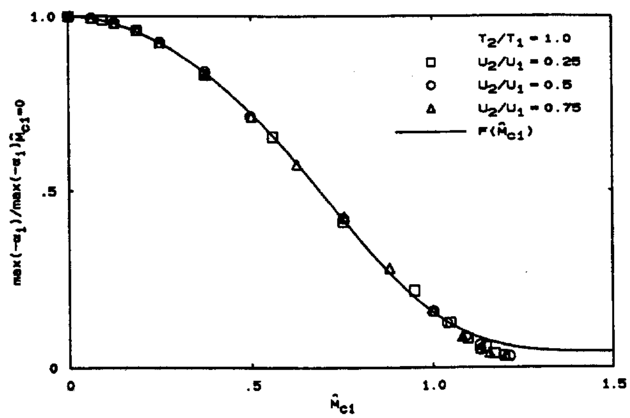
Note that $\delta_x(\hat{M}_{c1}\to\infty)/\delta_x(0) = 1 - p_0$, and that the coefficient p_2 is related to the second derivative at $\hat{M}_{c1}=0$, etc. Note also that these results suggest that $\mathcal{F}'(\hat{M}_{c1}=0) = 0$, as might have been argued a priori. The results, shown in Figs. 4 - 9, also suggest that the normalized maximum amplification rate decreases significantly with increasing $\hat{M}_{c1}$ in the region $\hat{M}_{c1} < 1$.

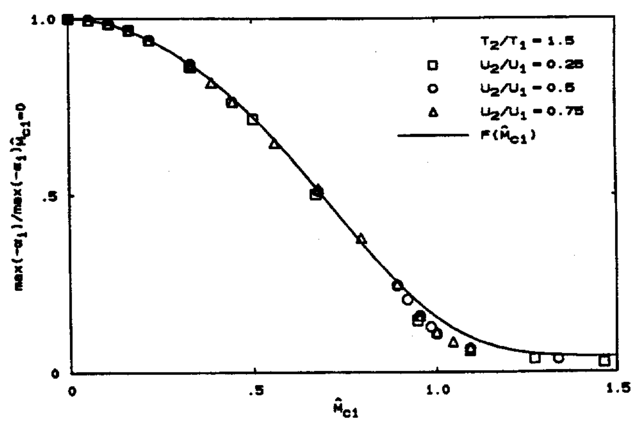












Figs. 4 - 9 Normalized maximum amplification rate vs $\hat{M}_{c1}$.

In the second set of calculations, the mean temperature profile was specified via Eq.(16), i.e. $\overline{T}(y) = 1/\overline{\rho}(y)$. The resulting temperature profiles for $T_R = 0.67$, 1, and 2 are plotted in Fig. 3. The velocity ratio $U_R = 0.5$ with each of these three temperature ratios was studied for the convective Mach number $\hat{M}_{c1}$ from 0 to about 1.5. The results, shown in Fig. 10, substantiate the convective Mach number as the relevant compressibility parameter and also display good agreement with the plot $\delta_x(\hat{M}_{c1})/\delta_x(0)$ vs. $\hat{M}_{c1}$ obtained from Eq.(18), even though these two mean temperature profiles are very different at supersonic convective Mach numbers (see Figs. 2, 3).

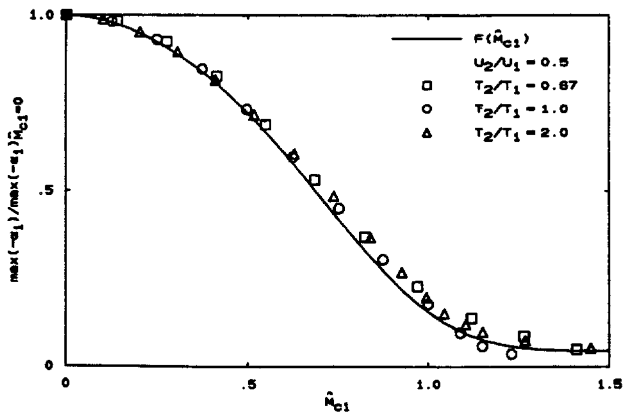


Fig. 10 Normalized maximum amplification rate vs $\hat{M}_{c1}$ for hyperbolic tangent mean temperature profiles comparison with F($\hat{M}_{c1}$)

With $\tilde{U}_c$ calculated from Eq.(4) and C_r obtained from the numerical calculations under the linear theory, $\tilde{M}_{c1}$ does not necessarily equal $\hat{M}_{c1}$. In fact, even the real phase velocity may not be unique for supersonic convective Mach number, because of the existence of a second mode. Blumen, Drazin & Billings[4] have noted this behavior for a shear layer of an inviscid fluid with two-dimensional temporally disturbances[4]. We can see that, for both temperature profiles (Eq.(15) and Eq.(16) with $\overline{T}(y) = 1/\overline{\rho}(y)$), there are very small differences between $\hat{M}_{c1}$ and $\tilde{M}_{c1}$ from the plot of $(\hat{M}_{c1} - \tilde{M}_{c1})/\hat{M}_{c1}$ vs. $\hat{M}_{c1}$ for $\hat{M}_{c1} \leq 1$, but the differences only become substantial when $\hat{M}_{c1} > 1$. See Figs. 11, 12. We only studied the cases for $\hat{M}_{c1} < 1.5$, since shock waves can exist in a shear layer at high convective Mach numbers and therefore, the validity of a linear description of these phenomena would be suspect.

A comparison of our estimate of $\delta_x(\hat{M}_{c1})/\delta_x(0)$ with Ragab's numerical data and with Papamoschou's experimental data is made in Fig. 13. The data from our calculations are very close to Ragab & Wu's. The difference between $\hat{M}_{c1}$ and $\tilde{M}_{c1}$, though not small in the region $\hat{M}_{c1} > 1$, does not affect the results, since the normalized amplification rates are very small in this region. According to Papamoschou & Roshko's experimental data, the growth rate of the shear layer tapers off as the convective Mach number becomes supersonic. As opposed to their findings, however, the growth rate of our calculations decreases to zero as $\hat{M}_{c1} \gg 1$. Preliminary calculations suggest that a larger value for the growth rate at large M_{c1} is exhibited by more complex velocity and/or density profiles.

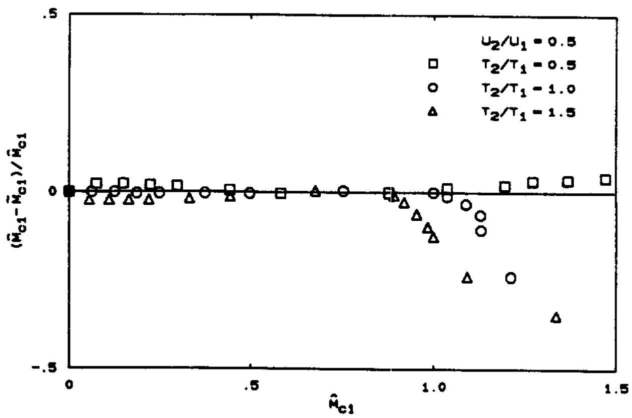


Fig. 11 Normalized difference between $\hat{M}_{c1}$ and $\tilde{M}_{c1}$ vs $\hat{M}_{c1}$.

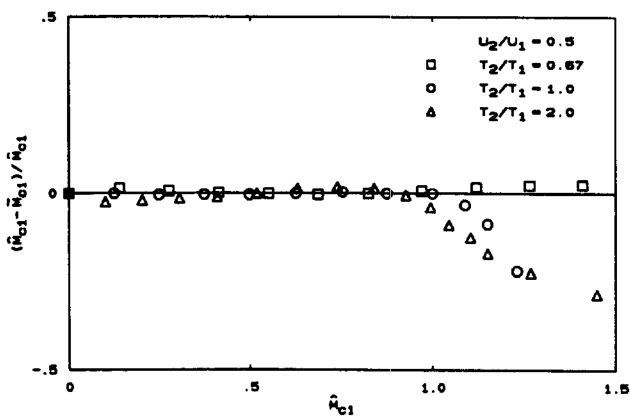


Fig. 12 Normalized difference between $\hat{M}_{c1}$ and $\tilde{M}_{c1}$ vs $\hat{M}_{c1}$ for hyperbolic tangent mean temperature profiles.

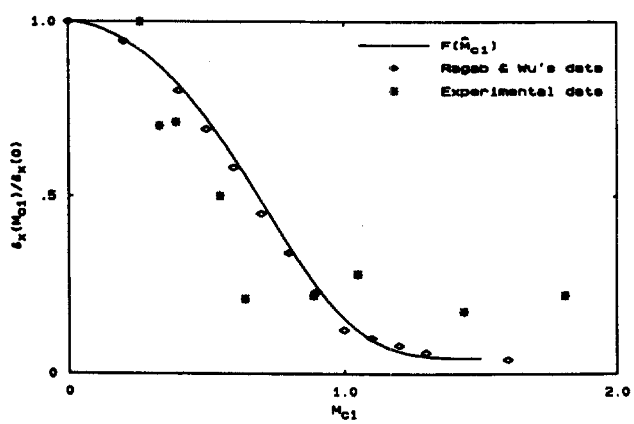


Fig. 13 A comparison of F($\hat{M}_{c1}$) with Ragab & Wu's numerical data and with Papamoschou & Roshko's experimental data.

Conclusion

The influences of the convective Mach number, the velocity and temperature ratios and the temperature profiles of the flow on the linear spatial instability characteristics of a plane shear layer, formed by the same gas, were investigated. It was found that there is a nearly universal dependence of the normalized maximum amplification rate on the convective Mach number, and this amplification rate decreases significantly with increasing $\hat{M}_{c1}$ in the region of $\hat{M}_{c1} < 1$.

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