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Sum Rules for Real Parts of Current-Particle Scattering Amplitudes*

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Sum rules for the real parts of causal amplitudes are derived in the quark-model light-cone algebra.

I. INTRODUCTION

It is now well known that the results of the experiments on deep-inelastic electron scattering[1] can be understood in terms of the nature of the light-cone singularity of the commutator of two electromagnetic currents.[2,3] The light-cone algebra abstracted from the quark model is compatible with the observed scaling and the suppression of the longitudinal-virtual-photon cross section.[4] It is an extension and generalization of the $U(6)\times U(6)$ equal-time current algebra. The light-cone algebra has been used to rederive fixed-mass sum rules for current-particle cross sections, which had originally been derived from the $U(6)\times U(6)$ equal-time algebra through the $|P|\to\infty$ technique.[5] Since the light-cone algebra is a much stronger set of algebraic assumptions, it does not require the singular $|P|\to\infty$ limit order to give fixed-mass sum rules. In some cases the rederivation yields explicit modifications of the old sum rules.

In the present work, we employ the light-cone algebra to derive fixed-mass sum rules for the forward current-particle causal amplitude.

The basic technique is to define a causal amplitude

$$T(p,q)=\int d^4x\, e^{iq\cdot x}\theta(x^0)\langle p|[J(x),J(0)]|p\rangle_C$$

and to consider the integral

$$\int_{-\infty}^{\infty}\frac{d(q^0+q^3)}{q^0+q^3+i\epsilon}T(p,q)$$

in the frame where p has only a time component. One argues that the entire contribution to this integral comes, in coordinate space, from the region $x^3\geq x^0$, and since the retarded commutator vanishes outside the forward light cone, only one ray of the forward light cone can contribute to the integral. The light-cone algebra is used to evaluate this contribution. The sum rules so obtained for the imaginary part of the causal amplitude are well known, but those for the real part are new. These sum rules were not derived from the equal-time current algebra because the application of the $|P|\to\infty$ technique would have led to mathematical ambiguities.

These sum rules for the real parts can in principle be verified by measuring the asymptotic energy dependences at fixed momentum transfer of total inelastic electron-proton and neutrino-proton scattering cross sections. Via dispersion relations the asymptotic behavior of the imaginary part of the causal amplitude, which is proportional to the total cross section, is related to the integral over the real part. The content of one sum rule is that the asymptotic forward virtual Compton scattering structure function $W_2(\nu, q^2)$ comes entirely from the virtual hadronic component of the off-shell photon, via the diffraction of the virtual hadrons.

The paper is organized as follows. Section II contains the kinematics, and the derivation of the sum rules. Our conclusions are summarized in Sec. III and some technical points are relegated to two appendixes.

II. DERIVATION OF SUM RULES

A. Kinematics and Definitions

In this section sum rules will be derived for the forward scattering of a current by a nucleon. SU(3) symmetry is assumed, and the currents are conserved. For the case of the electromagnetic current, the sum rules are for the forward virtual Compton scattering amplitude, whose absorptive part is measured in inelastic electron scattering experiments. We assume that the connected part of the current-particle scattering amplitude amy be written as the Fourier transform of the retarded commutator of the two currents:

$$T_{ab}^{\mu\nu}(p, q, s)=\frac{i}{\pi}\int d^4x\, e^{iq\cdot x}\theta(x^0)\langle ps|[J_a^\mu(x), J_b^\nu(0)]|ps\rangle_C . \tag{1a}$$

Noncovariant terms are assumed to be c numbers. The decompositon into invariant amplitudes is

$$\begin{aligned}T_{ab}^{\mu\nu}(p, q, s)=&\,T_1^{ab}(\nu, q^2)\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+T_2^{ab}(\nu,q^2)\frac{1}{M^2}\left[p^\mu p^\nu-\frac{p\cdot q}{q^2}(p^\mu q^\nu+q^\mu p^\nu)+\left(\frac{p\cdot q}{q^2}\right)^2 q^\mu q^\nu\right]\\&+S_1^{ab}(\nu, q^2)(iM\epsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma)+S_2^{ab}(\nu, q^2)\frac{i}{M}\epsilon^{\mu\nu\rho\sigma}q_\rho[s_\sigma(p\cdot q)-p_\sigma(s\cdot q)]\\&+\frac{i}{\pi}f_{abc}F_c\left[\frac{p\cdot q}{q^2}g^{\mu\nu}-\frac{1}{q^2}(p^\mu q^\nu+q^\mu p^\nu)\right].\end{aligned} \tag{1b}$$

The absorptive part of $T_{ab}^{\mu\nu}$ is

$$\begin{aligned}W_{ab}^{\mu\nu}(p, q, s)=&\,\frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\langle ps|[J_a^\mu(x), J_b^\nu(0)]|ps\rangle_C\\\equiv&\,W_1^{ab}(\nu, q^2)\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+W_2^{ab}(\nu, q^2)\frac{1}{M^2}\left[p^\mu q^\nu-\frac{p\cdot q}{q^2}(p^\mu q^\nu+q^\mu p^\nu)+\left(\frac{p\cdot q}{q^2}\right)^2 q^\mu q^\nu\right]\\&+G_1^{ab}(\nu, q^2)(iM\epsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma)+G_2^{ab}(\nu, q^2)\frac{i}{M}\epsilon^{\mu\nu\rho\sigma}q_\rho[s_\sigma(p\cdot q)-p_\sigma(s\cdot q)],\end{aligned} \tag{2}$$

where p, M, and s are the nucleon 4-momentum, mass, and covariant polarization, respectively, and q is the 4-momentum carried by the current. The invariant amplitudes are functions of the invariants q^2 and $p\cdot q=M\nu$; indices a, b, and c are SU(3) labels; and F_c is a generalized form factor at zero momentum transfer:

$$\langle ps|J_c^\nu(0)|ps\rangle \equiv p^\nu F_c .$$

This last term in Eq. (1b) results from the equal-time current algebra, which requires

$$\begin{aligned}q_\mu T_{ab}^{\mu\nu}(p, q, s)&=-\frac{1}{\pi}\int d^4x\, e^{iq\cdot x}\langle ps|[J_a^0(x), J_b^\nu(0)]|ps\rangle_C\delta(x^0)\\&=-\frac{i}{\pi}f_{abc}\langle ps|J_c^\nu(0)|ps\rangle .\end{aligned}$$

It gives rise to the $J=1$ fixed pole of the Adler–Dashen–Gell-Mann–Fubini sum rule.[6] Note that the kinematic form of this term is not unique since any conserved tensor may be added without violating the current-algebra requirements. However, the form chosen in Eq. (1b) gives the best asymptotic behavior of the structure functions in free-field theory.

It is convenient to introduce light-cone components of 4-vectors $q^\pm=q^0\pm q^3$, $q_\perp^i=(q^1, q^2)$, and to denote symmetrization or antisymmetrization of indices by { } or [ ], respectively. Specifically we define

$$\begin{aligned}T_{ab}^{\mu\nu}&=T_{ab}^{\{\mu\nu\}}+iT_{ab}^{[\mu\nu]}\\&=T_{\{ab\}}^{\{\mu\nu\}}+iT_{[ab]}^{\{\mu\nu\}}+iT_{\{ab\}}^{[\mu\nu]}-T_{[ab]}^{[\mu\nu]} .\end{aligned}$$

B. Formal Derivation of Sum Rules

To obtain fixed-mass sum rules, we choose the nucleon rest frame, set $q^-=0$, and evaluate

$$\Sigma_{ab}^{\mu\nu}=\int_{-\infty}^{\infty}\frac{d\nu}{\nu+i\epsilon}T_{ab}^{\mu\nu}(p, q, s) \tag{3}$$

using the quark light-cone algebra. Interchanging integrals and using

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\nu}{\nu+i\epsilon}e^{i\nu x^-}=\theta(-x^-)$$

gives

$$\Sigma_{ab}^{\mu\nu}=2\int d^4x\,\theta(-x^-)\theta(x^0)e^{-i\vec{q}_\perp\cdot\vec{x}_\perp}\langle ps|[J_a^\mu(x), J_b^\nu(0)]|ps\rangle_C . \tag{4}$$

For $x^2<0$ the commutator vanishes by causality, and consequently the step functions restrict the integration to the line $x^-=x_\perp^i=0$, which lies on the light-cone $x^2=0$. Thus $\Sigma_{ab}^{\mu\nu}$ may be evaluated using the form of the commutator on the light cone.

Since (as is shown in Appendix A) any light-cone singularity that is weaker than $\delta'(x^2)$ (the leading singularity in the free quark model) does not contribute to $\Sigma_{ab}^{\mu\nu}$ we will evaluate Eq. (4) by using the free quark model light-cone algebra[4]:

$$[J_a^\mu(x),J_b^\nu(0)]=\frac{1}{4\pi}\{\partial_\rho[\epsilon(x^0)\delta(x^2)]\}\{if_{abc}[s^{\mu\nu\rho\sigma}(\mathcal{F}_{c\sigma}(x,0)+\mathcal{F}_{c\sigma}(0,x))-i\epsilon^{\mu\nu\rho\sigma}(\mathcal{F}^5_{c\sigma}(x,0)-\mathcal{F}^5_{c\sigma}(0,x))]$$
$$+d_{abc}[s^{\mu\nu\rho\sigma}(\mathcal{F}_{c\sigma}(x,0)-\mathcal{F}_{c\sigma}(0,x))-i\epsilon^{\mu\nu\rho\sigma}(\mathcal{F}^5_{c\sigma}(x,0)+\mathcal{F}^5_{c\sigma}(0,x))]\}+\cdots, \quad (5)$$

where $\mathcal{F}_{c\sigma}(x,0)$ and $\mathcal{F}^5_{c\sigma}(x,0)$ are bilocal densities, $\cdots$ denotes terms less singular than $\delta'(x^2)$ at $x^2=0$, and

$$s^{\mu\nu\rho\sigma}=\tfrac{1}{4}\mathrm{Tr}[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma].$$

Formally we obtain

$$\Sigma_{ab}^{\mu\nu}=\frac{1}{8}\int_0^\infty dx^+\{s^{\mu\nu-\sigma}[if_{abc}\langle ps|(\mathcal{F}_{\sigma c}(x,0)+\mathcal{F}_{\sigma c}(0,x))|ps\rangle+d_{abc}\langle ps|(\mathcal{F}_{\sigma c}(x,0)-\mathcal{F}_{\sigma c}(0,x))|ps\rangle]$$
$$-i\epsilon^{\mu\nu-\sigma}[if_{abc}\langle ps|(\mathcal{F}^5_{\sigma c}(x,0)-\mathcal{F}^5_{\sigma c}(0,x))|ps\rangle+d_{abc}\langle ps|(\mathcal{F}^5_{\sigma c}(x,0)+\mathcal{F}^5_{\sigma c}(0,x))|ps\rangle]\}. \quad (6)$$

This is obtained by partial integration, using the fact that for any nonsingular function $f(x)$,

$$\int d^4x\,\delta(x^0)\delta(x^2)f(x)=0.$$

However, the integral over x^+ at $q^-=0$ in Eq. (6) is singular, and must be defined as the limit as $q^-\to 0$ of a convergent Fourier integral. Thus the quantity on the right-hand side of Eq. (6) is properly

$$L_{ab}^{\mu\nu}\equiv\lim_{q^-\to 0}\int_{-\infty}^{\infty}\frac{d\nu}{\nu+i\epsilon}T_{ab}^{\mu\nu}(p,q)\Big|_{q^-\ \mathrm{fixed}}, \quad (7)$$

and this may differ from $\Sigma_{ab}^{\mu\nu}$. We note that when $q^-\neq 0$, q^2 will vary with ν. Since we seek fixed q^2 sum rules, i.e., $\Sigma_{ab}^{\mu\nu}$, we must isolate those states which contribute differently to $\Sigma_{ab}^{\mu\nu}$ and $L_{ab}^{\mu\nu}$. These are the Class-II states of the $|P|\to\infty$ current-algebra sum rules[7,8] – the hadron-dominance states of Fig. 1. If the semihadronic amplitudes of Fig. 1 tend to a constant as $\nu\to\infty$ at fixed q^2, they may be shown to give a finite contribution to $\Sigma_{ab}^{\mu\nu}$, but no contribution to $L_{ab}^{\mu\nu}$. This is discussed in Appendix B. The general form of the sum rules is then

$$\Sigma_{ab}^{\mu\nu}=L_{ab}^{\mu\nu}+R_{ab}^{\mu\nu}, \quad (8)$$

where $L_{ab}^{\mu\nu}$ is evaluated using the light-cone algebra, and $R_{ab}^{\mu\nu}$ is the possible contribution of all Class-II states. Sum rules are obtained by taking special components of $\Sigma_{ab}^{\mu\nu}$ and expressing these in terms of the invariant amplitudes of Eq. (1).

We use Regge theory to indicate the convergence properties of the sum rules and to obtain the contributions of Class-II states. (The Pomeranchukon is assumed to have unit intercept, and the pole decouples from the spin-dependent amplitudes.)

The most convergent components $(\mu\nu)=\{--\}$ and $[-i]$ give dispersion relations for the invariant amplitudes. The light-cone algebra gives $L_{ab}^{-\nu}=0$, and the following sum rules are expected to be free of Class-II contributions:

$$\{-i\}:\ \int_{-\infty}^{\infty}d\nu\,W_2^{[ab]}(\nu,q^2)=Mf_{abc}F_c, \quad (9)$$

$$[-+]:\ \int_{-\infty}^{\infty}d\nu\,G_2^{\{ab\}}(\nu,q^2)=0, \quad (10)$$

$$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}S_2^{[ab]}(\nu,q^2)=0, \quad (11)$$

The following sum rules are expected to have divergent Class-II contributions:

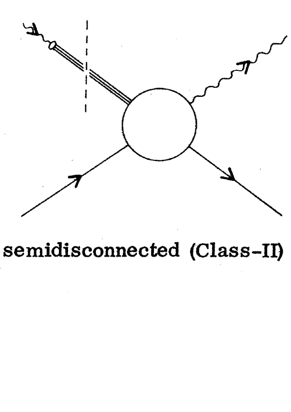

FIG. 1. A semidisconnected (Class-II) intermediate state.

$$[ij]:\quad \int_{-\infty}^{\infty} d\nu\, G_1^{[ab]}(\nu,q^2)=\frac{i}{2M^2}f_{abc}\left[\int_0^{\infty} du[\tilde{h}_1^c(u)+u\tilde{h}_2^c(u)]\right]+\left\{\int_{-\infty}^{\infty} d\nu\, G_1^{[ab]}(\nu,q^2)\bigg|_{\rm II}\right\}, \tag{12}$$

$$\begin{aligned}\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}S_1^{\{ab\}}(\nu,q^2)&=\frac{-1}{2M^2}d_{abc}\left[\int_0^{\infty} du[\tilde{f}_1^c(u)+u\tilde{f}_2^c(u)]\right]+\left\{\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}S_1^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right\}\\ &=\frac{\pi}{M^2}g_1^{\{ab\}}(0)+\left\{\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}S_1^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right\};\end{aligned} \tag{13}$$

$$\begin{aligned}[+i]:\quad &\int_{-\infty}^{\infty} d\nu\left(G_1^{[ab]}(\nu,q^2)+\frac{\nu}{M}G_2^{[ab]}(\nu,q^2)\right)\\ &=\frac{i}{2M^2}f_{abc}\left[\int_0^{\infty} du\, \tilde{h}_1^c(u)\right]+\left\{\int_{-\infty}^{\infty} d\nu\left(G_1^{[ab]}(\nu,q^2)\bigg|_{\rm II}+\frac{\nu}{M}G_2^{[ab]}(\nu,q^2)\bigg|_{\rm II}\right)\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}&\int_{-\infty}^{\infty} d\nu\left(\mathrm{Re}S_1^{\{ab\}}(\nu,q^2)+\frac{\nu}{M}\mathrm{Re}S_2^{\{ab\}}(\nu,q^2)\right)\\ &=\frac{-1}{2M^2}d_{abc}\left[\int_0^{\infty} du\, \tilde{f}_1^c(u)\right]+\left\{\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}\left(S_1^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}+\frac{\nu}{M}S_2^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right)\right\}\\ &=\frac{\pi}{M^2}[g_1^{\{ab\}}(0)+g_2^{\{ab\}}(0)]+\left\{\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}\left(S_1^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}+\frac{\nu}{M}S_2^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right)\right\},\end{aligned} \tag{15}$$

where we have defined

$$\langle ps|(\mathfrak{F}^5_{\sigma c}(x,0)+\mathfrak{F}^5_{\sigma c}(0,x))|ps\rangle\equiv 2s_\sigma\tilde{f}_1^c(x\cdot p)+2p_\sigma(s\cdot x)\tilde{f}_2^c(x\cdot p)+\cdots,$$

$$\langle ps|(\mathfrak{F}^5_{\sigma c}(x,0)-\mathfrak{F}^5_{\sigma c}(0,x))|ps\rangle\equiv 2s_\sigma\tilde{h}_1^c(x\cdot p)+2p_\sigma(s\cdot x)\tilde{h}_2^c(x\cdot p)+\cdots,$$

and

$$g_1^{\{ab\}}(\xi)=\lim_{\nu,-q^2\to\infty} M^2\nu G_1^{\{ab\}}(\nu,q^2),$$

$$g_2^{\{ab\}}(\xi)=\lim_{\nu,-q^2\to\infty} M\nu^2 G_2^{\{ab\}}(\nu,q^2),$$

with

$$\xi=-q^2/2M\nu \text{ fixed}.$$

The notation for the Class-II contributions is defined in Appendix B.

Equation (9) is the Adler–Dashen–Gell-Mann–Fubini sum rule,[6] and Eqs. (10), (12), and (14) are sum rules of Dicus *et al.*[5,9] Equations (13) and (15) are new sum rules for the real parts of scattering amplitudes, and indicate the presence of $J=0$ fixed poles in the imaginary parts of $S_1^{\{ab\}}$ and $S_2^{\{ab\}}$. The presence of these fixed poles could in principle be detected by measuring the ν dependence at fixed q^2 of the structure functions G_1 and G_2 in the inelastic scattering of polarized electrons by polarized nucleons.[10] However, Regge cuts are likely to lie above these poles. The sum rules (10) and (11) imply that S_2^{ab} has no $J=1$ fixed poles of either signature.

The remaining sum rule is for $\Sigma_{\{ab\}}^{\{-i\}}$. This sum rule is expected to have a finite contribution from Class-II states arising from Pomeranchukon exchange in the semihadronic amplitudes:

$$\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}T_2^{\{ab\}}(\nu,q^2)=\left\{\int_{-\infty}^{\infty} d\nu\, \mathrm{Re}T_2^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right\}. \tag{16}$$

This sum rule implies that in virtual Compton scattering the $1/\nu$ term in T_2 arises entirely from the hadronic component of the photon, i.e., from the Class-II states of Fig. 1.

Additional sum rules with worse convergence properties can be derived with these techniques.

C. Axial-Vector-Current Sum Rules

Sum rules may be derived in the same manner for the retarded commutators of a vector and an axial-vector current. For the case of spin-averaged matrix elements, the relevant causal amplitude is

$$\begin{aligned}T^{(5)\mu\nu}_{ab}(p,q)&=\frac{i}{\pi}\int d^4x\, e^{iq\cdot x}\theta(x^0)\langle p|[J_a^{5\mu}(x),J_b^\nu(0)]|p\rangle\\ &\equiv i\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma T_3^{ab}(\nu,q^2),\end{aligned} \tag{17}$$

with absorptive part

$$\begin{aligned}W^{(5)\mu\nu}_{ab}(p,q)&=\frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\langle p|[J_a^{5\mu}(x),J_b^\nu(0)]|p\rangle\\ &\equiv i\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma W_3^{ab}(\nu,q^2),\end{aligned} \tag{18}$$

where $J_a^{5\mu}(x)$ is an axial-vector current. These kinematic forms for the spin-averaged matrix elements are correct even for nonconserved currents.

The relevant quark light-cone commutator is

$$[J_a^{5\mu}(x),J_b^{\nu}(0)]=\frac{1}{4\pi}\{\partial_\rho[\epsilon(x^0)\delta(x^2)]\}\{if_{abc}[s^{\mu\nu\rho\sigma}(\mathfrak{F}^5_{\sigma c}(x,0)+\mathfrak{F}^5_{\sigma c}(0,x))-i\epsilon^{\mu\nu\rho\sigma}(\mathfrak{F}_{\sigma c}(x,0)-\mathfrak{F}_{\sigma c}(0,x))]$$
$$+d_{abc}[s^{\mu\nu\rho\sigma}(\mathfrak{F}^5_{\sigma c}(x,0)-\mathfrak{F}^5_{\sigma c}(0,x))-i\epsilon^{\mu\nu\rho\sigma}(\mathfrak{F}_{\sigma c}(x,0)+\mathfrak{F}_{\sigma c}(0,x))]\}+\cdots\,. \quad (19)$$

From the components $(\mu\nu)=[-i]$ and $[+i]$ one obtains dispersion relations, whereas $(\mu\nu)=[ij]$ gives the sum rules

$$\int_{-\infty}^{\infty}d\nu\,W_3^{[ab]}(\nu,q^2)=\frac{-i}{2M}f_{abc}\left[\int_0^\infty du\,\tilde{A}_c(u)\right]+\left\{\int_{-\infty}^{\infty}d\nu\,W_3^{[ab]}(\nu,q^2)\bigg|_{\rm II}\right\}, \quad (20)$$

$$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_3^{\{ab\}}(\nu,q^2)=\frac{1}{2M}d_{abc}\left[\int_0^\infty du\,\tilde{S}_c(u)\right]+\left\{\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_3^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right\}$$
$$=\pi F_3^{\{ab\}}(0)+\left\{\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_3^{\{ab\}}(\nu,q^2)\bigg|_{\rm II}\right\}, \quad (21)$$

where

$$\langle p|(\mathfrak{F}_{\sigma c}(x,0)+\mathfrak{F}_{\sigma c}(0,x))|p\rangle\equiv 2p_\sigma\tilde{S}_c(x\cdot p)+\cdots\,,$$
$$\langle p|(\mathfrak{F}_{\sigma c}(x,0)-\mathfrak{F}_{\sigma c}(0,x))|p\rangle\equiv 2p_\sigma\tilde{A}_c(x\cdot p)+\cdots\,,$$

and

$$F_3^{[ab]}(\xi)=\lim_{\nu,-q^2\to\infty}\nu W_3^{[ab]}(\nu,q^2),$$

with $\xi=-q^2/2M\nu$ fixed.

These sum rules, although expected to diverge, nevertheless may indicate the presence of $J=0$ fixed poles in the structure functions accessible in neutrino scattering.[10]

III. CONCLUSIONS

Our technique of considering the integral of a causal amplitude[11]

$$\int_{-\infty}^{\infty}\frac{d\nu}{\nu+i\epsilon}T(p,q)$$

enables us to use the quark light-cone algebra to obtain sum rules for the real and imaginary parts of the amplitude. The most convergent sum rules are dispersion relations evaluated at $\nu=0$. More convergent expressions like

$$\int_{-\infty}^{\infty}\frac{d\nu}{(\nu+i\epsilon)^2}T(p,q)$$

merely lead to dispersive evaluations of derivatives of $T(p,q)$ with respect to ν, at $\nu=0$.

The new sum rules presented here are for the real parts of the invariant amplitudes T_2, S_1, S_2, and T_3; they are summarized in Table I.

The sum rules C, D, and E suggest the presence of $J=0$ poles in the structure functions G_1,

TABLE I. Sum rules for real parts of forward current scattering amplitudes. The expected Regge convergence and the possible presence of fixed poles are also shown. All sum rules are satisfied in free field theory.

	Sum rule	Regge convergence	J-plane singularity (signature τ)
A	$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_2^{\{ab\}}(\nu,q^2)=\left\{\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_2^{\{ab\}}(\nu,q^2)\Big\|_{\rm II}\right\}$	Yes	Pomeranchukon
B	$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,S_2^{[ab]}(\nu,q^2)=0$	Yes	No ($J=1$; $\tau=-1$)
C	$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,S_1^{\{ab\}}(\nu,q^2)=\frac{\pi}{M^2}g_1^{\{ab\}}(0)+\left\{\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,S_1^{\{ab\}}(\nu,q^2)\Big\|_{\rm II}\right\}$	No	Yes ($J=0$; $\tau=+1$)
D	$\int_{-\infty}^{\infty}d\nu\,\frac{\nu}{M}\,\mathrm{Re}\,S_2^{\{ab\}}(\nu,q^2)=\frac{\pi}{M^2}g_2^{\{ab\}}(0)+\left\{\int_{-\infty}^{\infty}d\nu\,\frac{\nu}{M}\,\mathrm{Re}\,S_2^{\{ab\}}(\nu,q^2)\Big\|_{\rm II}\right\}$	No	Yes ($J=0$; $\tau=+1$)
E	$\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_3^{\{ab\}}(\nu,q^2)=\pi F_3^{\{ab\}}(0)+\left\{\int_{-\infty}^{\infty}d\nu\,\mathrm{Re}\,T_3^{\{ab\}}(\nu,q^2)\Big\|_{\rm II}\right\}$	No	Yes ($J=0$; $\tau=+1$)

G_2, and W_3 of inelastic lepton-nucleon scattering. Their experimental detection will be difficult because of higher Regge singularities.

From a theoretical standpoint the sum rule A for the real part of T_2 seems surprising, since it states that the diffractive contribution to $W_2(\nu, q^2)$, the $1/\nu$ term as $\nu\to\infty$ at fixed q^2, arises only from the virtual hadronic component of the off-shell photon. We remark that in the free quark model Class-II singularities are absent and the sum rules are valid.

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APPENDIX A

We outline a proof that any light-cone singularity weaker than $\delta'(x^2)$ gives no contribution to the sum rules.

The integral of interest is of the form

$$I_c=\int d^4x\,\theta(x^0)\theta(-x^-)E(x;c)f(x^+,x^-,\vec{x}_\perp), \quad \text{(A1)}$$

where $f(x)$ is a regular function. The general light-cone singular function is

$$E(x;c)=\frac{-i}{2^{4-c}\pi^2}\frac{\Gamma(\frac{1}{2}c)}{\Gamma(2-\frac{1}{2}c)}\times\left[\frac{1}{(-x^2-i\epsilon x^0)^{c/2}}-\frac{1}{(-x^2+i\epsilon x^0)^{c/2}}\right].$$

Changing variables as follows:

$$y^+=x^+,$$
$$y^-=Kx^-, \quad K>0$$
$$\vec{y}_\perp{}^2=K\vec{x}_\perp{}^2,$$

one has

$$I_c=K^{(c-4)/2}\int d^4y\,\theta(Ky^++y^-)\theta(-y^-)\times E(y;c)f\left(y^+,\frac{y^-}{K},\frac{\vec{y}_\perp}{K^{1/2}}\right). \quad \text{(A2)}$$

Using the causality property

$$E(y;c)=0 \quad \text{for } y^2<0$$

and taking the limit $K\to\infty$, one obtains

$$I_c=\lim_{K\to\infty}K^{(c-4)/2}\int d^4y\,\theta(y^+)\theta(-y^-)E(y;c)f(y^+,0,0)$$
$$=0 \quad \text{for } c<4. \quad \text{(A3)}$$

APPENDIX B

Consider the integral

$$D_{ab}^{\mu\nu}(q^-)=\int_{-\infty}^{\infty}\frac{dq^+}{q^++i\epsilon}T_{ab}^{\mu\nu}(p,q) \quad \text{(B1)}$$

at fixed q^-. Since $q^2=q^+q^--\vec{q}_\perp{}^2$, those intermediate states that are associated with singularities in q^2 contribute differently at $q^-=0$ and $q^-\neq 0$: In the latter case, q^2 is varying in the integral over q^+, but not in the former. These intermediate states are the semidisconnected (Class-II) states.[7,8]

As an example, consider a single pole at $q^2=\mu^2$. It gives a contribution to $T_{ab}^{\mu\nu}$ of the form

$$T_{ab}^{\mu\nu}(p,q)|_{\text{II}}=-\frac{1}{\pi}\frac{1}{q^2-\mu^2+i\epsilon q^0}A_{ab}^{\mu\nu}(p,q), \quad \text{(B2)}$$

where

$$A_{ab}^{\mu\nu}(p,q)=\langle 0|J_a^\mu(0)|\mu(q)\rangle\langle\mu(q);p|J_b^\nu(0)|p\rangle+\langle p|J_a^\mu(0)|\mu(q);p\rangle\langle\mu(q)|J_b^\nu(0)|0\rangle.$$

$A_{ab}^{\mu\nu}(p,q)$ is the semihadronic amplitude shown in Fig. 1. Then

$$D_{ab}^{\mu\nu}(q^-)|_{\text{II}}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dq^+}{q^++i\epsilon}\frac{A_{ab}^{\mu\nu}(p,q)}{q^+q^--\vec{q}_\perp{}^2-\mu^2+i\epsilon q^0}. \quad \text{(B3)}$$

Since $A_{ab}^{\mu\nu}(p,q)$ is analytic in the upper q^+ plane, the contribution of the Class-II states depends on the behavior of $A_{ab}^{\mu\nu}(p,q)$ as $q^+\to\infty$. For $q^-\neq 0$, there is an extra convergence factor of $1/q^+$; this is absent at $q^-=0$.

For the interesting case that

$$A_{ab}^{\mu\nu}(p,q)\to i\beta_{ab}^{\mu\nu}(\mu)=\text{const.} \quad \text{as } q^+\to\infty$$

we find

$$D_{ab}^{\mu\nu}(q^-)|_{\text{II}}=0 \quad \text{for } q^-\neq 0$$
$$=\frac{\beta_{ab}^{\mu\nu}(\mu)}{\vec{q}^2+\mu^2}=\frac{\beta_{ab}^{\mu\nu}(\mu)}{\mu^2-q^2} \quad \text{for } q^-=0. \quad \text{(B4)}$$

Therefore we see that

$$L_{ab}^{\mu\nu}\equiv\lim_{q^-\to 0}D_{ab}^{\mu\nu}(q^-)$$

differs from $\Sigma_{ab}^{\mu\nu}\equiv D_{ab}^{\mu\nu}(q^-=0)$ by the contribution $R_{ab}^{\mu\nu}$ of the Class-II states:

$$\Sigma_{ab}^{\mu\nu}=L_{ab}^{\mu\nu}+R_{ab}^{\mu\nu}, \quad \text{(B5)}$$

where $R_{ab}^{\mu\nu}$ has an analogous kinematic decomposition to $W_{ab}^{\mu\nu}$. For example, the Class-II contribution to sum-rule C is explicitly

$$\left\{\int_{-\infty}^{\infty} d\nu \, \mathrm{Re}S_1^{\{ab\}}(\nu, q^2)\Big|_{\mathrm{II}}\right\} \equiv \left[\int_{-\infty}^{\infty} d\nu \, \mathrm{Re}S_1^{\{ab\}}(\nu, q^2; q^- = 0)\Big|_{\mathrm{II}}\right] - \lim_{q^- \to 0}\left[\int_{-\infty}^{\infty} d\nu \, \mathrm{Re}S_1^{\{ab\}}(\nu, q^2; q^-)\Big|_{\mathrm{II}}\right].$$

If the Class-II contributions diverge, the sum rules to which they contribute are presumably invalid.

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[1]H. Kendall, in *Proceedings of the 1971 International Symposium on Electron and Photon Interactions at High Energies,* edited by N. B. Mistry (Cornell Univ. Press, Ithaca, N. Y., 1971).

[2]B. L. Ioffe, Phys. Letters 30B, 123 (1969); H. Leutwyler and J. Stern, Nucl. Phys. B20, 77 (1970); L. S. Brown, in *Lectures in Theoretical Physics,* edited by K. T. Mahanthappa and W. E. Brittin (Gordon and Breach, New York, 1971).

[3]R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); Y. Frishman, Phys. Rev. Letters 25, 966 (1970); Ann. Phys. (N.Y.) 66, 373 (1971); K. Wilson, Phys. Rev. 179, 1499 (1969).

[4]H. Fritzsch and M. Gell-Mann, in *Broken Scale Invariance and the Light Cone,* 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2, p. 1; J. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).

[5]D. Dicus, R. Jackiw, and V. L. Teplitz, Phys. Rev. D 4, 1733 (1971).

[6]S. L. Adler, Phys. Rev. 143, 1144 (1966); R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966,* edited by A. Perlmutter, J. Wojtaszek, E. C. G. Sudarshan, and B. Kurşunoğlu (Freeman, San Francisco, Calif., 1966); S. Fubini, Nuovo Cimento 34A, 475 (1966).

[7]S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968), Appendix C, p. 339.

[8]G. Calucci, R. Jengo, G. Furlan, and C. Rebbi, Phys. Letters 37B, 416 (1971). See also Appendix D of Ref. 5.

[9]Equation (10) was first derived by H. Burkhardt and W. N. Cottingham [Ann. Phys. (N.Y.) 56, 453 (1970)].

[10]The sum rule for $W_3^{[ab]}(\nu, q^2)$, Eq. (20), has been derived by D. Corrigan [Phys. Rev. D 5, 490 (1972)].

[11]The R-products we have used differ in momentum space from T-products by the reversal of signs of their imaginary parts when real $\nu < 0$. Crossing enables all integrals to be written over $\nu > 0$, so the sum rules for real parts, and their implications for the asymptotic behaviors of imaginary parts as $\nu \to +\infty$, hold for the matrix elements of both T- and R-products of currents in momentum space.

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Analytic Continuation of Reduced Pion-Nucleon Partial-Wave Amplitudes*

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It is shown that one can write the partial-wave amplitudes $h_l^+ = (2k^2/4\mu^2)^{-l} f_l^+$ and $h_{l+1}^- = (2k^2/4\mu^2)^{-l} f_{l+1}^-$ for meson-nucleon scattering as $h_l^\pm = h_{l,\mathrm{I}}^\pm + h_{l,\mathrm{II}}^\pm$, where $h_{l,\mathrm{I}}^\pm$ is analytic in the l plane except for fixed poles at negative integers, and $h_{l,\mathrm{II}}^\pm$ is analytic in the energy W complex plane except for cuts which do not disconnect the plane. Moreover, $h_{l,\mathrm{II}}^+(W) = -h_{l+1,\mathrm{II}}^-(-W)$. Consequences of this result are discussed, in particular its relevance for the theory of Regge poles and the problem of the uniqueness of Mandelstam amplitudes for meson-nucleon scattering.

The problem of determining uniqueness conditions for amplitudes satisfying Mandelstam's representation was thoroughly investigated by Martin.[1] One of his results is that the scattering amplitude is uniquely determined by its absorptive part in the elastic region of one channel. The original derivation was given for the scattering of two identical spinless particles. In a recent paper Cheung and Chen-Cheung[2] generalized this result to cover cases of scattering of particles with spin and isospin, such as pion-nucleon scattering. The argument for this generalization depends on the rela-