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AN ENERGY ESTIMATE FOR THE BIHARMONIC EQUATION AND ITS APPLICATION TO SAINT-VENANT'S PRINCIPLE IN PLANE ELASTOSTATICS

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A new energy estimate is given for a boundary value problem for the biharmonic equation. The result is applied to the estimation of stresses in a plane elasticity problem.

1. INTRODUCTION

Among the many relatively recent studies of the issue known in the theory of elasticity as Saint-Venant's principle, several have been devoted to the construction of certain kinds of energy inequalities for the biharmonic differential equation in plane domains*. Such inequalities are designed to estimate the spatial rate of decay of stored energy away from a portion of the boundary which carries a "self-equilibrated" load, the remainder of the boundary being traction-free. These energy estimates for two-dimensional, linear problems in plane strain—as well as generalizations of such estimates appropriate to anisotropic materials or to axisymmetric three-dimensional problems—have all been based on one or the other of two basic arguments involving differential inequalities. The first of these was given in Knowles (1966) and produced an explicit estimate of the spatial decay rate valid for bounded, simply connected domains of general shape; subsequently, Flavin (1974) showed that a slight modification of the analysis given in Knowles (1966) leads to an improved estimate of the decay rate. The second kind of argument is that given more recently by Oleinik and Yosifian (1978a, b)**; their results apply to bounded domains of general shape and account for the influence on the estimated decay rate of certain geometrical features of the underlying domain not reflected in the results of Knowles (1966) or Flavin (1974).

*For a discussion of work done since 1965 concerning principles of Saint-Venant type, see Horgan and Knowles (1982). A number of the important earlier results, together with appropriate references, may be found in (Gurtin 1972).

**See especially Theorem 1 of Oleinik and Yosifian (1978a & 1978b); see also Theorem 6 of Oleinik (1979 a) Theorem 1 of Oleinik (1979b).

The simplest problem of plane strain—perhaps, in fact, the canonical one—with which the question underlying Saint-Venant's principle can be illustrated is that for a semi-infinite strip whose long sides are traction-free and whose end carries a self-equlibrated load. The stress field is assumed to vanish at infinity. Explicit representations for the solution to the associated biharmonic boundary value problem make it possible in this case to exhibit the 'exact' rate of spatial decay of stored energy away from the end of the strip. Thus the 'estimated' decay rates predicted either by the argument in Knowles (1966), incorporating the modifications of Flavin (1974), or by the alternative procedure of Oleinik and Yosifian (1978a, b) can be tested against this exact result. [Although the results of Knowles (1966) and Oleinik and Yosifian (1978 a, b)] are directly applicable only to bounded domains, it is a simple matter to establish their validity for the semi-infinite strip. It turns out that, for the strip problem, both energy arguments yield the some estimated rate of decay, and it is conservative with respect to the exact value by a factor of nearly two.

In contrast, when energy arguments of the kind given in Knowles (1966) and Flavin (1974) or Oleinik and Yosifian (1978 a, b) are applied to analogous problems for Laplace's equation, they always yield the best possible estimate of the appropriate spatial decay rate, even in three or more dimensions. [See the discussion in Horgan and Knowles (1982) of a "model problem" for Laplace's equation]. In a number of of areas in elasticity theory where one wishes to make quantitative applications of Saint-Venant's principle, analyses based on energy decay inequalities have so far proved inadequate because of the excessively conservative estimated decay rates they predict; (see Horgan and Knowles 1982, Section III, c, 2). It is thus desirable to modify the energy procedures, if possible, to remedy this defect. In the present paper we give a third kind of argument to establish energy decay for the biharmonic problem in a semi-infinite strip. The technique given here is based on the consideration of a certain "higher order energy" in addition to the physical energy directly associated with the problem, and our procedure differs substantially in structure from either of the two schemes alluded to above. While it does indeed provide an improved estimate of the rate of decay, the result still falls disappointingly short of the best possible one.

2. The Boundary Value Problem

Let $\mathscr{R}$ be the semi-infinite strip for which $0 < x_1 < \infty, 0 < x_2 < h$, and consider the following boundary value problem on $\mathscr{R}$: find $\varphi = \varphi(x_1, x_2)$ satisfying the differential equation

$$\Delta\Delta\varphi \equiv \varphi_{,\alpha\alpha\beta\beta} = 0 \quad \text{on } \mathscr{R}. \qquad \ldots(2.1)$$

[Greek subscripts have the range 1, 2 and are summed over this range where repeated; a subscript preceded by a comma indicates partial differentiation with respect to the corresponding cartesian coordinate.]

φ also satisfies the boundary conditions

$$\varphi(0, x_2) = f(x_2),\ \varphi_{,1}(0, x_2) = g(x_2),\ 0 \leqslant x_2 \leqslant h \quad \ldots(2.2)$$

$$\varphi(x_1, 0) = \varphi_{,2}(x_1, 0) = \varphi(x_1, h) = \varphi_{,2}(x_1, h) = 0,\ 0 \leqslant x_1 < \infty \quad \ldots(2.3)$$

$$\varphi_{,\alpha\beta}(x_1, x_2) \to 0 \text{ as } x_1 \to \infty, \text{ uniformly for } 0 \leqslant x_2 \leqslant h. \quad \ldots(2.4)$$

Here f and g are given functions.

Regarded as a plane strain problem for a homogeneous, isotropic, elastic material*, (2.1)-(2.4) serve to determine the Airy stress function φ arising from a given set of tractions $t_\alpha(x_2)$ applied to the end of the strip. The latter are related f and g through

$$f(x_2) = \int_0^{x_2} (x_2 - \eta)\, t_1(\eta)\, d\eta,\ g(x_2) = -\int_0^{x_2} t_2(\eta)\, d\eta,\ 0 \leqslant x_2 \leqslant h. \quad \ldots(2.5)$$

The requirement that t_1, t_2 be self-equilibrated means that the associated components of resultant force in the x_1-and x_2-directions, as well as the resultant moment about the origin, must vanish :

$$\int_0^h t_1\, d\eta = \int_0^h t_2\, d\eta = \int_0^h \eta t_1\, d\eta = 0. \quad \ldots(2.6)$$

These requirements in turn imply through (2.5) that

$$f(0) = f(h) = f'(0) = f'(h) = g(0) = g(h) = 0. \quad \ldots(2.7)$$

Once φ has been determined, the components of stress $\sigma_{\alpha\beta}(x_1, x_2)$ in the strip are found from the relations

$$\sigma_{\alpha\beta} = \epsilon_{\alpha\lambda}\, \epsilon_{\beta\mu}\, \varphi_{,\lambda\mu} \quad \ldots(2.8)$$

(see Gurtin 1972), where $\epsilon_{\alpha\lambda}$ is the two-dimensional alternator : $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$.

We shall assume the existence of a solution φ of (2.1)-(2.4) which is four times continuously differentiable** on the closure $\overline{\mathscr{R}}$ of $\mathscr{R}$. This smoothness assumption necessarily requires that $f \in C^4([0, h])$, $g \in C^3([0, h])$, and it imposes conditions on the behaviour of the derivatives of f and g at the end-points $x_2 = 0$ and $x_2 = h$ beyond those listed in (2.7). Since we make no explicit use of the latter conditions, we shall not record them.

The boundary value problem (2.1)-(2.4) has a long history which we cannot discuss here†. When one seeks exponential solutions $\varphi = e^{-\lambda x_1}\psi(x_2)$ of (2.1) that

*For a discussion of plane strain in the linear theory of elasticity, see § D–VII of Gurtin (1972).

**This smoothness assumption is somewhat more restrictive than necessary.

†See Bogy (1975) for an extensive list of references.

also satisfy the homogeneous boundary conditions (2.3), (2.4) with the intent to superpose them in the hope of satisfying (2.2), one is led to a non-self-adjoint problem for $\psi(x_2)$ whose complex eigenvalues λ satisfy the transcendental equation $\sin^2\lambda - \lambda^2 = 0$. A full treatment of the completeness question for the resulting eigenfunctions has been given only recently by Gregory (1980). We shall make no direct use of explicit representations of the solution φ of (2.1)-(2.4), except to note that the eigenvalues λ of smallest positive real part in general govern the exact rate of spatial decay of φ, and that they satisfy

$$\mathscr{R}e\, \lambda \doteq \frac{4.2}{h}. \qquad \ldots(2.9)$$

3. First- and second-order Energies. Past and Present Results

For any $z \geqslant 0$, let $\mathscr{R}_z$ stand for the semi-infinite strip in which $x_1 > z$, $0 < x_2 < h$; $\mathscr{R}_0 \equiv \mathscr{R}$. We shall refer to

$$E_1(z) = \int_{\mathscr{R}_z} \varphi_{,\alpha\beta}\, \varphi_{,\alpha\beta}\, dA, \; z \geqslant 0 \qquad \ldots(3.1)$$

as the first-order energy* contained in $\mathscr{R}_z$. One can show directly that, for a solution φ of the smoothness assumed in (2.1)-(2.4), $E_1(z)$ is finite (see Appendix A).

When adapted to the semi-infinite strip, the results reported in Knowles (1966), Flavin (1974) and Oleinik and Yosifian (1978 a, b) show that $E_1(z)$ satisfies an inequality of the form

$$E_1(z) \leqslant 2E_1(0)e^{-2kz}, \; 0 \leqslant z < \infty \qquad \ldots(3.2)$$

for some positive constant k. In Knowles (1966), k was found to be

$$k = \left(\frac{\sqrt{2} - 1}{2}\right)^{1/2} \frac{\pi}{h} \doteq \frac{1.7}{h}, \text{ [estimated decay rate in Knowles (1966)]}. \qquad \ldots(3.3)$$

When Flavin's modification [Flavin (1974)] of the analysis in Knowles (1966) is invoked, or when the results of Oleinik and Yosifian (1978 a, b) are suitably specialized, it is found that

$$k = \frac{1}{\sqrt{2}} \frac{\pi}{h} \doteq \frac{2.2}{h}, \text{ [estimated decay rate in Flavin (1974), Oleinik and Yosifian (1978 a, b)]}. \qquad \ldots(3.4)$$

Considerations based on a representation of φ in terms of the eigenfunctions discussed briefly at the end of the preceding section show that

* If *Poisson's ratio* for the material vanishes, the physical strain energy $U(z)$ stored in $\mathscr{R}_z$ is proportional to $E_1(z)$. Otherwise, $U(z)$ differs slightly from $E_1(z)$, but there are positive constants c_1 and c_2, dependent only on the material, such that $c_1 E_1(z) \leqslant U(z) < c_2 E_1(z)$.

$$k \doteq \frac{4.2}{h}, \text{ (actual decay rate);} \qquad \ldots(3.5)$$

see (2.9). Thus (3.4) is conservative by a factor of almost two.

Mieth (1975) has shown that (3.2) can be replaced by

$$E_1(z) \leqslant \frac{2E_1(0)}{1 + e^{-4kz}} e^{-2kz}, \; z \geqslant 0 \qquad \ldots(3.6)$$

thus repairing an obvious shortcoming in (3.2) at $z = 0$. In (3.6), k may be taken as in (3.4).

For present purposes, it is essential to introduce a second-order energy* $E_2(z)$ through

$$E_2(z) = \int_{\mathscr{R}_z} \varphi_{,1\alpha\beta}\, \varphi_{,1\alpha\beta}\, dA, \; z \geqslant 0. \qquad \ldots(3.7)$$

Note that $E_2(z)$ may be regarded as the first-order energy associated with the derivative $\varphi_{,1}$ of φ. Again, one can prove that $E_2(z)$ is finite; see Appendix A.

By considerations leading to a differential inequality, we shall prove that there are positive constants m and k such that

$$\left.\begin{aligned} E_1(z) &\leqslant [E_1(0) + \frac{1}{m} E_2(0)]\, e^{-2kz}, \; z \geqslant 0 \\ E_2(z) &\leqslant [E_2(0) + mE_1(0)]\, e^{-2kz}, \; z \geqslant 0 \end{aligned}\right\}. \qquad \ldots(3.8)$$

Indeed, we shall show that

$$k \doteq \frac{2.7}{h}, \; m \doteq \frac{22.4}{h^2}. \qquad \ldots(3.9)$$

The estimated decay rate k in (3.9) clearly improves upon that of (3.4), but falls considerably short of the exact result in (3.5).

We shall also discuss the use of (3.8) in estimating stresses directly.

4. A Differential Inequality

We begin by obtaining alternate representations for the energies $E_1(z)$ and $E_2(z)$ of (3.1), (3.7). For any $z \geqslant 0$, let S_z stand for the line segment $x_1 = z$, $0 \leqslant x_2 \leqslant h$; for any $\zeta > z$, denote by $\mathscr{R}_{z,\zeta}$ the rectangle for which $z < x_1 < \zeta$, $0 < x_2 < h$. Let

$$e_1(z, \zeta) = \int_{\mathscr{R}_{z,\zeta}} \varphi_{,\alpha\beta}\, \varphi_{,\alpha\beta}\, dA, \; e_2(z, \zeta) = \int_{\mathscr{R}_{z,\zeta}} \varphi_{,1\alpha\beta}\, \varphi_{,1\alpha\beta}\, dA \qquad \ldots(4.1)$$

be the first-and second-order energies contained in $\mathscr{R}_{z,\zeta}$.

* Higher-order energies were introduced in connection with principles of Saint-Venant type for second-order elliptic problems in Knowles (1967). They had been used earlier by Shield (1965) in a similar way but in a quite different setting pertaining to dynamic stability.

With the help of the differential equation (2.1), the integrand in the first of (4.1) can be expressed as a divergence, so that $e_1(z, \zeta)$ may be represented as a line integral around the boundary of $\mathscr{R}_{z\zeta}$. The boundary conditions (2.3) on the long sides of the strip may then be invoked to show that the contributions to this line integral from those pieces of the boundary of $\mathscr{R}_{z,\zeta}$ for which $x_2 = 0$ or $x_2 = h$ will in fact vanish. This procedure yields the representation

$$e_1(z, \zeta) = f_1(\zeta) - f_1(z), \ \zeta \geqslant z \geqslant 0 \qquad ...(4.2)$$

where

$$f_1(z) = \int_{S_z} (\varphi_{,\alpha} \varphi_{,\alpha 1} - \varphi\varphi_{,111} - \varphi\varphi_{,122})\, dx_2, \ z \geqslant 0. \qquad ...(4.3)$$

An integration by parts applied to the last term in the integrand in (4.3), together with an appeal to the boundary conditions (2.3), leads to the following formula for f_1:

$$f_1(z) = g_1'(z), \ z \geqslant 0 \qquad ...(4.4)$$

where

$$g_1(z) = \int_{S_z} (\varphi_{,\alpha} \varphi_{,\alpha} - \varphi\varphi_{,11}) dx_2, \ z \geqslant 0 \qquad ...(4.5)$$

and the prime in (4.4) indicates differentiation with respect to z. Thus by (4.4), (4.2)

$$e_1(z, \zeta) = g_1'(\zeta) - g_1'(z), \ \zeta \geqslant z \geqslant 0. \qquad ...(4.6)$$

Since $\varphi_{,1}$ 'also' satisfies the biharmonic equation (2.1) and the boundary condition (2.3), a representation for $e_2(z, \zeta)$ analogous to (4.6) is available. By an argument similar to that used above, but differing slightly in the final details, one finds that

$$e_2(z, \zeta) = g_2'(\zeta) - g_2'(z), \ \zeta \geqslant z \geqslant 0 \qquad ...(4.7)$$

where

$$g_2(z) = \frac{1}{2} \int_{S_z} (\varphi_{,11}^2 + \varphi_{,22}^2)\, dx_2, \ z \geqslant 0. \qquad ...(4.8)$$

It is shown in Appendix A that the conditions (2.4) imposed on the solution φ at infinity are strong enough to imply that

$$g_1'(z) \to 0, \ g_2'(z) \to 0 \text{ as } z \to \infty. \qquad ...(4.9)$$

By (4.6), (4.7), one concludes that $e_\alpha(z, \zeta)$ tends to $-g_\alpha'(z)$ as $\zeta \to \infty$, and hence from (3.1), (3.7), (4.1), (4.5) and (4.8) that the energies $E_\alpha(z)$ are finite and may be represented in the form

$$E_1(z) = \lim_{\zeta \to \infty} e_1(z, \zeta) = -\frac{d}{dz} \int_{S_z} (\varphi_{,\alpha} \varphi_{,\alpha} - \varphi\varphi_{,11})\, dx_2, \ z \geqslant 0 \qquad ...(4.10)$$

$$E_2(z) = \lim_{\zeta \to \infty} e_2(z, \zeta) = -\frac{d}{dz} \int_{S_z} (\varphi_{,11}^2 + \varphi_{,22}^2)\, dx_2,\ z \geqslant 0. \qquad \text{...(4.11)}$$

The final ingredients needed in the derivation of the main differential inequality are the following expressions for the derivatives of E_1, E_2 :

$$E_1'(z) = -\int_{S_z} \varphi_{,\alpha\beta}\, \varphi_{,\alpha\beta}\, dx_2,\ z \geqslant 0 \qquad \text{...(4.12)}$$

$$E_2'(z) = -\int_{S_z} \varphi_{,1\alpha\beta}\, \varphi_{,1\alpha\beta}\, dx_2,\ z \geqslant 0. \qquad \text{...(4.13)}$$

These come immediately from the definitions (3.1) and (3.7).

Let m and k be as yet undetermined but fixed positive constants, and set

$$D(z) = [E_2(z) + mE_1(z)]' + 2k\,[E_2(z) + mE_1(z)]. \qquad \text{...(4.14)}$$

We now represent $D(z)$ as a line integral over S_z by using (4.10), (4.11)-with the z-differentiations carried out under the integrals—and (4.12), (4.13). After some algebra, one finds that, for $z \geqslant 0$,

$$D(z) = -\int_{S_z} \{(\varphi_{,111} + k\varphi_{,11} - mk\varphi)^2 + (m - k^2)\,\varphi_{,12}^2 - 2mk^2\,\varphi_{,2}^2$$
$$- m^2k^2\varphi^2 + (\varphi_{,122} + k\varphi_{,22})^2 + 2m\,(\varphi_{,12} + k\varphi_{,2})^2$$
$$+ 2\varphi_{,112}^2 + (m - k^2)\,\varphi_{,11}^2 + 2mk\,(\varphi_{,1} + k\varphi)\,\varphi_{,11}\}\, dx_2. \qquad \text{...(4.15)}$$

Let C_0^2 be the class of real-valued functions, each of which is twice continuously differentiable on $[0, h]$ and vanishes together with its first derivative at $x_2 = 0$ and $x_2 = h$. For any u, v, w in C_0^2, define

$$\mathscr{D}\{u; v; w\} = \int_0^h \{(m - k^2)\ddot{u}^2 - 2mk^2\dot{u}^2 - m^2k^2u^2$$
$$+ \dot{v}^2 + 2m\,v^2 + 2\dot{w}^2 + (m - k^2)\,w^2 + 2mkvw\}\, dx_2 \qquad \text{...(4.16)}$$

here the superposed dot indicates differentiation with respect to x_2. If z is fixed and we choose

$$u(x_2) = \varphi(z, x_2),\ v(x_2) = \varphi_{,1}(z, x_2) + k\varphi(z, x_2),\ w(x_2) = \varphi_{,11}(z, x_2) \qquad \text{...(4.17)*}$$

then (4.15) may be written

$$D(z) = -\int_{S_z} (\varphi_{,111} + k\varphi_{,11} - mk\varphi)^2\, dx_2 - \mathscr{D}\{u; v; w\} \qquad \text{...(4.18)}$$

* Note that, in view of the smoothness assumed of φ and the boundary conditions (2.3), u, v and w in (4.17) are each in C_0^2.

from which we may immediately conclude that

$$D(z) \leqslant - \mathscr{D}\{u; v; w\}, \; z \geqslant 0. \qquad ...(4.19)$$

In the next Section, we shall give a necessary and sufficient condition on the pair of constants (m, k) in order that

$$\mathscr{D}\{u; v; w\} \geqslant 0 \qquad ...(4.20)$$

for all $u, v, w \in C_0^2$. Temporarily taking this for granted, we conclude from (4.19), (4.14) that, for any (m, k) satisfying this condition, the differential inequality

$$[E_2(z) + mE_1(z)]' + 2k\,[E_2(z) + mE_1(z)] \leqslant 0, \; z \geqslant 0 \qquad ...(4.21)$$

holds. It then follows immediately that

$$E_2(z) + mE_1(z) \leqslant [E_2(0) + mE_1(0)]\, e^{-2kz}, \; z \leqslant 0. \qquad ...(4.22)$$

Since both $E_1(z)$ and $E_2(z)$ are nonnegative, we further conclude that

$$E_1(z) \leqslant [E_1(0) + \frac{1}{m} E_2(0)]e^{-2kz}, \; z \geqslant 0 \qquad ...(4.23)$$

$$E_2(z) \leqslant [E_2(0) + mE_1(0)]\, e^{-2kz}, \; z \geqslant 0. \qquad ...(4.24)$$

Among all pairs (m, k) which satisfy the condition assuring that the functional $\mathscr{D}$ has the positivity property (4.20), we shall choose the (unique) one which maximizes the estimated decay rate k in (4.23), (4.24). This choice of (m, k) is optimal for the present procedure in the sense that, once the first term on the right side of (4.18) is discarded, no large choice of k is available. This special value of the pair (m, k) leads directly to the results stated in (3.8), (3.9).

5. Positivity of $\mathscr{D}$; the estimated decay rate

In studying the functional $\mathscr{D}$ on C_0^2, we shall make use of the following inequality : for any function $u \in C_0^2$ and for any real number $t \geqslant 0$,

$$\int_0^h \ddot{u}^2\, dx_2 \geqslant \lambda(t) \int_0^h (\dot{u}^2 + tu^2)\, dx_2 \qquad ...(5.1)$$

where

$$\lambda(t) = \frac{4}{h^2} \frac{r^4(\tau)}{\tau + r^2(\tau)}, \quad \tau = \frac{th^2}{4} \qquad ...(5.2)$$

and $r(\tau)$ is the smallest positive root of the equation

$$\tan r = -\sqrt{\frac{\tau}{\tau + r^2}} \tanh\left(r\sqrt{\frac{\tau}{\tau + r^2}} \right), \; \tau \geqslant 0. \qquad ...(5.3)$$

Moreover $\lambda(t)$ as given by (5.2), (5.3) is the largest possible constant in (5.1) in the sense that if, for a given t, $\lambda(t)$ is replaced by a smaller constant, there is a $u \in C_0^2$ for which (5.1) fails to hold.

The proof of the result stated above is outlined briefly in Appendix B.

One shows easily from (5.2) that $r(\tau)$ decreases monotonically with increasing τ, and that

$$r(0+) = \pi, \; r(\infty) = r^* \qquad \ldots(5.4)$$

where $r^* \doteq 2.36$ is the smallest positive root of

$$\tan r = -\tanh r. \qquad \ldots(5.5)$$

It then follows from (5.2), (5.4) that $\lambda(t)$ is monotone decreasing as t increases, and that

$$\lambda(0^+) = \frac{4\pi^2}{h^2}, \; \lambda(t) \sim \left(\frac{2r^*}{h}\right)^4 t^{-1} \text{ as } t \to \infty . \qquad \ldots(5.6)$$

We shall now prove that

$$\mathscr{D}\{u; v; w\} \geqslant 0 \text{ for all } u, v, w \in C_0^2 \qquad \ldots(5.7)$$

if and only if $m > 0$ and $k > 0$ satisfy

$$k \leqslant \sqrt{m\,\lambda\left(\frac{m}{2}\right)\Big/\left(2m+\lambda\left(\frac{m}{2}\right)\right)} \equiv \Lambda(m). \qquad \ldots(5.8)$$

To show that (5.8) is necessary, we first note from (4.16) that

$$\mathscr{D}\{u; 0; 0\} = \int_0^h [(m-k^2)\ddot{u}^2 - 2mk^2\dot{u}^2 - m^2k^2u^2]\,dx_2 \qquad \ldots(5.9)$$

if $m - k^2 \leqslant 0$, one can clearly choose $u \in C_0^2$ such that $\mathscr{D}\{u; 0; 0\} < 0$ and hence such that (5.7) is violated. Thus $m - k^2 > 0$ is necessary for (5.7). If (5.7) holds, we may also infer that

$$\int_0^h \ddot{u}^2\,dx_2 \geqslant \frac{2mk^2}{m-k^2}\int_0^h \left(\dot{u}^2 + \frac{m}{2}u^2\right)dx_2 \qquad \ldots(5.10)$$

the all $u \in C_0^2$. Since $\lambda(t)$ is the largest constant such that (5.1) holds for all $u \in C_0^2$, we conclude that

$$\frac{2mk^2}{m-k^2} \leqslant \lambda\left(\frac{m}{2}\right) \qquad \ldots(5.11)$$

from which (5.8) follows immediately.

To show that (5.8) is sufficient for (5.7), we first note that (5.8) implies $m - k^2 > 0$, so that, by (5.9), (5.1), (5.8),

$$\mathscr{D}\{u; 0; 0\} \geqslant (m-k^2)\left[\lambda\left(\frac{m}{2}\right) - \frac{2mk^2}{m-k^2}\right]\int_0^h \left(\dot{u}^2 + \frac{m}{2}u^2\right)dx_2 \geqslant 0 \qquad \ldots(5.12)$$

for all $u \in C_0^2$. Next, (4.16), (5.9), (5.12) show that

$$\mathscr{D}\{u; v; w\} = \mathscr{D}\{u; 0; 0\} + \mathscr{D}\{0; v; w\} \geqslant \mathscr{D}\{0; v; w\}. \qquad \ldots(5.13)$$

But, by (4.16),

$$\mathcal{D}\ \{0;\, v;\, w\} = \int_0^h \left(\dot{v}^2 + 2\, m\, \dot{v}^2 - \frac{m\ k^2}{m-k^2}\ v^2 \right) dx_2$$

$$+ \int_0^h \left[\, 2\, w^2 + (m-k^2) \left(w + \frac{m\ k}{m-k^2} \right)^2 \right] dx_2. \qquad \ldots(5.14)$$

The second integral in (5.14) is nonnegative; discarding it and invoking (5.1) in the first integral, one obtains

$$\mathcal{D}\ \{0;\, v;\, w\} \geqslant \lambda \left(\frac{m}{2} \right) \int_0^h \left(\dot{v}^2 + \frac{m}{2}\ v^2 \right) dx_2 + 2m \int_0^h \dot{v}^2\, dx_2$$

$$- \frac{2mk^2}{m-k^2} \int_0^h \frac{m}{2}\ v^2\, dx_2. \qquad \ldots(5.15)$$

Appealing to (5.8) (or, equivalently, (5.11)), we find from (5.15) that

$$\mathcal{D}\ \{0;\, v;\, w\} \geqslant 0 \qquad \ldots(5.16)$$

and thus, because of (5.13), the sufficiency of (5.8) is established.

In view of the discussion in the previous Section, we have now proved that the energy decay inequalities (4.23), (4.24) hold for any $k > 0$ and $m > 0$ satisfying (5 8) A more convenient representation for the function $\Lambda\,(m)$ of (5.8) can be obtained by substituting from (5.2); this leads to

$$\Lambda\,(m) = \frac{1}{h}\ \mathrm{K}\,(\tau) \qquad \ldots(5.17)$$

where

$$\mathrm{K}\,(\tau) = \frac{2\sqrt{2}\ \tau r^2\,(\tau)}{2\tau + r^2\,(\tau)}\ ,\quad \tau = \frac{m}{8}\ h^2 > 0. \qquad \ldots(5.18)$$

Because of (5.4), we infer that $\mathrm{K}\,(\tau) \to 0$ as $\tau \to 0+$ and as $\tau \to \infty$, so that $\mathrm{K}\,(\tau)$ has an absolute maximum, say at $\tau = \tau^* > 0$. The optimal choice of the estimated decay rate k consistent with the constraint (5.8) is thus

$$k = \frac{1}{h}\ \mathrm{K}\,(\tau^*) = \frac{1}{h} \max_{0 < \tau < \infty} \mathrm{K}\,(\tau) \qquad \ldots(5.19)$$

the corresponding value of m, by (5.18), is

$$m = \frac{8\tau^*}{h^2}. \qquad \ldots(5.20)$$

It is easy to show from (5.4), (5.18) that

$$r^* < K(\tau^*) < \pi \qquad \text{...(5.21)}$$

numerical calculations give $\tau^* \doteq 2.8$, $K(\tau^*) = 2.7$, leading through (5.20), (5.19) to the results stated in (3.8), (3.9).

6. Stress Estimates

Although our concern here has been primarily with an alternative procedure for estimating the energy decay rate, we shall comment briefly on the application of the results (4.23), (4.24) to the estimation of streses.

Cross-sectional mean-squares of the normal (diagonal) stress components σ_{11}, σ_{22} can be obtained immediately from the energy inequality (4.24). By (2.8) and the representation (4.11) for the second-order energy $E_2(z)$, we have

$$\int_{S_z} (\sigma_{22}^2 + \sigma_{11}^2)\, dx_2 = \int_{S_z} (\varphi_{,11}^2 + \varphi_{,22}^2)\, dx_2 = -\int_z^{\infty} E_2(\zeta)\, d\zeta \qquad \text{...(6.1)}$$

so that, by (4.22),

$$\int_{S_z} (\sigma_{22}^2 + \sigma_{11}^2)\, dx_2 \leqslant \frac{1}{2k}\, [E_2(0) + mE_1(0)]\, e^{-2kz}. \qquad \text{...(6.2)}$$

Here k and m are, of course, given by (5.19), (5.20).

To obtain a cross-sectional mean-square estimate for the shearing stress $\sigma_{12} = \sigma_{21} = -\varphi_{,12}$ we let

$$s(z) = \int_{S_z} \sigma_{12}^2\, dx_2 = \int_{S_z} \varphi_{,12}^2\, dx_2 \qquad \text{...(6.3)}$$

and observe that, by (2.4), $s(z) \to 0$ as $z \to \infty$. Thus

$$s(z) = -\int_z^{\infty} s'(\zeta)\, d\zeta = -2 \int_{\mathscr{R}_z} \varphi_{,12}\, \varphi_{,121}\, dA \qquad \text{...(6.4)}$$

from which it follows that

$$s^2(z) \leqslant 4 \int_{\mathscr{R}_z} \varphi_{,12}^2\, dA \int_{\mathscr{R}_z} \varphi_{,121}^2\, dA. \qquad \text{...(6.5)}$$

Appealing to the definitions (3.1) add (3.7) of the energies $E_1(z)$ and $E_2(z)$, we find from (6.5) that

$$s^2(z) \leqslant E_1(z)\, E_2(z). \qquad \text{...(6.6)}$$

Finally, from (6.6), (6.3), (4.23) and (4.24), we obtain

$$\int_{S_z} \sigma_{12}^2\, dx_2 \leqslant \frac{1}{\sqrt{m}}\, [E_2(0) + mE_1(0)]\, e^{-2kz}. \qquad \text{...(6.7)}$$

Pointwise estimates of the stresses can be obtained at interior points of the strip by making use of a mean value theorem for biharmonic functions. The construction of such estimates requires a decay inequality only for the first-order energy $E_1(z)$ and has been carried out in detail in Knowles (1966). Pointwise bounds on the stresses obtained in this way deteriorate fatally near the boundary of $\mathscr{R}$; results which are free of this drawback can be obtained by considering a third-order energy $E_3(z)$, but we shall not discuss such a procedure here.

In order to render the estimates (6.2), (6.7) fully explicit, it would be necessary to obtain upper bounds on the total energies $E_1(0)$, $E_2(0)$ in terms of the given data f, g entering the original boundary value problem (2.1)-(2.4). A technique based on a minimum principle for constructing such an estimate for $E_1(0)$ has been given in Knowles (1966). A corresponding upper bound for $E_2(0)$ is not known.

Fiually it should be remarked that the extension to more general plane domains —even to the rectangle of finite length—of the energy decay arguments given in Section 4 and 5 is by no means routine.

References

Bogy, D. B. (1975). Solution of the plane end problem for a semi-infinite elastic strip. *Z. angew. Math. Phys.*, **26**, 749–69.

Courant R., and Hilbert, D. (1953). Methods of Mathematical Physics, Vol. I. Intersience, New York.

Flavin, J. N. (1974). On Knowles' version of Saint-Venant's principle in two-dimensional elastostatics. *Arch. Ration. Mech. Anal.*, **53**, 366–75.

Gregory, R. D. (1980). The traction boundary-value problem for the elastostatic semi-infinite strip; existence of solution and completeness of the Papkovich-Fadle eigenfunctions. *J. Elasticity*, **10**, 295–327.

Gurtin, M. E. (1972). The linear theory of elasticity. in *Handbuch der Physik*, vol. VI a/2, (S. Flügge, ed.) Springer, Berlin, pp. 1–295.

Hardy, G. H., Littlewood, J.E., and Pòlya, G. (1967). Inequalities, Cambridge University Press, Cambridge, p. 185.

Horgan, C. O., and Knowles, J. K. (1983). Recent developments concerning Saint-Venant's principle. *Advances in Applied Mechanics*, **24**, Academic Press.

Knowles, J. K. (1966). On Saint-Venant's principle in the two-dimensional linear theory of elasticity. *Arch. Ration. Mech. Anal.*, **21**, 1–22.

———(1967). A Saint-Venant principle for a class of second-order elliptic boundary value problems. *Z. angew. Math. Phys.*, **18**, 473–90.

Mieth, H. J. (1975) Über abklingende Lösungen elliptischer Randwertprobleme (*Prinzip von Saint-Venant*), Dissertation, Technische Hochscule Darmstadt.

Oleinik, O. A. (1979 a). Energetic estimates analogous to the Saint-Venant principle and their applications, in Equadiff IV (J. Fabera, ed.), *Lecture Notes in Mathematics*, vol. 703 Springer, Berlin, pp. 328–339.

———(1979 b). Applications of the energy estimates analogous to Saint-Venant's principle to problems of elasticity and hydrodynamics, in *Lecture Notes in Physics*, vol. 90. Springer, Berlin, pp. 422–32.

Oleinik, O. A., and Yosifian, G. A. (1978 a). On Saint-Venant's principle in plane elasticity theory. *Dokl., Akad. Nauk. SSSR*, **239**, 530–33; translated in *Soviet Math. Dokl.*, **19**, 364–68.

———(1978 b). The Saint-Venant principle in the two-dimensional theory of elasticity and boundary problems for a biharmonic equation in unbounded domains. *Sibirsk. Mat. Z.*, **19**, 1154–65; translated in *Siberian Math. J.*, **19**, 813–22.

Shield, R. T. (1965). On the stability of linear continuous systems. *Z. angew. Math. Phys.*, **16**, 649–86.

Appendix A : Finiteness of $E_1(z)$, $E_2(z)$

We show here that the functions $g_1(z)$, $g_2(z)$ defined respectively by (4.5), (4.8) are such that $g_1'(z) \to 0$, $g_2'(z) \to 0$ as $z \to \infty$. Because of (4.6), (4.7), (4.1) (3.1) and (3.7), this will establish the finiteness of $E_1(z)$, $E_2(z)$ as well as the representations (4.10), (4.11).

We first prove a more general proposition from which the above assertions will follow. Let $g \in C^2([0, \infty))$, and suppose that g is bounded and convex on $[0, \infty)$. Then $g'(z) \to 0$ as $z \to \infty$. To show this, we note first that, by the convexity of g, $g'(z)$ is monotone nondecreasing on $[0, \infty)$. Thus for $\zeta \geqslant z \geqslant 0$,

$$g(\zeta) = g(z) + \int_z^\zeta g'(s)\,ds \geqslant g(z) + (\zeta - z)\,g'(z). \qquad \ldots(A.1)$$

Divide by ζ, let $\zeta \to \infty$ and use the boundedness of g to conclude that

$$g'(z) \leqslant 0, \; z \geqslant 0 \qquad \ldots(A.2)$$

Since $g'(z)$ is nondecreasing, $d \equiv \lim_{z \to \infty} g'(z)$ exists, and, by (A.2), $d \leqslant 0$. Thus by the equality in (A.1),

$$g(\zeta) \leqslant g(z) + d(\zeta - z), \; \zeta \geqslant z \geqslant 0. \qquad \ldots(A.3)$$

Suppose that $d < 0$. Then (A.3) implies that $g(\zeta) \to -\infty$ as $\zeta \to \infty$, contradicting the boundedness of g. Hence $d = 0$, and the proposition follows.

Observe from (4.1) that e_1, e_2 are nonnegative, so that by (4.6), (4.7) and the smoothness of φ, both g_1 and g_2 are twice continuously differentiable on $[0, \infty)$ and convex. Insofar as g_2 is concerned, (2.4) and (4.8) show that $g_2(z) \to 0$ as $z \to \infty$, so that g_2 is bounded on $[0, \infty)$. It follows from the proposition proved about that $g_2'(z) \to 0$ as $z \to \infty$.

As to $g_1(z)$, we note from (4.5) and the Schwarz inequality that

$$|g_1(z)| \leqslant \int_{S_z} (\varphi_{,1}^2 + \varphi_{,2}^2)\,dx_2 + \left(\int_{S_z} \varphi^2\,dx_2 \int_{S_z} \varphi_{,11}^2\,dx_2 \right)^{1/2}. \qquad \ldots(A.4)$$

But if $f \in C'([0, h])$ and $f(0) = f(h) = 0$, we have (Hardy *et al.* 1967)

$$\int_0^h f^2\,dx_2 \leqslant \frac{h^2}{\pi^2} \int_0^h \dot{f}^2\,dx_2. \qquad \ldots(A.5)$$

Bearing in mind the boundary conditions (2.3), we may apply (A.5) to estimate the integrals in (A.4) which involve $\varphi_{,1}$, $\varphi_{,2}$ and φ. This leads to

$$|g_1(z)| \leqslant \frac{h^2}{\pi^2}\left\{\int_{S_z}(\varphi_{,12}^2+\varphi_{,22}^2)\,dx_2+\left(\int_{S_z}\varphi_{,22}^2\,dx_2\int_{S_z}\varphi_{,11}^2\,dx_2\right)^{1/2}\right\} \qquad \text{...(A.6)}$$

so that, by (2.4), $g_1(z) \to 0$ as $z \to \infty$. Thus g_1 is bounded on $[0, \infty)$, and again the general proposition implies that $g_1'(z) \to 0$ as $z \to \infty$.

Appendix B : The Inequality (5.1)

We sketch briefly here an outline of the proof of (5.1). The variational problem of finding the extremals in C_0^2 of the ratio

$$\mathcal{G}\{u\} = \frac{\int_0^h \ddot{u}^2\,dx_2}{\int_0^h (\dot{u}^2+tu^2)\,dx_2} \qquad \text{...(B.1)}$$

for fixed $t \geqslant 0$ leads formally to the eigenvalue problem

$$u^{(iv)} + \lambda\ddot{u} - \lambda t u = 0 \text{ on } [0, h] \qquad \text{...(B.2)}$$

$$u(0) = \dot{u}(0) = u(h) = \dot{u}(h) = 0. \qquad \text{...(B.3)}$$

By treating (B.2), (B.3) explicitly, one finds that the eigenvalues λ are given by

$$\lambda(t) = \frac{4}{h^2}\,\frac{r^4(\tau)}{\tau+r^2(\tau)},\quad \tau = \frac{th^2}{4} \qquad \text{...(B.4)}$$

where r is a positive root of either of the equations

$$\tan r = \sqrt{\frac{\tau+r^2}{\tau}}\tanh\left(r\sqrt{\frac{\tau}{\tau+r^2}}\right) \qquad \text{...(B.5)}$$

$$\tan r = -\sqrt{\frac{\tau}{\tau+r^2}}\tanh\left(r\sqrt{\frac{\tau}{\tau+r^2}}\right). \qquad \text{...(B.6)}$$

Detailed analysis show that the smallest eigenvalue $\lambda_1(t)$—which is the candidate for the absolute minimum of $\mathcal{G}\{u\}$—corresponds through (B.4) to the smallest positive root $r_1(\tau)$ of (B.6). The eigenfunction $u_1(x_2)$ corresponding to $\lambda_1(t)$ can be found explicitly and can be shown to have no zero in $(0, h)$. Moreover,

$$\mathcal{G}\{u_1\} = \lambda_1(t). \qquad \text{...(B.7)}$$

To show that $\lambda_1(t)$ indeed supplies the absolute minimum of $\mathcal{G}\{u\}$, one may adapt the "method of multiplicative variation" due apparently to Jacobi (Courant and Hilbert 1953, p. 458). Since u_1 has no zeros in $(0, h)$, we may represent any $u \in C_0^2$ in the form

$$u(x_2) = u_1(x_2)\, v(x_2),\; 0 \leqslant x_2 \leqslant h \qquad \text{...(B.8)}$$

where v is smooth enongh to permit a direct calculation leading to

$$\int_0^h \ddot{u}^2\, dx_2 - \lambda_1(t) \int_0^h (\dot{u}^2 + tu^2)\, dx_2 = \int_0^h \ddot{v}^2(x_2)\, u_1^2(x_2)\, dx_2$$

$$+ \int_0^h \dot{v}^2(x_2)\, U(x_2)\, dx_2 \qquad \text{...(B.9)}$$

provided

$$U(x_2) = 2\,\dot{u}_1^2 - 4\ddot{u}_1\, u_1 - \lambda_1(t)\, u_1^2\,. \qquad \text{...(B.10)}$$

The explicit representation of the fundamental eigenfunction u_1 allows one to show that $U(x_2) > 0$ for $0 < x_2 < h$. It then follows from (B.9), (B.1) that

$$\mathscr{T}\{u\} \geqslant \lambda_1(t) \qquad \text{...(B.11)}$$

for any $u \in C_0^2$, thus establishing (5.1)-(5.3)*.

Finally, equality holds in (B.11) if and only if v in (B.9) is constant, so that, by (B.8), u is a constant multiple of u_1. Thus $\lambda_1(t)$ is indeed the infimum of $\mathscr{T}\{u\}$ over C_0^2.

* In (5.1) the subscript 1 on $\lambda_1(t)$ has been dropped.