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Rate Equation Analysis of Flowing Lasing Systems

Part I. Uniform Pumping

by

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ABSTRACT

A simple analytical model for flowing laser systems has been developed. The lasing species is modeled as a two-state system with specified pumping and relaxation rates. Threshold requirements and output efficiency are expressed in the form of universal dimensionless functions. In terms of these universal functions, the behavior of particular systems can be studied in a parametric way. The analysis shows that flow would not improve the performance of all laser systems. The transformation $\frac{\partial}{\partial t} \to v \frac{\partial}{\partial x}$ allows one to predict the performance of flowing systems from the behavior of pulsed systems.

INTRODUCTION

In many high speed gas lasers, it is found that the pumping rate is approximately uncoupled from the lasing and the deactivation rates. When this approximation is adequate, the analytical description of the operation of the laser is reduced to the solution of two simultaneous linear first-order differential equations.

This approach allows the laser to be studied quite generally with respect to the various rates that affect it, without restricting the analysis to a specific system. The performance of high speed lasers can then be predicted to a fair approximation by merely substituting into general solutions the particular values of the atomic constants, optical cavity parameters, and the pumping and deactivation rates (these are known or can be determined by independent means).

The first part of this discussion deals with those lasing systems whose local volume pumping rate can be considered constant throughout the lasing region. Examples of such lasers are premixed chemical and electrical discharge lasers. The second part (which will follow) deals with the interesting special case in which the pumping is the result of a chemical reaction, or energy transfer, as limited by the rate of turbulent mixing of two streams of unequal velocities.

An effort has been made to render the main body of the discussion complete in its description of the various aspects of the laser operation in the hope that it can be read as one unit. To this end, the problems and results of the optical considerations (in connection with the gain of the amplifying medium and power extraction) are discussed in Appendix 1. In the interest of preserving the contiguity of the discussion, most of the algebra is deferred from the main body and included in Appendix 2.

RATE EQUATIONS

In the present model, lasing is assumed to take place between two levels: the upper state U, and the lower state L. These are single states (as defined by a complete set of quantum numbers) if the lasing occurs on a single line, or composite "states" if the lasing occurs on many lines simultaneously. The processes that are considered in this model are summarized in Fig. 1. The various quantities that appear are defined as follows:

R	- volume pumping rate into the upper lasing level ($cm^{-3} \cdot sec^{-1}$),
n_U, n_L	- upper and lower lasing level populations (particles per cm^3),
n^*	- photon density in the lasing mode (photons per cm^3),
g_U, g_L	- degeneracies of the upper and lower lasing levels,
$\frac{1}{\tau_{UO}}$	- transition rate from the upper lasing level to any final state other than the lower lasing level (sec^{-1}),
$\frac{1}{\tau_{UL}}$	- transition rate from the upper lasing level to the lower lasing level by all mechanisms other than stimulated emission (sec^{-1}),
$\frac{1}{\tau_L}$	- transition rate out of the lower lasing level (sec^{-1}),
κ^*	- stimulated emission rate constant ($cm^3 \cdot sec^{-1}$).

We can define the linear gain coefficient (see Appendix 1) in the amplifying medium α (cm^{-1}), as the product of an optical cross section, σ^* (cm^2), and the population inversion, Δn (particles per cm^3). That is

$$\alpha = \sigma^* \Delta n$$

where

$$\sigma^* \equiv \frac{\lambda^2}{8\pi} \cdot \frac{g(\nu - \nu_o)}{\tau^r_{UL}} , \qquad (1)$$

$$\Delta n \equiv (n_U - \frac{g_U}{g_L} n_L),$$

λ = wavelength of the transition radiation,

$g(\nu - \nu_0)$ = normalized line shape function,

and

$\frac{1}{\tau_{UL}^r}$ = spontaneous radiation rate (Einstein coefficient A_{UL}).

In terms of σ^*, κ^* is given by $c\sigma^*$ where c is the speed of light. The lasing condition on the population inversion can be expressed as a gain equals loss condition (see Appendix 1), or

$$r_0 e^{-\varepsilon_0} \, r_L e^{-\varepsilon_L} \, e^{2\alpha L} = 1 \tag{2}$$

where r_0 and r_L are the mirror reflectivities, ε_0 and ε_L are the fractional losses (diffraction, absorption, scattering, etc.) of the two mirrors, and L is the length of the amplifying medium.

If we denote the solution of Eq. 2 by α_{th}, then we have

$$\alpha_{th} = \frac{\varepsilon + \delta}{2L}$$

where

$$\varepsilon \equiv \varepsilon_0 + \varepsilon_L$$

and

$$\delta \equiv \delta_0 + \delta_L \equiv \ln \frac{1}{r_0} + \ln \frac{1}{r_L} \, .$$

Since $\alpha_{th} \equiv \sigma^* \Delta n_{th}$, we have

$$\Delta n_{th} = \frac{\varepsilon + \delta}{2L\sigma^*}$$

or

$$\Delta n_{th} = \frac{c}{2L}\left(\frac{\varepsilon + \delta}{\kappa^*}\right). \tag{3}$$

We can write the rate equations for the populations of upper and lower lasing levels from Fig. 1. In a frame at rest with respect to the lasing medium, we have

$$\frac{dn_U}{dt} = R - \kappa^* n^* \Delta n - \frac{n_U}{\tau_U} \tag{4a}$$

$$\frac{dn_L}{dt} = \frac{n_U}{\tau_{UL}} + \kappa^* n^* \Delta n - \frac{n_L}{\tau_L} \tag{4b}$$

where

$$\frac{1}{\tau_U} \equiv \frac{1}{\tau_{U0}} + \frac{1}{\tau_{UL}} \tag{5}$$

In the discussions that follow, the assumption is made that the various quantities of interest (n_U, n_L, and n^*) depend only on x. The dependence on z is small and is neglected. This assumption is valid under very mild restrictions (see Appendix 1).

The rate equations for a flowing system are obtained by performing the transformation

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$$

where we have assumed that the velocity v is along the x-axis. At steady state $\frac{\partial}{\partial t} = 0$, and consequently

$$v \frac{\partial}{\partial x} n_U + \kappa^* n^* \Delta n + \frac{n_U}{\tau_U} = R \tag{6a}$$

and

$$v \frac{\partial}{\partial x} n_L - \kappa^* n^* \Delta n + \frac{n_L}{\tau_L} = \frac{n_U}{\tau_{UL}} . \tag{6b}$$

The geometrical configuration is sketched in Fig. 2. The pumping region fills the volume

$$0 < x < x_3 \ ,$$
$$0 < y < b \ ,$$
$$0 < z < L \ .$$

The pumping rate R is constant in this volume and zero outside. The optical cavity forms a Fabry-Perot interferometer whose optical axis is parallel to the z-axis.

We now assume that the initial conditions are given at x = 0 as

$$n_U(0) = 0$$

and

$$n_L(0) = 0.$$

In other words, all of the lasing species that populate these two levels result from the pumping mechanism. Initially the photon density $n^*(x)$ is zero and remains zero, in this approximation, until the population inversion $\Delta n(x)$ equals the threshold value Δn_{th}. Thus in the build-up region, $0 < x < x_1$ (where $n^*(x) = 0$), the system is described by the equations

$$v \frac{\partial n_U}{\partial x} + \frac{n_U}{\tau_U} = R \ , \tag{7a}$$

$$v \frac{\partial n_L}{\partial x} + \frac{n_L}{\tau_L} = \frac{n_U}{\tau_{UL}} \ , \tag{7b}$$

and

$$n_U(0) = n_L(0) = 0 \ . \tag{7c}$$

The solution of Eq. 7 is given by:

$$n_U(x) = \tau_U \, R \left(1 - e^{-x/v\tau_U}\right) \tag{8a}$$

and

$$n_L(x) = \frac{\tau_U \tau_L}{\tau_{UL}} R \left(1 - \frac{e^{-x/v\tau_L}}{1 - \tau_U/\tau_L} + \frac{e^{-x/v\tau_U}}{\tau_L/\tau_U - 1} \right) . \tag{8b}$$

From Eq. 8, we can calculate the population inversion at each station x in the build-up region. If $\Delta n(x)$ is normalized by $\tau_L R$, we obtain

$$\frac{\Delta n(x)}{\tau_L R} = \frac{\tau_U}{\tau_L} - \frac{\tau_U}{\tau_{UL}} \left(\frac{g_U}{g_L} \right) - \left[\frac{\tau_U}{\tau_L} + \frac{\tau_U/\tau_L}{1 - \tau_U/\tau_L} \left(\frac{\tau_U}{\tau_{UL}} \right) \left(\frac{g_U}{g_L} \right) \right] e^{-x/v\tau_U} \tag{9}$$

$$+ \frac{\tau_U}{\tau_{UL}} \left(\frac{g_U}{g_L} \right) \frac{e^{-x/v\tau_L}}{1 - \tau_U/\tau_L} .$$

Equation 9 provides an implicit equation for x_1, the end of the build-up region, since $\Delta n(x_1) = \Delta n_{th}$.

For $x > x_1$, the population inversion is clamped to Δn_{th} by the lasing condition, and hence the equations become

$$v \frac{\partial n_U}{\partial x} + \frac{n_U}{\tau_U} + (\kappa^* \Delta n_{th}) n^* = R , \tag{10a}$$

$$v \frac{\partial n_L}{\partial x} + \frac{n_L}{\tau_L} - (\kappa^* \Delta n_{th}) n^* = \frac{n_U}{\tau_{UL}} , \tag{10b}$$

and

$$n_U = \Delta n_{th} + \frac{g_U}{g_L} n_L . \tag{10c}$$

The initial conditions for $n_U(x_1)$ and $n_L(x_1)$ are given by Eq. 8 with $x = x_1$. Adding Eqs. 10a and 10b and substituting 10c yields

$$\left(1 + \frac{g_U}{g_L} \right) v \frac{\partial n_L}{\partial x} + \frac{n_L}{\tau} = R - \frac{\Delta n_{th}}{\tau_{U0}} \tag{11}$$

where

$$\frac{1}{\tau} \equiv \frac{1}{\tau_L} + \frac{1}{\tau_{U0}}\left(\frac{g_U}{g_L}\right) . \tag{12}$$

Therefore, for $x > x_1$

$$n_L(x) = n_L(x_1)\, e^{-\frac{x-x_1}{\ell}} + \tau\left(R - \frac{\Delta n_{th}}{\tau_{U0}}\right)\left(1 - e^{-\frac{x-x_1}{\ell}}\right) \tag{13a}$$

and

$$n_U(x) = \Delta n_{th} + \frac{g_U}{g_L}\left\{n_L(x_1)\, e^{-\frac{x-x_1}{\ell}} + \tau\left(R - \frac{\Delta n_{th}}{\tau_{U0}}\right)\left(1 - e^{-\frac{x-x_1}{\ell}}\right)\right\} \tag{13b}$$

where

$$\ell \equiv \left[1 + \left(\frac{g_U}{g_L}\right)\right] v\tau . \tag{14}$$

The photon density $n^*(x)$ can now be calculated by substituting Eq. 13b into Eq. 10a. This yields

$$n^*(x) = (n_1^* - n_\infty^*)\, e^{-\frac{x-x_1}{\ell}} + n_\infty^* . \tag{15a}$$

If we make the substitution

$$\kappa^* \, \Delta n_{th} = \frac{c(\varepsilon + \delta)}{2L}$$

from Eq. 3, we can express n_1^* and n_∞^* as follows:

$$n_1^* = n^*(x_1^+) = \frac{2LR}{c(\varepsilon + \delta)\left(1 + \frac{g_U}{g_L}\right)}\left\{1 - \left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right)\left(1 - e^{-\frac{x_1}{v\tau_U}}\right) - \frac{\Delta n_{th}}{\tau_L R}\right\} \tag{15b}$$

and

$$n_\infty^* = n^*(\infty) = \frac{2LR}{c(\varepsilon + \delta)} \left[1 - \frac{\tau}{\tau_U}\left(\frac{g_U}{g_L}\right) - \frac{\tau}{\tau_U} \cdot \frac{\Delta n_{th}}{\tau_L R} \right] . \tag{15c}$$

Thus n_1^* is the photon density inside the cavity at the station where the laser turns on ($x = x_1^+$) and n_∞^* is the photon density at infinity.

If $n_\infty^* > 0$, Eqs. 13a through 15c describe the operation of the laser up to $x = x_3$ where the pumping region ends. If $n_\infty^* < 0$, then there exists an x_2 such that $n^*(x_2) = 0$, or

$$\frac{x_2 - x_1}{\ell} = \ln\left(\frac{n_1^*}{-n_\infty^*} + 1\right) . \tag{16}$$

The lasing then ends at x_2 or x_3, whichever comes first. If $x_2 < x_3$, then $n^*(x) = 0$ for $x_2 < x < x_3$, and the populations are once more described by Eqs. 7a and 7b (with initial conditions given by Eqs. 8 and 9 evaluated at x_2). The evolution of the solution in the three regions is summarized in Fig. 3 for a hypothetical set of parameters for which $\frac{\tau_U}{\tau_L} < 1$ and $n_\infty^* < 0$.

Note that in the approximations of this analysis, the photon density is discontinuous at x_1, i.e.,

$$n^*(x_1^-) = 0$$

and

$$n^*(x_1^+) = n_1^* .$$

This is a consequence of neglecting spontaneous emission in the lasing mode, any optical mode structure, and the photon dynamics as the intensity builds up after x_1. These approximations result from using the threshold condition as given by Eq. 3, instead of solving a third coupled rate equation for the local photon density. Such a refinement would have introduced a "soft" start of the lasing at x_1 tending asymptotically to the expression for the photon density as given by Eq. 15 for $x > x_1$ and to zero for $x < x_1$.

Several interesting conclusions can be drawn from the analysis so far:

a) n_∞^* is the photon density at $x - x_1 >> \ell$ or, equivalently, the photon density that would have resulted if the laser was operated as a stationary laser. If n_∞^* is negative, the laser will not operate as a stationary laser. From equation 15c it can be seen that for n_∞^* to be positive, the quantity

$$1 - \frac{\tau}{\tau_U}\left(\frac{g_U}{g_L}\right) \equiv \tau\left[\frac{1}{\tau_L} - \frac{1}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\right]$$

must be positive (this is necessary, but not sufficient). This result is intuitively appealing since it says that for a stationary laser the relaxation rate out of the lower lasing level must be high enough for the inversion to be maintainable.

b) The characteristic length ℓ, which scales the lasing region, does not depend on the rate τ_{UL}^{-1}. This rather surprising result is perhaps difficult to reconcile with the intuitive notion from stationary laser performance that a high τ_{UL}^{-1} rate would cause a lower level bottleneck and terminate the lasing. In fact, this independence from τ_{UL} suggests that the criteria for a good stationary laser (n_∞^* as high as possible) and for an efficient flowing laser (ℓ as large as possible) might even be conflicting. A large n_∞^* requires a large τ_L^{-1} rate. Since

$\ell = \left(1 + \frac{g_U}{g_L}\right)v\tau < \left(1 + \frac{g_U}{g_L}\right)v\tau_L$, a high τ_L^{-1} rate would give a very

short lasing region.

c) Even neglecting the possibility that ℓ might be too small, it should be noted that not all lasers would benefit by being flowed. There exist combinations of the parameters, for which $n_1^* < n_\infty^*$. Such lasing systems would work better as stationary lasers ($v = 0$) and should not be flowed. This point will be discussed later in the context of minimum pumping rate required for laser operation.

d) The Galilean transformation $x/v \rightarrow t$ in Eqs. 8 through 13 describes the transient response of a stationary laser. All the results of the present analysis carry over to time-dependent pulsed laser performance. The equivalence being one-to-one, we can predict the behavior of CW flowing systems from the pulsed behavior of the same system (and vice-versa) to the extent that we can duplicate the various relaxation rates when the system is stationary.

THRESHOLD PUMPING RATE

Using Eq. 9, we can calculate x_1 for a given pumping rate R, threshold population inversion Δn_{th}, deactivation rates τ_L, τ_{UL}, and τ_{UO}, and spectroscopic constants. We can then ask for the minimum pumping rate that would be required for the laser to turn on at any x_1. This can be calculated by requiring that n_1^* (the lasing mode photon density at x_1^+) be non-negative. If we denote the minimum or threshold pumping rate that satisfies this condition by R_{th}, we have from Eq. 15b

$$\frac{\Delta n_{th}}{\tau_L R_{th}} = 1 - \left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right)\left(1 - e^{-x_{th}/v\tau_U}\right) \tag{17}$$

where x_{th} is equal to x_1 for $R = R_{th}$. Note that $\Delta n(x_{th}) = \Delta n_{th}$ and therefore x_{th} satisfies Eq. 9 with $R = R_{th}$, i.e.,

$$\frac{\Delta n_{th}}{\tau_L R_{th}} = \frac{\tau_U}{\tau_L} - \frac{\tau_U}{\tau_{UL}} \cdot \left(\frac{g_U}{g_L}\right) - \left[\frac{\tau_U}{\tau_L} + \frac{\tau_U/\tau_L}{1 - \tau_U/\tau_L} \cdot \left(\frac{\tau_U}{\tau_{UL}}\right) \cdot \frac{g_U}{g_L} \; e^{-x_{th}/v\tau_U}\right] \tag{18}$$

$$+ \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\frac{e^{-x_{th}/v\tau_L}}{1 - \tau_U/\tau_L} .$$

Eqs. 17 and 18 are simultaneous equations for R_{th} and x_{th}. To facilitate their solution, we make the following substitutions:

$$\theta_L = \frac{\tau_L R_{th}}{\Delta n_{th}} , \tag{19a}$$

$$w = \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right) \equiv \frac{g_U/g_L}{1 + \frac{\tau_{UL}}{\tau_{UO}}} , \tag{19b}$$

$$s = \frac{\tau_U}{\tau_L}, \qquad (19c)$$

and

$$q_{th} = e^{-x_{th}/v\tau_U}. \qquad (19d)$$

After some algebra, the transformed equations become

$$\phi = (1 - s + w)\, q_{th} \qquad (17')$$

and

$$\phi = \frac{w}{1-s} q_{th}^{s} - \frac{s}{1-s}(1 - s + w)\, q_{th}, \qquad (18')$$

where

$$\phi \equiv \frac{1}{\theta_L} - s + w\ . \qquad (19e)$$

From Eq. 17', we have

$$q_{th} = \frac{\phi}{1 - s + w} \qquad (20)$$

where we note that since $0 \leq q_{th} \leq 1$, the sign of ϕ must be the sign of $(1 - s + w)$.

Substituting Eq. 20 into 18', we have

$$\phi - w\left(\frac{\phi}{1 - s + w}\right)^{s} = 0. \qquad (21)$$

For $1 - s + w > 0$, Eq. 21 has two solutions given by

$$\phi = 0$$

and

$$\phi = w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}} .$$

It can be verified that in this case the minimum pumping rate corresponds to the latter. For $1 - s + w < 0$, however, only the $\phi = 0$ solution is possible. Thus, depending on the sign of

$$1 - s + w \equiv 1 - \tau_U \left[\frac{1}{\tau_L} - \frac{1}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\right] \tag{22}$$

we have two types of lasers whose threshold behavior is summarized in the following table.

TYPE I	TYPE II
$1 - s + w > 0$	$1 - s + w < 0$
$\frac{1}{\theta_L} = w\left(1 + \frac{1-s}{w}\right)^{-\frac{1}{1-s}} + s - w$	$\frac{1}{\theta_L} = s - w$
$q_{th} = \left(1 + \frac{1-s}{w}\right)^{-\frac{1}{1-s}}$	$q_{th} = 0$

(23)

It can be seen from the expression for q_{th} that a type II laser reaches the threshold condition at $x_{th}/v\tau_U = \infty$ when pumped by the minimum pumping rate R_{th}. Thus, for a type II laser, the optimum threshold behavior is realized when $v = 0$. In other words, a type II laser should not be flowed.

For type I lasers,

$$\frac{1}{\theta_L} = w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}} + s - w. \tag{23a}$$

For this type of laser $s < 1 + w$; in fact, it is frequently true that $s \ll 1$. It is useful, therefore, to examine the behavior of Eq. 23a for small s. Making a Taylor expansion about $s = 0$ yields:

$$\frac{1}{\theta_L} \sim s\left[1 - w \ln\left(1 + \frac{1}{w}\right)\right] \tag{23b}$$

If we define

$$\theta_U(s,w) \equiv \frac{\tau_U R_{th}}{\Delta n_{th}} = s\theta_L \tag{24}$$

we have, for small s,

$$\frac{1}{\theta_U} \sim 1 - w \ln\left(1 + \frac{1}{w}\right). \tag{25a}$$

It can be verified that, for $s < 0.1$, Eq. 25a is good to about 2%.

From Eq. 25a, we find that

$$\theta_U(0,w) \sim \begin{cases} 1, & \text{for } w < 0.1. \\ 2w + \frac{4}{3}, & \text{for } w > 0.1. \end{cases} \tag{25b}$$

The function $\theta_U(0,w)$ and the two asymptotic expressions for small and large w are plotted in Fig. 4. From Eq. 23a, the entire function $\theta_U(s,w)$ is given by:

$$\frac{1}{\theta_U(s,w)} = \begin{cases} 1 - \frac{w}{s} + \frac{w}{s}\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}}, & \text{for } s < 1 + w. \\ 1 - \frac{w}{s}, & \text{for } s > 1 + w. \end{cases} \tag{26}$$

(Note that $0 < w < \frac{g_U}{g_L}$.) This function is tabulated in Table I and plotted in Fig. 5; $x_{th}/v\tau_U$ is tabulated in Table II and plotted in Fig. 6.

LASER OUTPUT AND EFFICIENCY

From the photon density of the lasing mode, as given by Eq. 15, we can calculate the lasing intensity inside the cavity at each station x (see Appendix 1):

$$I(x) = ch\nu n^*(x) \tag{27}$$

where

I = lasing intensity

h = Planck's constant

c = speed of light

ν = lasing frequency.

Let us now assume that the laser has one-sided output optics. In other words, the $z > L$ mirror is 100% reflecting ($\delta_L \simeq 0$) and loss-less ($\varepsilon_L \simeq 0$). The output intensity from the $z < 0$ mirror is then given by (see Appendix 1)

$$I_{out}(x) = \frac{1 - e^{-\delta_0}}{2} \cdot \frac{(\delta_0 + \varepsilon_0)\, e^{\frac{1}{2}(\delta_0 - \varepsilon_0)}}{2 \sinh\left(\frac{\delta_0 + \varepsilon_0}{2}\right)} \cdot I(x) \tag{28}$$

and the total power output of the laser by

$$P_{out} = b \int_{x_1}^{x_{2,3}} I_{out}(x)\, dx, \tag{29}$$

where

$$x_{2,3} = \begin{cases} x_2 & \text{if } x_2 < x_3, \\ x_3 & \text{if } x_3 < x_2, \end{cases}$$

and b is the width of the pumping (and the lasing) region in the y-direction (see Fig. 2). Substituting for $I_{out}(x)$, we have

$$P_{out} = ch\nu b \, \frac{1 - e^{-\delta_o}}{2} \, \frac{(\delta_o + \varepsilon_o)\, e^{\frac{1}{2}(\delta_o - \varepsilon_o)}}{2\sinh\left(\frac{\delta_o + \varepsilon_o}{2}\right)} \int_{x_1}^{x_{2,3}} n^*(x)\, dx \,, \qquad (30)$$

where

$$\int_{x_1}^{x_{2,3}} n^*(x)\, dx = \ell \left[\left(1 - e^{-\xi}\right)\left(n_1^* - n_\infty^*\right) + \xi\, n_\infty^*\right]$$

and

$$\xi = \frac{x_{2,3} - x_1}{\ell} \,.$$

The power supplied to the upper lasing level is given by

P_U = (Energy per upper level particle)

x (Upper level particles generated per second per cm^3)

x (Pumping volume)

or

$$P_U = E_U\, R\, (x_3\, b\, L). \qquad (31)$$

In terms of P_U and P_{out}, we can define the lasing efficiency η_L, as

$$\eta_L \equiv \frac{P_{out}}{P_U} . \qquad (32)$$

We then find that the lasing efficiency can be written as the product of three parts

$$\eta_L = \eta_Q \eta_O \eta_S \,, \qquad (33)$$

where η_Q is the quantum efficiency given by

$$\eta_Q = \frac{h\nu}{E_U} , \tag{34}$$

η_O is the "optical efficiency" and is given by

$$\eta_O = \frac{(1 - e^{-\delta_o}) \, e^{\frac{1}{2}(\delta_o - \varepsilon_o)}}{2\sinh\left(\frac{\delta_o + \varepsilon_o}{2}\right)} , \tag{35}$$

and η_S is the "system efficiency" given by

$$\eta_S = \frac{\ell}{(1 + g_U/g_L)\, x_3} \left\{ \left[1 - \left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right)\left(1 - e^{-x_1/v\tau_U}\right) - \frac{\Delta n_{th}}{\tau_L R} \right.\right.$$

$$\left. + \left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1 + \frac{\tau}{\tau_U}\frac{\Delta n_{th}}{\tau_L R}\right)\right] \left(1 - e^{-\xi}\right) \tag{36}$$

$$\left. - \xi \left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1 + \frac{\tau}{\tau_U}\frac{\Delta n_{th}}{\tau_L R}\right)\right\} .$$

It is interesting to study the system efficiency in the limit of $\frac{R_{th}}{R} \to 0$. Let

$$q = e^{-x_1/v\tau_U} ,$$

then from Eq. 9 we have

$$q^s + \frac{s}{w}(1 - s + w)(1 - q) = 1 + \frac{1}{\theta_L}\left(\frac{1 - s}{w}\right)\frac{R_{th}}{R} . \tag{37}$$

It can be shown that the solution that corresponds to the beginning of the lasing region is given by

$$q = 1 - \frac{1}{\theta_U}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right)^2 . \tag{38}$$

In other words as $\frac{R_{th}}{R} \to 0$ the laser "starts" at $x_1 \to 0$. Therefore, in the limit of $R >> R_{th}$, we have

$$\lim_{R\to\infty} \left\{\eta_S\right\} = \eta_{S\infty} = \frac{1}{1 + \frac{g_U}{g_L}} \left(\frac{x_{2,3}}{x_3}\right) \left\{\left(1 + \zeta\right)\left(\frac{1 - e^{-\xi}}{\xi}\right) - \zeta\right\}, \tag{39a}$$

where

$$\zeta = \left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1\right) \equiv \left(1 + \frac{g_U}{g_L}\right)\left(\frac{w - s}{\frac{g_U}{g_L} - w + s}\right) \tag{39b}$$

$\left(\text{note that } x_1 \to 0 \text{ as } R \to \infty \text{ and hence } \lim_{R\to\infty}\left\{\xi\right\} = \frac{x_{2,3}}{\ell}\right)$.

Several interesting cases described by Eq. 39 are worth examining.

a) $\underline{\xi << 1, (x_{2,3} = x_3)}$. This corresponds to a flowing laser system in which the pumping terminates just after the laser turns on. See Fig. 7a. Then

$$\lim_{\xi\to 0} \left\{\eta_{S\infty}(\xi;\zeta)\right\} = \eta_{S\infty}(0;\zeta) = \frac{1}{1 + \frac{g_U}{g_L}} . \tag{40}$$

b) $\xi \gg 1$, $(x_{2,3} = x_3)$. This corresponds to a pumping region extending far downstream of ℓ. See Fig. 7b. Note that such a pumping scheme would only make sense if $n_\infty^* > 0$. In this case

$$\lim_{\xi\to\infty} \left\{ \eta_{S\infty} (\xi;\zeta) \right\} = \eta_{S\infty} (\infty;\zeta) = \frac{-\zeta}{1 + \frac{g_U}{g_L}}$$

or

$$\eta_{S\infty} (\infty;\zeta) = 1 - \frac{\tau}{\tau_U}\left(\frac{g_U}{g_L}\right) \equiv \tau\left[\frac{1}{\tau_L} - \frac{1}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\right] . \tag{41}$$

It should be noted, however, that in this case we should probably not be flowing at all since the same limiting efficiency would be extractable from a stationary system (to the extent that the values of the other parameters could be maintained as $v \to 0$).

c) $\xi = \ln\left(\frac{1}{\zeta} + 1\right)$, $(x_{2,3} = x_3)$. This corresponds to the case where $n_\infty^* < 0$ and the end of the pumping region coincides with the end of the lasing region (i.e., $x_2 = x_3$). See Fig. 7c. In this case

$$\xi = \frac{x_2 - x_1}{\ell} = \ln\left(\frac{n_1^*}{-n_\infty^*} + 1\right),$$

but for $\frac{R_{th}}{R} \to 0$

$$\frac{n_1^*}{-n_\infty^*} \to \frac{(1 + g_U/g_L)^{-1}}{\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1} = \frac{1}{\zeta} .$$

Therefore, for this case

$$\xi = \ln\left(\frac{1}{\zeta} + 1\right)$$

and substituting in Eq. 39a

$$\eta_{S\infty}\left[\ln\left(\frac{1}{\zeta}+1\right);\zeta\right] = \frac{1}{1+\frac{g_U}{g_L}}\left\{\frac{1}{\ln\left(\frac{1}{\zeta}+1\right)} - \zeta\right\} . \tag{42}$$

It is useful to observe that the expression in the brackets has an asympotic behavior for large positive ζ given by

$$\frac{1}{\ln\left(1+\frac{1}{\zeta}\right)} - \zeta \equiv f(\zeta) \sim \frac{1}{2} - \frac{1}{12\zeta} + 0\left(\frac{1}{\zeta^2}\right). \tag{43}$$

Hence, for $\zeta \gtrsim 1$

$$\eta_{S\infty}\left[\ln\left(\frac{1}{\zeta}+1\right);\zeta\right] \sim \frac{1}{2(1+g_U/g_L)} . \tag{42'}$$

The function $f(\zeta)$ is plotted in Fig. 8.

Since $g_U/g_L \sim 1$ and ζ is often of the order of 1 or greater, an engineering estimate for the system efficiency of $\frac{1}{4} \lesssim \eta_{S\infty} \lesssim \frac{1}{2}$ could be a good approximation. The upper limit is attained when the pumping region is terminated soon after the laser turns on, whereas the lower limit is attained when the pumping region persists to the end of the lasing region. Thus, we could write for the lasing efficiency, if $\frac{R_{th}}{R} \ll 1$ and if $\zeta \gtrsim 1$ (see Eq. 39b),

$$\lim_{R\to\infty}\left\{\eta_L\right\} \sim \frac{1}{4}\eta_Q\eta_0.$$

Fig. 9 presents a plot of η_0 versus δ_0 and ε_0.

It should be emphasized that the lasing efficiency η_L is defined with respect to the power supplied to the upper lasing level. This power may or may not be equal to the input power to the laser P_{in}, depending on whether there exist intermediate pumping steps, alternative competing processes, branching ratios, etc. Thus if pumping the upper lasing level is an indirect process, the overall efficiency η, defined with respect to the total input power to the laser is given by

$$\eta = \frac{P_{out}}{P_{in}} \tag{45}$$

or

$$\eta = \eta_Q \, \eta_O \, \eta_S \, \eta_P \ , \tag{46}$$

where η_P is the pumping efficiency defined by

$$\eta_P = \frac{P_U}{P_{in}} \ . \tag{47}$$

CALCULATION OF PUMPING RATE

It is interesting to calculate the pumping rate R and the pumping efficiency η_p for some of the common ways that are used in pumping the lasing species to the upper lasing level.

a) Chemical Pumping (premixed reactants).

Reactants A and B, with concentrations n_A and n_B, are assumed to react with a reaction constant $k_r(T)$ depending on temperature. The upper lasing level appears as one of the products with a certain branching ratio a_U. The excitation energy E_U is part of the available heat of reaction $-\Delta H$. That is,

$$A + B \xrightarrow{k_r} N_U + C \qquad -\Delta H.$$

The pumping rate is then given by

$$R = a_U k_r n_A n_B \, , \tag{46}$$

while the pumping efficiency is given by

$$\eta_p = a_U \cdot \frac{E_U}{-\Delta H} \tag{47a}$$

with respect to an input power

$$P_{in} = k_r n_A n_B \, (-\Delta H)(b \; L \; x_3) \, . \tag{47b}$$

In this case, where the chemicals are premixed, the concentrations of the reactants will actually decrease as the lasing proceeds. Some average value for n_A and n_B can then be used for the purpose of obtaining estimates.

b) Collisional Energy Transfer from Metastable Excited (Premixed) Species

A species M is assumed to be created at a rate R_m in an excited metastable state (see Fig. 10). The metastable excited state can then transfer its excitation to the lasing species with a transfer reaction constant $k_t(T)$ or it can be deactivated at a rate τ_d^{-1} due to all other possible mechanisms. The rate equation for the excited metastable is then

$$\frac{d}{dt} m_U = R_m - \left(k_t n + \frac{1}{\tau_d}\right) m_U$$

where m_U is the excited metastable density (particles per cm^3) and n is the ground state density (particles per cm^3) of the lasing species. If we make the transformation

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$$

as before, we have at steady state

$$v \frac{\partial}{\partial x} m_U + \frac{m_u}{\tau_m} = R_m \tag{48a}$$

where

$$\frac{1}{\tau_m} = \frac{1}{\tau_d} + k_t n \; . \tag{48b}$$

For positions downstream such that $x/v >> \tau_m$, we have

$$m_U \simeq \tau_m R_m \; .$$

Therefore

$$R \simeq a_U \ k_t n \ \tau_m \ R_m$$

or

$$R \simeq a_U \ \frac{k_t n \ \tau_d}{1 + k_t n \ \tau_d} \ R_m \tag{49}$$

where a_U is the branching ratio to the upper lasing level. The pumping efficiency is then given by

$$\eta_p = a_U \ \frac{k_t n \ \tau_d}{1 + k_t n \ \tau_d} \ \frac{E_U}{E_m} \tag{50a}$$

with respect to an input power given by

$$P_{in} = E_m \ R_m \ (b \ L \ x_3) \tag{50b}$$

where E_m is the energy of the excited metastable. In this calculation, as in the previous case, we have ignored the reverse reaction process.

c) Electric Discharge Excitation

The pumping region is assumed to be filled with a uniform discharge characterized by an electric field $\underline{\mathcal{E}}$, a current density $\underline{j}$, and an electron density n_e. The upper lasing level is then excited from the ground state by electron impact. For a wide class of discharges it is found that the excitation rate can be written as proportional to n_e with a constant of proportionality conventionally denoted by ν_x:

$$R = \nu_x n_e \ . \tag{51a}$$

For a small degree of ionization and an electron mean free path much smaller than the characteristic dimensions of the apparatus, one can show by similarity arguments that ν_x (when normalized by the ground state density) is only a function of the ratio of the electric field to the ground state density. That is

$$\frac{\nu_x}{n} = F\left(\frac{\mathcal{E}}{n}\right). \tag{51b}$$

The pumping efficiency for this process is given by

$$\eta_p = \frac{\nu_x n_e \; E_U}{\underline{\mathcal{E}} \cdot \underline{j}} \tag{52a}$$

with respect to an input power given by

$$P_{in} = \underline{\mathcal{E}} \cdot \underline{j} \; (b \; L \; x_3) \; . \tag{52b}$$

These methods are often combined to provide composite pumping methods. For example, one can dissociate SF_6 in a discharge by electron impact to liberate F atoms which in turn react with H_2 molecules to chemically pump an HF laser.

RELAXATION RATES

The relaxation rates τ_{UO}^{-1}, τ_{UL}^{-1}, and τ_L^{-1} were introduced phenomenologically into the rate equations for the upper and lower lasing levels. These rates can be the result of several different processes that can remove an atom (or molecule) of the lasing species from the corresponding level. To the extent that these processes act independently on the corresponding level, we can assign a rate due to each one separately. The total rate is then the sum of these independent rates (insofar as the transition probabilities per unit time add). For example

$$\frac{1}{\tau_L} = \frac{1}{\tau_L^{\,r}} + \frac{1}{\tau_L^{\,c}} + \frac{1}{\tau_L^{\,d}} + \frac{1}{\tau_L^{\,D}} + \cdots \tag{53}$$

where

$\frac{1}{\tau_L^{\,r}}$ = radiative decay rate of the lower lasing level,

$\frac{1}{\tau_L^{\,c}}$ = inelastic collision deactivation rate of the lower lasing level,

$\frac{1}{\tau_L^{\,d}}$ = dissociation rate (of the molecule, in this case) in the lower lasing level,

$\frac{1}{\tau_L^{\,D}}$ = diffusion rate out of the lasing volume.

In turn, each of these can be composite rates. The radiative decay rate, $\frac{1}{\tau_L^{\,r}}$, will be the sum of the rates to all the possible final states that are radiatively connected with the lower lasing level. Hence

$$\frac{1}{\tau_L^{\,r}} = \sum_i A_{Li} \tag{53a}$$

where the A_{Li} are the radiative Einstein coefficients to go from the lower lasing level to the i^{th} final state. The inelastic collision deactivation rate, $\frac{1}{\tau_L^c}$, can be due to a sum of the collision frequencies with each of the possible deactivators present. That is

$$\frac{1}{\tau_L^c} = \sum_i k_{Li} n_i \tag{53b}$$

where the k_{Li} is the collision rate constant of the lasing species in the lower lasing level with the i^{th} deactivator and n_i is the number density of the i^{th} deactivator. The dissociation rate, $\frac{1}{\tau_L^d}$, can be due to radiative dissociation or collisional dissociation.

The diffusion rate out of the lower lasing level, $\frac{1}{\tau_L^d}$, is usually not important in a transverse flow geometry, but with the advent of waveguide lasers, the possibility might arise to consider such a loss. The other rates, τ_{UO}^{-1} and τ_{UL}^{-1}, are defined similarly.

CONCLUSIONS

The results of this analysis can be summarized as follows:

1) There exist two types of lasers:

Type I: $$1 - \tau_U\left[\frac{1}{\tau_L} - \frac{1}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\right] > 0,$$

Type II: $$1 - \tau_U\left[\frac{1}{\tau_L} - \frac{1}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\right] < 0.$$

The performance of a type I lasing system is improved by flow. A type II laser works better as a stationary laser and, in general, should not be flowed.

2) The effects of flow on lasing are confined to a region scaled by ℓ, where

$$\ell = \left(1 + \frac{g_U}{g_L}\right) v\tau$$

and

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_{U0}}\left(\frac{g_U}{g_L}\right).$$

Thus, for an upper state deactivation time of 10 μsec and a flow velocity of 10^5 cm/sec, we would have a lasing region ($g_U/g_L \simeq 1$) of the order of

$$\ell \simeq 2 \times 10^5 \times 10^{-5} = 2 \text{ cm}.$$

3) The minimum pumping rate for lasing to occur is given by

$$R_{th} \gtrsim \frac{\Delta n_{th}}{\tau_U}$$

for a wide range of conditions.

4) For a pumping rate R such that $R >> R_{th}$, the efficiency (with respect to the power supplied to the upper lasing level) is approximately given by

$$\eta_L \lesssim \frac{1}{4}\,\eta_Q\eta_0$$

where η_Q is the quantum efficiency, and η_0 is the optical efficiency (of the order of 1).

APPENDIX 1

Radiation that results from a transition between an upper and a lower state of an atomic or molecular species can be amplified or absorbed. In particular, it is found that the intensity of a beam of frequency ν in the z direction is given by

$$\frac{d}{dz} I_\nu(z) = \alpha(\nu)\ I_\nu(z) \tag{A1.1}$$

where $\alpha(\nu)$ is the gain per unit length (gain coefficient). The gain coefficient is given by

$$\alpha(\nu) = \frac{\lambda^2}{8\pi}\ \frac{g(\nu\ - \nu_0)}{\tau^r_{UL}} \left(n_u - \frac{g_U}{g_L} n_L\right) \tag{A1.2}$$

where

λ = wavelength of the transition

$g(\nu-\nu_0)$ = normalized line-shape function

i.e.,

$$\int_{-\infty}^{\infty} g(\nu - \nu_0)\ d\nu = 1 \tag{A1.3}$$

$\left(\tau^r_{UL}\right)^{-1}$ = spontaneous radiation transition rate from the upper to the lower level

n_U, n_L = densities of particles in the upper and lower states, respectively

g_U, g_L = spectroscopic degeneracies of upper and lower level, respectively.

The gain coefficient can be written as the product of the optical cross section $\sigma^*(\nu - \nu_o)$ and the population inversion Δn, i.e.,

$$\alpha(\nu) \equiv \sigma^* (\nu - \nu_o)\ \Delta n \tag{A1.4}$$

where

$$\sigma^*(\nu - \nu_o) = \frac{\lambda^2}{8\pi} \ \frac{g(\nu - \nu_o)}{\tau^r_{UL}} \tag{A1.5}$$

and

$$\Delta n = n_U - \frac{g_U}{g_L} n_L \ . \tag{A1.6}$$

If the intensity of the radiation is considered as a flux of particles of velocity c (the speed of light) and energy $h\nu$ (where h is Planck's constant and ν is the frequency of the radiation) then we can define the photon density n^*_ν, at the frequency ν, by

$$I_\nu = ch\nu n^*_\nu \ . \tag{A1.7}$$

The stimulated transition rate from the upper to the lower lasing level can then be written as the emission rate minus the absorption rate or,

$$\dot{n}_U = -\dot{n}_L = -\kappa^* n^*_\nu \left(n_U - \frac{g_U}{g_L} n_L\right) \tag{A1.8}$$

where κ^* is the stimulated transition rate given by

$$\kappa^* (\nu - \nu_o) = c\sigma^* (\nu - \nu_o) \ . \tag{A1.9}$$

It is instructive to compare $\kappa^* (\nu - \nu_o)$, as given by Eq. A1.9, with a classical collision rate coefficient, given by

$$k = \int \sigma(v)\ v\ f(\underline{v})\ d\underline{v} = \overline{v\sigma(v)}. \tag{A1.10}$$

Consider now an amplifying medium between $0 < z < L$ (in which $\alpha(\nu)$ can be considered constant) with a pair of mirrors facing each other at $z < 0$ and at $z > L$ forming an optical resonator. See Fig. A1.1. Let the mirrors have reflectivities r_0 and r_L, respectively, and total losses (absorption, diffraction, optical imperfections, etc.) given by $e^{-\varepsilon_0}$ and $e^{-\varepsilon_L}$. If I_i is the incident intensity on the mirror at $z < 0$, then the reflected intensity would be given by

$$I_r = r_0 e^{-\varepsilon_0} I_i$$

while the transmitted intensity would be given by

$$I_t = (1 - r_0)\, e^{-\varepsilon_0} I_i.$$

It is assumed here that the losses are due to mirror surface imperfections and edge diffraction losses as opposed to transition losses.

Let us now start at $z = 0$ with an intensity $I^+(0)$ directed to the right. Integrating Eq. A1.1, we find at $z = L$ the intensity has increased ($\alpha > 0$) to

$$I^+(L) = e^{\alpha L}\, I^+(0). \tag{A1.11a}$$

Part of the radiation is transmitted to the right as output radiation,

$$I^+_{out} = (1 - r_L) e^{-\varepsilon_L}\, I^+(L) \tag{A1.11b}$$

and part of it is reflected back

$$I^-(L) = r_L e^{-\varepsilon_L}\, e^{2\alpha L}\, I^+(L) \tag{A1.11c}$$

$$= r_L e^{-\varepsilon_L}\, e^{\alpha L}\, I^+(0).$$

This is further amplified on its way back, and at $z = 0$, we have

$$I^-(0) = r_L e^{-\varepsilon_L} e^{2\alpha L} I^+(0). \qquad \text{(A1.11d)}$$

Part of this is transmitted to the left as output,

$$I_{out}^- = (1 - r_o) e^{-\varepsilon_0} I^-(0) \qquad \text{(A1.11e)}$$

$$= (1 - r_o) e^{-\varepsilon_0} r_L e^{-\varepsilon_L} e^{2\alpha L} I^+(0) \qquad \text{(A1.11e')}$$

and part of it is reflected back,

$$I^+(0) = r_o e^{-\varepsilon_0} r_L e^{-\varepsilon_L} e^{2\alpha L} I^+(0). \qquad \text{(A1.11f)}$$

We can consider the sequence of steps for a complete round trip inside the cavity, starting from $z = 0$, as a sequence in time. Then it is clear that unless

$$r_o e^{-\varepsilon_0} r_L e^{-\varepsilon_L} e^{2\alpha L} = 1 \qquad \text{(A1.12)}$$

the laser intensity cannot maintain itself at a constant level in time. It must necessarily increase or decrease corresponding to whether the product of the loss and gain factors, on the left hand side of Eq. A1.12, is greater than or less than unity. Thus, Eq. A1.12 is a condition on α for the steady state operation of the laser. As this is also the threshold value of α, below which lasing cannot be sustained, we denote the solution of Eq. A1.12 by α_{th}, where

$$\alpha_{th} = \frac{\varepsilon + \delta}{2L} \qquad \text{(A1.13a)}$$

and

$$\varepsilon = \varepsilon_0 + \varepsilon_L, \tag{A1.13b}$$

$$\delta = \delta_0 + \delta_L \tag{A1.13c}$$

$$= \ln\left(\frac{1}{r_o}\right) + \ln\left(\frac{1}{r_L}\right).$$

The intensity in the interior of the cavity will be given by

$$I(z) = I^+(z) + I^-(z),$$

where

$$I^+(z) = I^+(0)\ e^{\alpha z}$$

and

$$I^-(z) = I^-(0)\ e^{-\alpha z}.$$

Now, since

$$I^+(0) = r_o e^{-\varepsilon_0}\ I^-(0)$$

we have

$$I(z) = \left(r_o e^{-\varepsilon_0}\ e^{\alpha z} + e^{-\alpha z}\right) I^-(0).$$

If we define the total loss factor for the mirror at $z < 0$ by

$$\gamma_0 = \varepsilon_0 + \delta_0,$$

we have

$$I(z) = 2e^{-\frac{1}{2}\gamma_0} \cosh\left(\alpha z - \frac{\gamma_0}{2}\right) I^-(0)$$

or

$$I(z) = 2e^{-\frac{1}{2}\gamma_0} \cosh\left[\frac{1}{2}\left(\gamma\frac{z}{L} - \gamma_0\right)\right] I^-(0), \tag{A1.14}$$

where

$$\gamma_L = \varepsilon_L + \delta_L .$$

Thus, in the limit of a loss-less cavity, the intensity inside the cavity approaches a constant:

$$\lim_{\gamma_0,\gamma_L \to 0} \{I(z)\} = 2I^-(0).$$

For reflectivities which differ appreciably from unity, a situation often encountered in high power lasers, we can see from Eq. A1.14 that I(z) can deviate somewhat from being a constant. In that case, the assumption that the stimulated transition rate, as given by Eq. A1.8 and as used in the rate equations, is only a function of x should be considered as an approximation. The corresponding photon density n^* is actually meant in an average sense, i.e.,

$$n^* = \frac{1}{ch\nu} I_{av}$$

where

$$I_{av} = \frac{1}{L}\int_0^L I(z)\, dz. \qquad \text{(A1.15)}$$

Substituting Eq. A1.14 for I(z), we then obtain

$$I_{av} = 4e^{-\frac{1}{2}\gamma_0} \left(\frac{\sinh\frac{\gamma_0}{2} + \sinh\frac{\gamma_L}{2}}{\gamma_0 + \gamma_L} \right) I^-(0), \qquad \text{(A1.16)}$$

and therefore,

$$I(z) = \frac{1}{2}\left(\frac{\gamma_0 + \gamma_L}{\sinh\frac{\gamma_0}{2} + \sinh\frac{\gamma_L}{2}} \right) \cosh\left[\frac{1}{2}\left(\gamma \frac{z}{L} - \gamma_0 \right)\right] I_{av}. \qquad \text{(A1.17)}$$

The unsymmetric looking argument of the hyperbolic cosine is an artifact of the choice of the origin on the side of the $z < 0$ mirror.

$I(z)/I_{av}$ is plotted in Fig. A1.2 versus z, for various values of γ_0 for the commonly encountered case of a one-sided output ($\gamma_L \approx 0$). As can be seen from Fig. A1.2, the approximation $I(z) \simeq$ constant is a very good one (note displaced origin).

From Eq. A1.16, we have

$$I^-(0) = \frac{(\gamma_0 + \gamma_L)}{4(\sinh\frac{\gamma_0}{2} + \sinh\frac{\gamma_L}{2})} e^{\frac{1}{2}\gamma_0} I_{av},$$

and from Eq. A1.11e, for a one-sided output, we can express the output intensity from the laser in terms of the input intensity and the mirror characteristics. That is (recall $r_0 = e^{-\delta_0}$),

$$I^-_{out} = \frac{1}{\cdot}\left(1 - e^{-\delta_0}\right)\left[\frac{(\delta_0 + \varepsilon_0)}{2\sinh\left(\frac{\delta_0 + \varepsilon_0}{2}\right)} e^{\frac{1}{2}(\delta_0 - \varepsilon_0)}\right] I_{av}. \tag{A1.18}$$

The quantity in the brackets approaches unity as δ_0 and ε_0 go to zero. In particular

$$\frac{(\delta_0 + \varepsilon_0)}{2\sinh\left(\frac{\delta_0 + \varepsilon_0}{2}\right)} e^{\frac{1}{2}(\delta_0 - \varepsilon_0)} \sim 1 + \frac{1}{2}(\delta_0 - \varepsilon_0) + \text{second order}.$$

Therefore, for small δ_0 and ε_0

$$I^-_{out} \to \left\{\frac{\delta_0}{2}\left(1 - \frac{\varepsilon_0}{2}\right) + \text{higher order}\right\} I_{av}.$$

I(x), as used in the main body of this discussion, corresponds to I_{av}, evaluated at each station x.

In this appendix, the algebra that is omitted from the main body of the discussion is presented. Equation numbers on the left refer to the numbering sequence in the main body.

Equation 8

From Eq. 7a, we have

$$\frac{\partial}{\partial x} n_U + \frac{n_U}{v\tau_U} = \frac{R}{v} \tag{A2.1}$$

or

$$\frac{\partial}{\partial x}\left(n_U\, e^{\frac{x}{v\tau_U}}\right) = \frac{R}{v}\, e^{\frac{x}{v\tau_U}} ,$$

and therefore, since $n_U(0) = 0$,

$$n_U\, e^{\frac{x}{v\tau_U}} = \frac{R}{v} \int_0^x e^{\frac{x'}{v\tau_U}}\, dx'$$

$$= \tau_U R\left(e^{\frac{x}{v\tau_U}} - 1\right).$$

Therefore

(8a) $$n_U(x) = \tau_U\, R\left(1 - e^{-x/v\tau_U}\right).$$

Substituting 8a into 7b yields

$$\frac{\partial}{\partial x} n_L + \frac{n_L}{v\tau_L} = \frac{\tau_U R}{v\tau_{UL}}\left(1 - e^{-x/v\tau_U}\right).$$

or

$$\frac{\partial}{\partial x}\left(n_L\, e^{x/v\tau_L}\right) = \frac{\tau_U R}{v\tau_{UL}}\left(1 - e^{-x/v\tau}\right) e^{x/v\tau_L}.$$

Since $n_L(0) = 0$,

$$n_L(x)\, e^{x/v\tau_L} = \frac{\tau_U R}{v\tau_{UL}}\left\{\int_0^x e^{x'/v\tau_L}\,dx' - \int_0^x e^{\frac{x'}{v}\left(\frac{1}{\tau_L} - \frac{1}{\tau_U}\right)}\,dx'\right\}$$

$$= \frac{\tau_U R}{\tau_{UL}}\left\{\tau_L\left(e^{x/v\tau_L} - 1\right) - \frac{e^{\frac{x}{v}\left(\frac{1}{\tau_L} - \frac{1}{\tau_U}\right)} - 1}{\frac{1}{\tau_L} - \frac{1}{\tau_U}}\right\}$$

$$= \frac{\tau_U\tau_L\; R}{\tau_{UL}}\left\{e^{\frac{x}{v\tau_L}} - 1 - \frac{e^{\frac{x}{v}\left(\frac{1}{\tau_L} - \frac{1}{\tau_U}\right)} - 1}{1 - \tau_L/\tau_U}\right\}.$$

Therefore

$$n_L(x) = \frac{\tau_U\tau_L}{\tau_{UL}} R\left\{1 - e^{-x/v\tau_U} - \frac{e^{-x/v\tau_U} - e^{-x/v\tau_L}}{1 - \tau_L/\tau_U}\right\};$$

combining the exponential then yields

(8b) $$n_L(x) = \frac{\tau_U\tau_L}{\tau_{UL}} R\left\{1 - \frac{e^{-\frac{x}{v\tau_L}}}{1 - \tau_U/\tau_L} + \frac{e^{-\frac{x}{v\tau_U}}}{\tau_L/\tau_U - 1}\right\}.$$

<u>Equation 9</u>

$$\frac{n_U(x)}{\tau_L R} = \frac{\tau_U}{\tau_L}\left(1 - e^{-x/v\tau_U}\right) \qquad \text{(A2.3)}$$

while from Eq. 8b

$$\left(\frac{g_U}{g_L}\right)\frac{n_L(x)}{\tau_L R} = \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\left\{1 - \frac{e^{-x/v\tau_L}}{1 - \tau_U/\tau_L} + \frac{\tau_U/\tau_L}{1 - \tau_U/\tau_L} e^{-x/v\tau_U}\right\} . \qquad (A2.4)$$

Subtracting (A2.4) from (A2.3) and collecting terms yields

(9)

$$\frac{\Delta n(x)}{\tau_U R} = \frac{n_U(x) - g_U/g_L \; n_L(x)}{\tau_L R}$$

$$= \frac{\tau_U}{\tau_L} - \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)$$

$$- \left[\frac{\tau_U}{\tau_{UL}} + \frac{\tau_U/\tau_L}{1 - \tau_U/\tau_L}\left(\frac{\tau_U}{\tau_{UL}}\right)\left(\frac{g_U}{g_L}\right)\right] e^{-x/v\tau_U}$$

$$+ \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right)\frac{e^{-x/v\tau_L}}{1 - \tau_U/\tau_L} .$$

Equations 11 and 12

Adding Eq. 10a and Eq. 10b yields

$$v \frac{\partial}{\partial x}\left(n_U + n_L\right) + \frac{n_U}{\tau_U} + \frac{n_L}{\tau_L} - \frac{n_U}{\tau_{UL}} = R. \qquad (A2.5)$$

From defining Eq. 5 for τ_U, we have

$$\frac{1}{\tau_U} - \frac{1}{\tau_{UL}} = \frac{1}{\tau_{U0}} ,$$

therefore

$$v \frac{\partial}{\partial x}\left(n_U + n_L\right) + \frac{n_U}{\tau_{U0}} + \frac{n_L}{\tau_L} = R. \tag{A2.6}$$

Substituting Eq. 10c into A2.6 yields

$$\left(1 + \frac{g_U}{g_L}\right) v \frac{\partial}{\partial x} n_L + v \frac{\partial}{\partial x} \Delta n_{th} + \frac{\Delta n_{th}}{\tau_{U0}} + \left[\frac{1}{\tau_L} + \frac{1}{\tau_{U0}}\left(\frac{g_U}{g_L}\right)\right] n_L = R. \tag{A2.7}$$

For plane parallel optics $\frac{\partial}{\partial x} \Delta n_{th} = 0$, and hence

(11)

$$\left(1 + g_U/g_L\right) v \frac{n_L}{\partial x} + \frac{n_L}{\tau} = R - \frac{\Delta n_{th}}{\tau_{U0}} \tag{A2.8}$$

where, from Eq. A2.7

(12)

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_{U0}}\left(\frac{g_U}{g_L}\right) .$$

<u>Equations 13 and 14</u>

From Eq. A2.8 we have

$$\frac{\partial n_L}{\partial x} + \frac{n_L}{\ell} = \frac{R - \Delta n_{th}/\tau_{U0}}{(1 + g_U/g_L) v} , \tag{A2.9}$$

where

(14)

$$\ell \equiv (1 + g_U/g_L)\ v\tau.$$

From Eq. A2.9 we have

$$\frac{\partial}{\partial x}\left(n_L e^{x/\ell}\right) = \frac{R - \Delta n_{th}/\tau_{U0}}{(1 + g_U/g_L) v}\, e^{x/\ell} ,$$

or integrating from x_1 to x,

$$n_L(x)\, e^{x/\ell} - n_L(x_1)\, e^{x_1/\ell} = \frac{R - \Delta n_{th}/\tau_{U0}}{(1 + g_U/g_L)v}\, \ell \left(e^{x/\ell} - e^{x_1/\ell}\right).$$

Therefore, using Eq. 14 for ℓ, we have

$$\text{(13a)} \qquad n_L(x) = n_L(x_1)\, e^{-\frac{x-x_1}{\ell}} + \tau\left(R - \frac{\Delta n_{th}}{\tau_{U0}}\right)\left(1 - e^{-\frac{x-x_1}{\ell}}\right).$$

Substituting into Eq. 10c then yields

$$\text{(13b)} \quad n_U(x) = \Delta n_{th} + \frac{g_U}{g_L}\left\{\Delta n_L(x_1)\, e^{-\frac{x-x_1}{\ell}} + \tau\left(R - \frac{\Delta n_{th}}{\tau_{U0}}\right)\left(1 - e^{-\frac{x-x_1}{\ell}}\right)\right\}.$$

<u>Equation 15</u>

From Eq. 10a, we have

$$n^*(x) = \frac{1}{\kappa^* \Delta n_{th}}\left\{ R - \left(v \frac{\partial}{\partial x} n_U + \frac{n_U}{\tau_U}\right)\right\}. \tag{A2.10}$$

From the functional form for $n_U(x)$, as given by Eq. 13b and Eq. A2.10 for $n^*(x)$, we can see that

$$\text{(15a)} \qquad n^*(x) = \left(n_1^* - n_\infty^*\right)e^{-\frac{x-x_1}{\ell}} + n_\infty^*.$$

We can easily calculate n_∞^* from Eq. A2.10 by observing that $\frac{\partial}{\partial x} n_U(x) \to 0$ as $x \to \infty$, i.e.,

$$n^*(\infty) = n^* = \frac{1}{\kappa^* \Delta n_{th}}\left\{ R - \frac{1}{\tau_U} n_U(\infty) \right\}. \tag{A2.11}$$

From Eq. 13b we have

$$n_U(\infty) = \Delta n_{th} + \left(\frac{g_U}{g_L}\right) \tau \left(R - \frac{\Delta n_{th}}{\tau_{U0}}\right),$$

and therefore

$$R - \frac{n_U(\infty)}{\tau_U} = R - \frac{g_U}{g_L}\frac{\tau}{\tau_U} R - \frac{\Delta n_{th}}{\tau_U}\left(1 - \frac{\tau}{\tau_{U0}}\frac{g_U}{g_L}\right)$$

$$= R\left(1 - \frac{\tau}{\tau_U}\frac{g_U}{g_L}\right) - \frac{\Delta n_{th}}{\tau_U}\left(1 - \frac{\tau}{\tau_{U0}}\frac{g_U}{g_L}\right).$$

Now, from Eq. 12

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_{U0}}\left(\frac{g_U}{g_L}\right),$$

hence

$$1 = \frac{\tau}{\tau_L} + \frac{\tau}{\tau_{U0}}\left(\frac{g_U}{g_L}\right)$$

and

$$1 - \frac{\tau}{\tau_{U0}}\left(\frac{g_U}{g_L}\right) = \frac{\tau}{\tau_L} .$$

Therefore

$$R - \frac{1}{\tau_U} n_U(\infty) = \left[1 - \frac{\tau}{\tau_U}\left(\frac{g_U}{g_L}\right)\right] R - \frac{\Delta n_{th}}{\tau_U \tau_L} . \qquad \text{(A2.12)}$$

Substituting A2.12 into A2.11 yields

$$n_\infty^* = \frac{R}{\kappa^* \Delta n_{th}} \left\{ 1 - \frac{\tau}{\tau_U}\left(\frac{g_U}{g_L}\right) - \frac{\tau}{\tau_U}\frac{\Delta n_{th}}{\tau_L R} \right\} . \qquad \text{(A2.13)}$$

From the lasing condition, as given by Eq. 3, we have

$$\kappa^* \Delta n_{th} = \frac{c}{2L} \; (\varepsilon + \delta), \tag{A2.14}$$

and finally

(15c) $$n^*_\infty = \frac{2LR}{c(\varepsilon + \delta)} \left\{ 1 - \frac{\tau}{\tau_U} \left(\frac{g_U}{g_L}\right) - \frac{\tau}{\tau_U} \frac{\Delta n_{th}}{\tau_L R} \right\} .$$

From Eq. A1.10 we have

$$n^*(x_1^+) = \frac{R}{\kappa^* \Delta n_{th}} \left\{ 1 - \left[\frac{v}{R} \left(\frac{\partial n_U}{\partial x}\right)_{x_1} + \left(\frac{n_U}{\tau_U R}\right)_{x_1} \right] \right\} . \tag{A2.15}$$

Now, from Eq. 13b

$$\left(\frac{\partial n_U}{\partial x}\right)_{x_1^+} = \frac{g_U}{g_L} \left\{ - \frac{n_1(x_1)}{\ell} + \frac{\tau}{\ell} \left(R - \frac{\Delta n_{th}}{\tau_{U0}} \right) \right\} ,$$

and from the threshold condition attained at x_1

$$\frac{g_U}{g_L} n_L(x_1) = n_U(x_1) - \Delta n_{th} .$$

Therefore,

$$\left(\frac{\partial n_U}{\partial x}\right)_{x_1^+} = - \frac{1}{\ell} \left[n_U(x_1) - \Delta n_{th} \right] + \frac{\tau}{\ell} \left(\frac{g_U}{g_L}\right) \left(R - \frac{\Delta n_{th}}{\tau_{U0}} \right)$$

or

$$\frac{v}{R}\left(\frac{\partial n_U}{\partial x}\right)_{x_1^+} = \frac{1}{1 + g_U/g_L}\left\{ \frac{g_U}{g_L}\left(1 - \frac{\Delta n_{th}}{\tau_{U0}R}\right) - \frac{n_U(x_1)}{\tau R} + \frac{\Delta n_{th}}{\tau R} \right\}$$

$$= \frac{1}{1 + g_U/g_L}\left\{ \frac{g_U}{g_L} - \frac{n_U(x_1)}{\tau R} + \frac{\Delta n_{th}}{\tau_L R} \right\}.$$

Hence

$$\frac{v}{R}\left(\frac{\partial n_U}{\partial x}\right)_{x_1^+} + \left(\frac{n_U}{\tau_U R}\right)_{x_1} = \frac{1}{1 + g_U/g_L}\left\{ \frac{g_U}{g_L} + \frac{n_U(x_1)}{\tau_U R}\left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right) + \frac{\Delta n_{th}}{\tau_L R} \right\},$$

and since (from Eq. 8a)

$$\frac{n_U(x_1)}{\tau_U R} = 1 - e^{-x_1/v\tau_U},$$

we have

$$1 - \frac{v}{R}\left(\frac{\partial n_U}{\partial x}\right)_{x_1^+} - \left(\frac{n_U}{\tau_U R}\right)_{x_1} = \frac{1}{1 + g_U/g_L}\left\{ 1 - \left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right)\left(1 - e^{-x_1/v\tau_U}\right) - \frac{\Delta n_{th}}{\tau_L R} \right\}. \quad \text{(A2.16)}$$

Now, from the threshold condition as given by Eq. A2.14 and from Eqs. A2.15 and A2.16, we obtain

(15b) $$n_1^* = n^*(x_1^+) = \frac{2LR}{c(\varepsilon + \delta)(1 + g_U/g_L)}\left\{ 1 - \left(1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau}\right)\left(1 - e^{-x_1/v\tau_U}\right) - \frac{\Delta n_{th}}{\tau_L R} \right\}.$$

Equation 16

From $n^*(x_2) = 0$, or

$$\left(n_1^* - n_\infty^*\right) e^{-\frac{x_2 - x_1}{\ell}} + n_\infty^* = 0$$

we have (recall that in this case $n_\infty^* < 0$)

$$-\frac{n_1^* - n_\infty^*}{-n_\infty^*} = e^{-\frac{x_2 - x_1}{\ell}} .$$

Therefore

$$\frac{x_2 - x_1}{\ell} = \ln\left(\frac{n_1^*}{-n_\infty^*} + 1\right) . \qquad (16)$$

Equations 17 and 18

Equation 17 follows from Eq. 15b by making the substitutions $R = R_{th}$, $x = x_{th}$, and requiring that $n_1^* = 0$. Eq. 18 follows from Eq. 9 by substituting $\Delta n(x) = \Delta n_{th}$, $R = R_{th}$, $x = x_{th}$.

Equations 17' and 18'

From the coefficient of $\left(1 - e^{-x_{th}/v\tau_U}\right)$ in Eq. 17, we have

$$1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau} = 1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau_L} - \frac{\tau_U}{\tau_{U0}}\left(\frac{g_U}{g_L}\right)$$

$$= 1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau_L} - \left(1 - \frac{\tau_U}{\tau_{UL}}\right)\frac{g_U}{g_L}$$

$$= 1 - \frac{\tau_U}{\tau_L} + \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right).$$

Substituting

$$w = \frac{\tau_U}{\tau_{UL}}\left(\frac{g_U}{g_L}\right) \qquad (19)$$

and

(19c) $$s = \frac{\tau_U}{\tau_L},$$

we have

$$1 + \frac{g_U}{g_L} - \frac{\tau_U}{\tau_L} = 1 - s + w. \tag{A2.17}$$

Therefore, from Eq. 17, we obtain

$$\frac{1}{\theta_L} = 1 - (1 - s + w)(1 - q_{th})$$

$$= 1 - 1 + s - w + (1 - s + w)\, q_{th}.$$

Hence

(17') $$\phi = (1 - s + w)\, q_{th}$$

where

(19e) $$\phi = \frac{1}{\theta_L} - s + w.$$

Similarly, from Eq. 18, we have

$$\frac{1}{\theta_L} = s - w - \left(s + \frac{s}{1-s} w\right) q_{th} + \frac{w}{1-s} q_{th}^{s},$$

from which it follows that

(18') $$\phi = \frac{w}{1-s} q_{th}^{s} - \frac{s}{1-s} (1 - s + w)\, q_{th}.$$

Equation 21

Substituting Eq. 20 into 18', we have

$$\phi = \frac{w}{1-s}\left(\frac{\phi}{1-s+w}\right)^{s} - \frac{s}{1-s}\left(1 - s + w\right)\left(\frac{\phi}{1-s+w}\right)$$

$$\phi = \frac{1}{1-s}\left\{ w\left(\frac{\phi}{1-s+w}\right)^{s} - s\phi \right\}$$

or

(21) $$\phi - w\left(\frac{\phi}{1-s+w}\right)^{s} = 0.$$

Eq. 21 has two solutions if $1 - s + w > 0$. These are

$$\phi_1 = 0$$

and

$$\phi_2 = w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}},$$

corresponding to

$$\frac{1}{\theta_L} = s - w$$

and

$$\frac{1}{\theta_L} = w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}} + s - w.$$

In this case

$$w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}} + s - w > s - w,$$

and consequently, the minimum pumping rate corresponds to ϕ_2.

Equation 23b

From Eq. 23a

$$\phi = w\left(1 + \frac{1-s}{w}\right)^{-\frac{s}{1-s}}, \qquad \text{(A2.18)}$$

and hence for small s

$$\phi(s, w) = \phi(0, w) + \frac{\partial}{\partial s} \phi(0,w) + 0(s^2). \tag{A2.19}$$

From Eq. A2.18, we have that

$$\phi(0, w) = w$$

and

$$\left(\frac{\partial \phi}{\partial s}\right)_{s=0} = \left(\phi \frac{\partial}{\partial s} \ell n\ \phi\right)_{s=0}$$

$$= w\ \ell n \left(1 + \frac{1}{w}\right).$$

Therefore

$$\phi(w, s) = w - sw.\ \ell n \left(1 + \frac{1}{w}\right) + 0\ s^2 \quad .$$

Substituting for ϕ

$$\frac{1}{\theta_L} - s + \cancel{w} = \cancel{w} - sw\ \ell n \left(1 + \frac{1}{w}\right) + 0\left(s^2\right),$$

or finally,

$$\frac{1}{\theta_L} \sim s \left[1 - w\ \ell n \left(1 + \frac{1}{w}\right)\right]. \tag{23b}$$

<u>Equation 25b</u>

From Eq. 25a, we have that as $w \to 0$

$$\frac{1}{\theta_U} \to 1 + w\ \ell n\ w.$$

Since

$$\lim_{w \to 0} \{w \ln w\} = 0$$

we have

$$\lim_{w \to 0} \left\{\frac{1}{\theta_u}\right\} = 1. \qquad \text{(A2.20)}$$

For large w,

$$\frac{1}{\theta_u} \sim 1 - w\left[\frac{1}{w} - \frac{1}{2}\left(\frac{1}{w}\right)^2 + \frac{1}{3}\left(\frac{1}{w}\right)^3 - \cdots\right]$$

$$= 1 - \left[1 - \frac{1}{2w} + \frac{1}{3w^2} - \cdots\right]$$

$$= \frac{1}{2w} - \frac{1}{3w^2} + \cdots$$

$$= \frac{1}{2w}\left(1 - \frac{2}{3w} + \cdots\right).$$

Therefore

$$\theta_U (0, w) \sim \frac{2w}{1 - \frac{2}{3w} + \cdots}$$

$$= 2w\left[1 + \frac{2}{3w} + 0\ (w^2)\right]$$

$$= 2w + \frac{4}{3} + 0\left(\frac{1}{w}\right).$$

As $w \to \infty$

$$\theta_U (0, w) \sim 2w + \frac{4}{3}. \qquad \text{(A2.21)}$$

In fact, Eq. A2.21 is good for $w > 0.1$ (see Fig. 4).

Equation 26

Since $\theta_U = s\theta_L$, Eq. 26 follows from 23 by dividing the entries for θ_L^{-1} by s.

Equation 30

Eq. 28 follows from Eq. A1.18 by substitution of I(x) for I_{av}. Eq. 29 assumes that the mirrors extend sufficiently to justify neglecting any variations in the y-direction. Eq. 30a follows by substituting 27 and 28 into 29. From Eq. 15a

$$\int_{x_1}^{x_{2,3}} n^*(x)dx = \left(n_1^* - n_\infty^*\right) \int_{x_1}^{x_{2,3}} e^{-\frac{x - x_1}{\ell}} dx + n_\infty^* \int_{x_1}^{x_{2,3}} dx$$

$$= \ell \left[\left(n_1^* - n_\infty^*\right)\left(1 - e^{-\xi}\right) + \xi n_\infty^*\right],$$

where

$$\xi = \frac{x_{2,3} - x_1}{\ell}.$$

Equation 37

Since $\Delta n(x_1) = \Delta n_{th}$, we have

$$\frac{\Delta n(x_1)}{\tau_L R} = \frac{\Delta n_{th}}{\tau_L R}$$

$$= \frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right).$$

Therefore, from Eq. 9, we have

$$\frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right) = s - w - \left(s + \frac{s}{1-s} w\right) q + \frac{s}{1-s} q^s$$

$$= s - w - \frac{s}{1-s}\left(1 - s + w\right) q + \frac{w}{1-s} q^s$$

$$= \frac{1}{1-s}\left[\left(1 - s\right)\left(s - w\right) - s\left(1 - s + w\right) q + wq^s\right]$$

$$= \frac{1}{1-s}\left[s - w - s\left(s-w\right) - s\left(1 - s + w\right) q + wq^s\right]$$

$$= \frac{1}{1-s}\left[w\left(q^s - 1\right) - s\left(1 - s + w\right)\left(q - 1\right)\right]$$

or

$$\left(q^s - 1\right) + \frac{s}{w}\left(1 - s + w\right)\left(1 - q\right) = \left(\frac{1 - s}{w}\right)\frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right) , \qquad \text{(A2.22)}$$

from which Eq. 37 follows trivially.

Equation 38

As $R_{th}/R \to 0$, Eq. A2.22 becomes of the form

$$q^s + sa\ (1 - q) - 1 = 0, \qquad \text{(A2.23a)}$$

where

$$a = 1 + \frac{1 - s}{w} . \qquad \text{(A2.23b)}$$

The solution of Eq. A2.23a may be viewed as the intersection of two curves:

$$y_1 = q^s \qquad \text{(A2.24a)}$$

and

$$y_2 = saq + (1 - sa). \tag{A2.24b}$$

Independently of s and a, $q = 1$ is always a solution of Eq. A2.23a. In addition, a second solution may exist for $0 < q < 1$. Since $q = e^{-x_1/v\tau_U}$, the $q = 1$ solution means that as $\frac{R_{th}}{R} \to 0$ the laser turns on immediately (i.e., $x_1 \to 0$). For $s < 1$, the $0 < q < 1$ solution corresponds to the position downstream where $\Delta n(x)$ would again be equal to Δn_{th} if the lasing did not start. Recall that for $s < 1$, $\tau_U < \tau_L$ and the laser eventually bottlenecks. Thus, in all cases $x_1 \to 0$ as $\frac{R_{th}}{R} \to 0$.

To see what happens for large but finite pumping rates, we make an expansion about the $R = \infty$ solution, i.e.,

$$q = 1 - \chi.$$

Then

$$\begin{aligned} q^s &= (1 - \chi)^s \\ &= 1 - s \quad + 0\,(\chi^2), \end{aligned}$$

and substituting into Eq. A2.22, we have

$$- s\chi + \frac{s}{w}(1 - s + w)\,\chi = \left(\frac{1 - s}{w}\right)\frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right) + 0\,(\chi^2)$$

or

$$\left(\frac{1 - s}{w}\right) s\chi = \left(\frac{1 - s}{w}\right)\frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right)^2.$$

If $s \neq 1$

$$s\chi = \frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right)^2$$

or

$$\chi = \frac{1}{s\,\theta_L}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right)^2.$$

Therefore, since $\theta_U = s\theta_L$

(38) $$q = 1 - \frac{1}{\theta_U}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right)^2.$$

Equation 39

From Eq. 38, for $\frac{R_{th}}{R} \to 0$,

$$1 - e^{-x_1/v\tau_U} \doteq 1 - q = \frac{1}{\theta_U}\left(\frac{R_{th}}{R}\right) + 0\left(\frac{R_{th}}{R}\right),$$

therefore,

$$\lim_{R\to\infty}\left\{1 - e^{-x_1/v\tau_U}\right\} = 0.$$

Similarly

$$\frac{\Delta n_{th}}{\tau_L R} = \frac{1}{\theta_L}\left(\frac{R_{th}}{R}\right)$$

and therefore, for finite θ_L and $\frac{R_{th}}{R} \to 0$,

$$\lim_{R\to\infty}\left\{\frac{\Delta n_{th}}{\tau_L R}\right\} = 0.$$

Therefore, from Eq. 36

$$\eta_{S\infty} \equiv \lim_{R\to\infty}\left\{\eta_S\right\} = \frac{\ell}{\left(1 + \frac{g_U}{g_L}\right)x_3}\left\{\left[1 + \left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1\right)\left(1 - e^{-\xi}\right)\right.\right.$$

$$\left.\left. - \xi\left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1\right)\right]\right\}. \qquad \text{(A2.25)}$$

If we now define

(39b) $$\zeta = \left(1 + \frac{g_U}{g_L}\right)\left(\frac{\tau}{\tau_U}\frac{g_U}{g_L} - 1\right),$$

then Eq. A2.25 can be written

(39a) $$\eta_{S\infty} = \frac{x_{2,3}}{\left(1 + \frac{g_U}{g_L}\right)x_3}\left\{(1 + \zeta)\left(\frac{1 - e^{-\xi}}{\xi}\right) - \zeta\right\}$$

where, since $x_1 \to 0$ as $R \to \infty$, we have used

$$\lim_{R\to\infty}\{\xi\} = \lim_{R\to\infty}\left\{\frac{x_{2,3} - x_1}{\ell}\right\} = \frac{x_{2,3}}{\ell}. \quad \text{(A2.26)}$$

Equation 40

For $\xi << 1$, we have

$$\frac{1 - e^{-\xi}}{\xi} = \frac{1}{\xi}\left[1 - \left(1 - \xi + \frac{1}{2}\xi^2 - \cdots\right)\right]$$

$$= \frac{1}{\xi}\left[\xi - \frac{1}{2}\xi^2 + \cdots\right]$$

$$= 1 - \frac{1}{2}\xi + 0(\xi^2).$$

Therefore, for Eq. 39a for $x_{2,3} = x_3$,

$$\lim_{\xi\to 0}\left\{\eta_{S\infty}(\xi;\zeta)\right\} = \eta_{S\infty}(0;\zeta) = \frac{1}{1 + \frac{g_U}{g_L}}\left\{(1 + \zeta) - \zeta\right\}$$

$$= \frac{1}{1 + \frac{g_U}{g_L}}.$$

Equation 41

For $\xi >> 1$, we have

$$\frac{1 - e^{-\xi}}{\xi} \sim \frac{1}{\xi}$$

and therefore, from Eq. 39a for $x_{2,3} = x_3$

$$\lim_{\xi\to\infty} \left\{ \eta_{S\infty} (\xi;\zeta) \right\} = \eta_{S\infty} (\infty;\zeta) = \frac{-\zeta}{1 + \frac{g_U}{g_L}} . \tag{A2.26}$$

Using the defining Eq. 39b, we then have

(41) $$\eta_{S\infty} = 1 - \frac{\tau}{\tau_U} \left(\frac{g_U}{g_L}\right) .$$

To show the second part of Eq. 41, we observe that

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_{U0}} \left(\frac{g_U}{g_L}\right) ,$$

and therefore

$$1 = \frac{\tau}{\tau_L} + \frac{\tau}{\tau_{U0}} \frac{g_U}{g_L}$$

$$= \frac{\tau}{\tau_L} + \tau \left(\frac{1}{\tau_U} - \frac{1}{\tau_{UL}}\right) \frac{g_U}{g_L} .$$

Consequently

$$1 = \frac{\tau}{\tau_U} \left(\frac{g_U}{g_L}\right) = \frac{\tau}{\tau_L} - \frac{\tau}{\tau_{UL}} \left(\frac{g_U}{g_L}\right) . \tag{A2.27}$$

Equation 42

This equation is essentially derived in the main body.

Equation 43

From the power series expansion for the natural logarithm, we have

$$\ell n\left(1+\frac{1}{\zeta}\right)=\frac{1}{\zeta}-\frac{1}{2\zeta^2}+\frac{1}{3\zeta^3}-\cdots$$

$$=\frac{1}{\zeta}\left[1-\frac{1}{2\zeta}+\frac{1}{3\zeta^2}+0\left(\frac{1}{\zeta^3}\right)\right].$$

Therefore

$$\frac{1}{\ell n\left(1+\frac{1}{\zeta}\right)}=\frac{\zeta}{1-\frac{1}{2\zeta}+\frac{1}{3\zeta^2}+0\left(\frac{1}{\zeta^3}\right)}$$

$$=\zeta\left[1+\frac{1}{2\zeta}-\frac{1}{3\zeta^2}+\frac{1}{4\zeta^2}+0\left(\frac{1}{\zeta^3}\right)\right]$$

$$=\zeta\left[1+\frac{1}{2\zeta}-\frac{1}{12\zeta^2}+0\left(\frac{1}{\zeta^3}\right)\right]$$

$$=\zeta+\frac{1}{2}-\frac{1}{12\zeta}+0\left(\frac{1}{\zeta^2}\right),$$

and hence

(43) $$\frac{1}{\ell n\left(1+\frac{1}{\zeta}\right)}-\zeta=\frac{1}{2}-\frac{1}{12\zeta}+0\left(\frac{1}{\zeta^2}\right).$$

Equation 42' follows by using the first term.

TABLE I

$$\theta_U = \frac{\tau_U R_{th}}{\Delta n_{th}} \text{ as a function of: } \quad s \equiv \frac{\tau_U}{\tau_L} \text{ and } w \equiv \frac{g_U/g_L}{1 + \tau_{UL}/\tau_{UO}}$$

Boxed values of θ_U are those corresponding to type II lasers.

s \ w	.001	.003	.01	.03	.1	.3	1.0	3.0	10.0
.001	1.007	1.017	1.048	1.119	1.316	1.786	3.257	7.299	21.28
.003	1.007	1.017	1.048	1.119	1.314	1.786	3.257	7.299	21.28
.01	1.007	1.017	1.047	1.117	1.312	1.783	3.257	7.299	21.28
.03	1.006	1.016	1.046	1.115	1.309	1.773	3.247	7.299	21.28
.1	1.005	1.014	1.041	1.105	1.292	1.748	3.205	7.246	21.28
.3	1.003	1.009	1.029	1.080	1.245	1.675	3.106	7.143	21.28
1.0	1.001	1.003	1.009	1.028	1.100	1.370	2.660	6.623	20.41
3.0	1.0	1.001	1.003	1.01	1.034	1.111	1.499	5.181	19.23
10.0	1.0	1.0	1.001	1.003	1.01	1.03	1.111	1.429	12.99
30.0	1.0	1.0	1.0	1.001	1.003	1.01	1.034	1.111	1.499
100.	1.0	1.0	1.0	1.0	1.001	1.003	1.01	1.03	1.111

TABLE II

$$\frac{x_{th}}{v\tau_U} \text{ as a function of: } \quad s \equiv \frac{\tau_U}{\tau_L} \text{ and } w \equiv \frac{g_U/g_L}{1 + \tau_{UL}/\tau_{U0}}$$

s \ w	.001	.003	.01	.03	.1	.3	1.0	3.0	10.0
.001	6.92	5.82	4.62	3.54	2.40	1.47	.693	.288	.095
.003	6.93	5.83	4.63	3.54	2.40	1.47	.694	.288	.095
.01	6.97	5.86	4.65	3.56	2.41	1.47	.695	.288	.095
.03	7.09	5.96	4.73	3.62	2.44	1.49	.699	.289	.095
.1	7.56	6.34	5.01	3.82	2.56	1.54	.713	.292	.096
.3	9.36	7.80	6.09	4.56	2.97	1.72	.758	.300	.097
1.0	1000	333	100	33	10	3.3	1.0	.333	.100
3.0	∞	∞	∞	∞	∞	∞	∞	.549	.112
10.0	∞	∞	∞	∞	∞	∞	∞	∞	.256

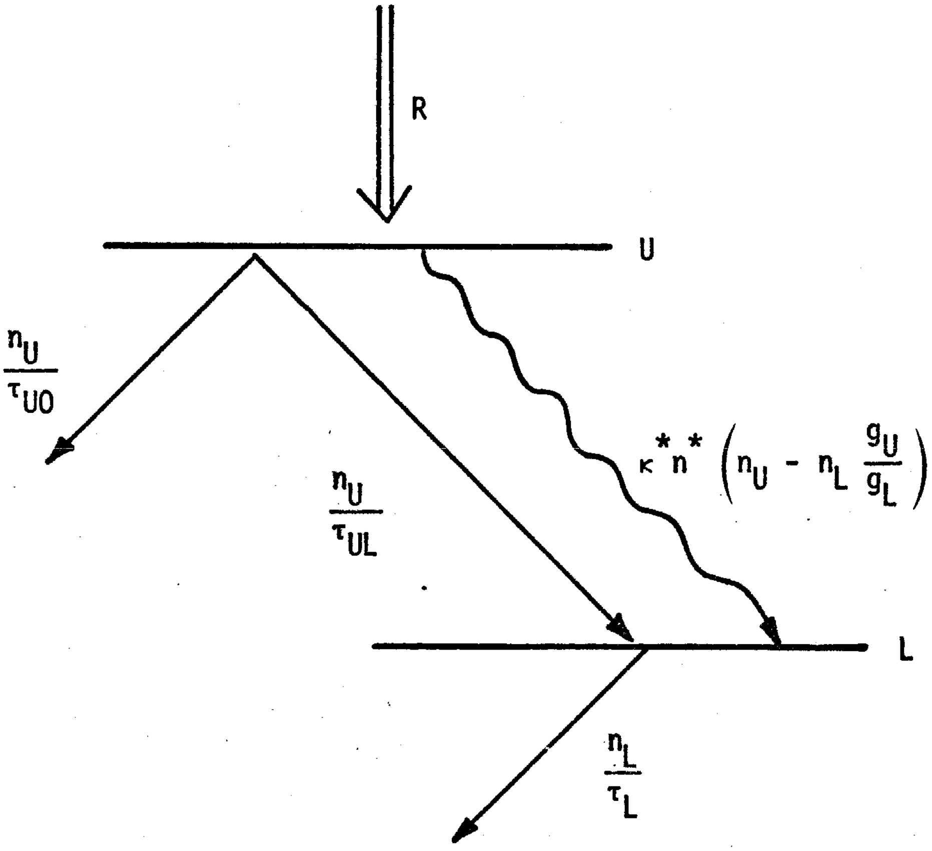

Figure 1. Energy Level Model for Lasing Species

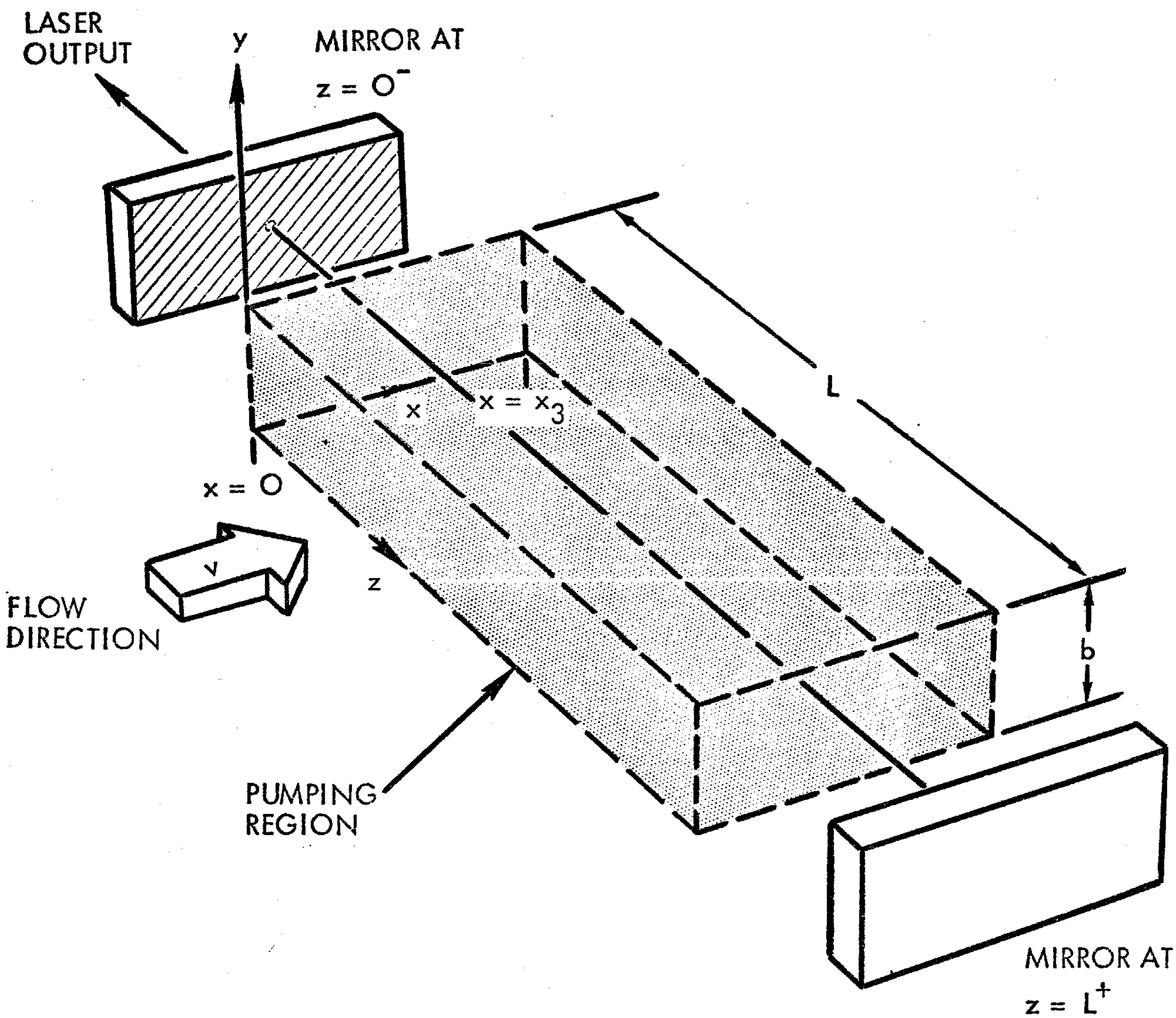

Figure 2. Laser System Geometry

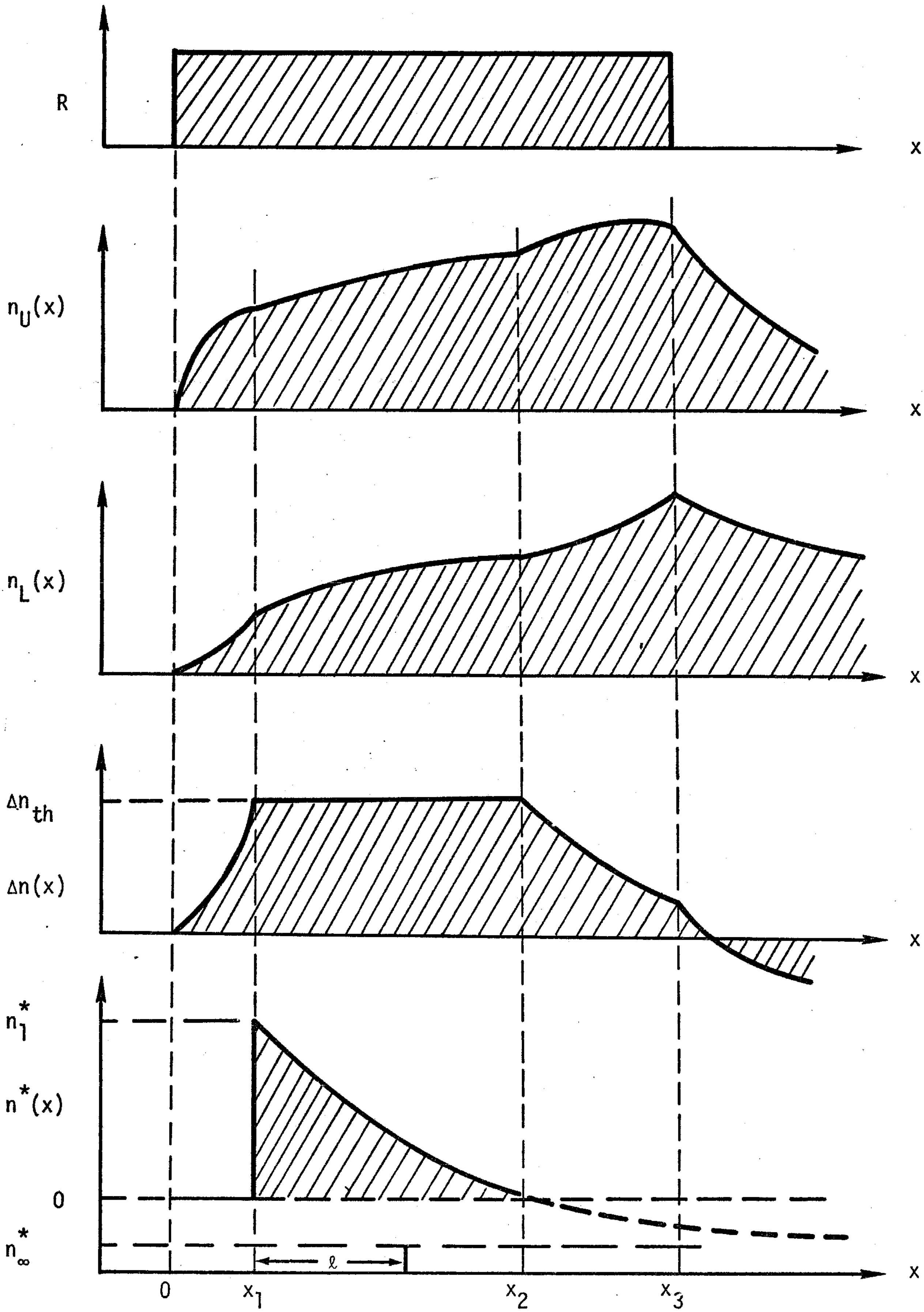

Figure 3. Rate Equation Solution

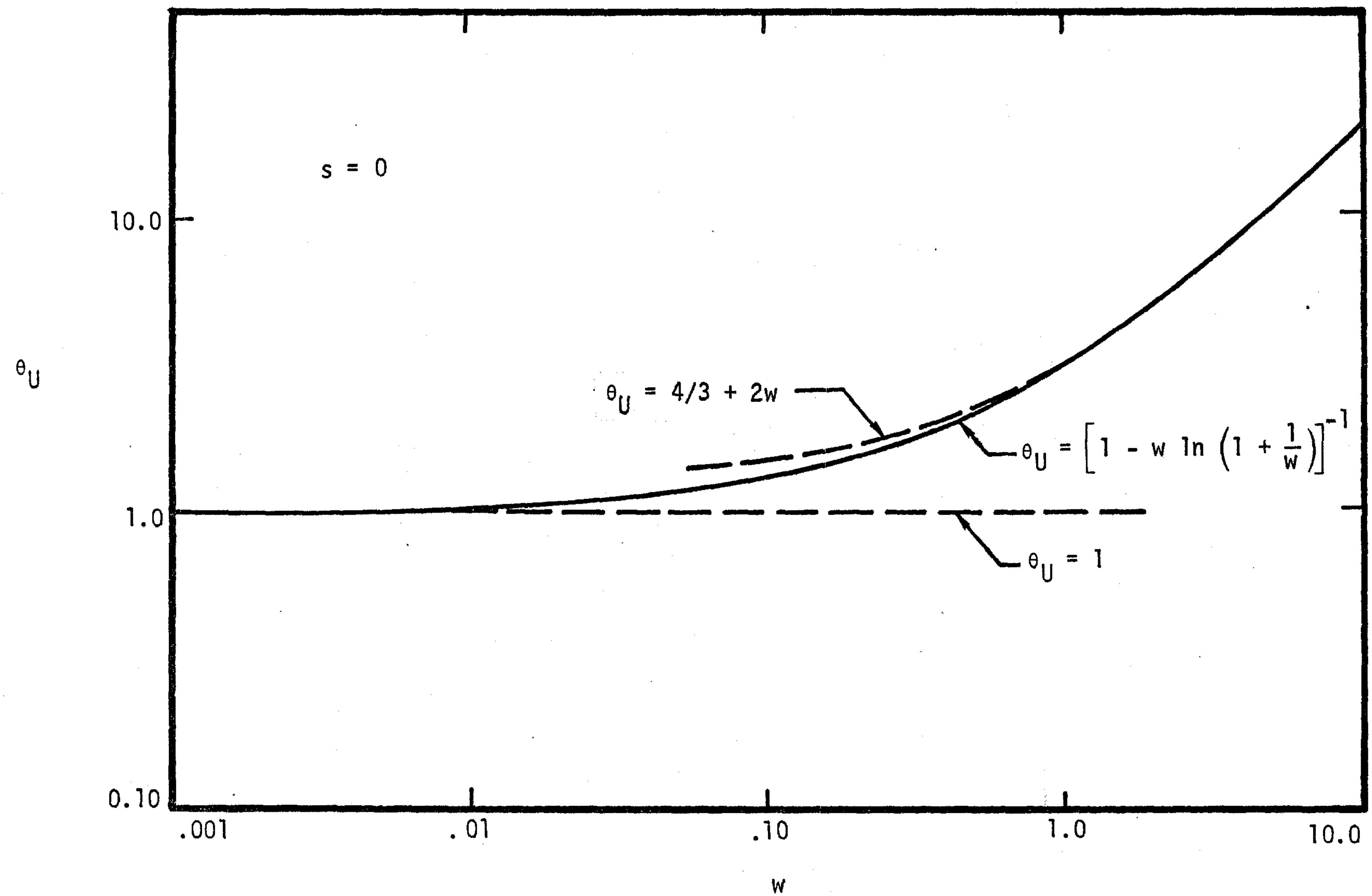

Figure 4. The Threshold Function $\theta_U(s,w)$ for s = 0

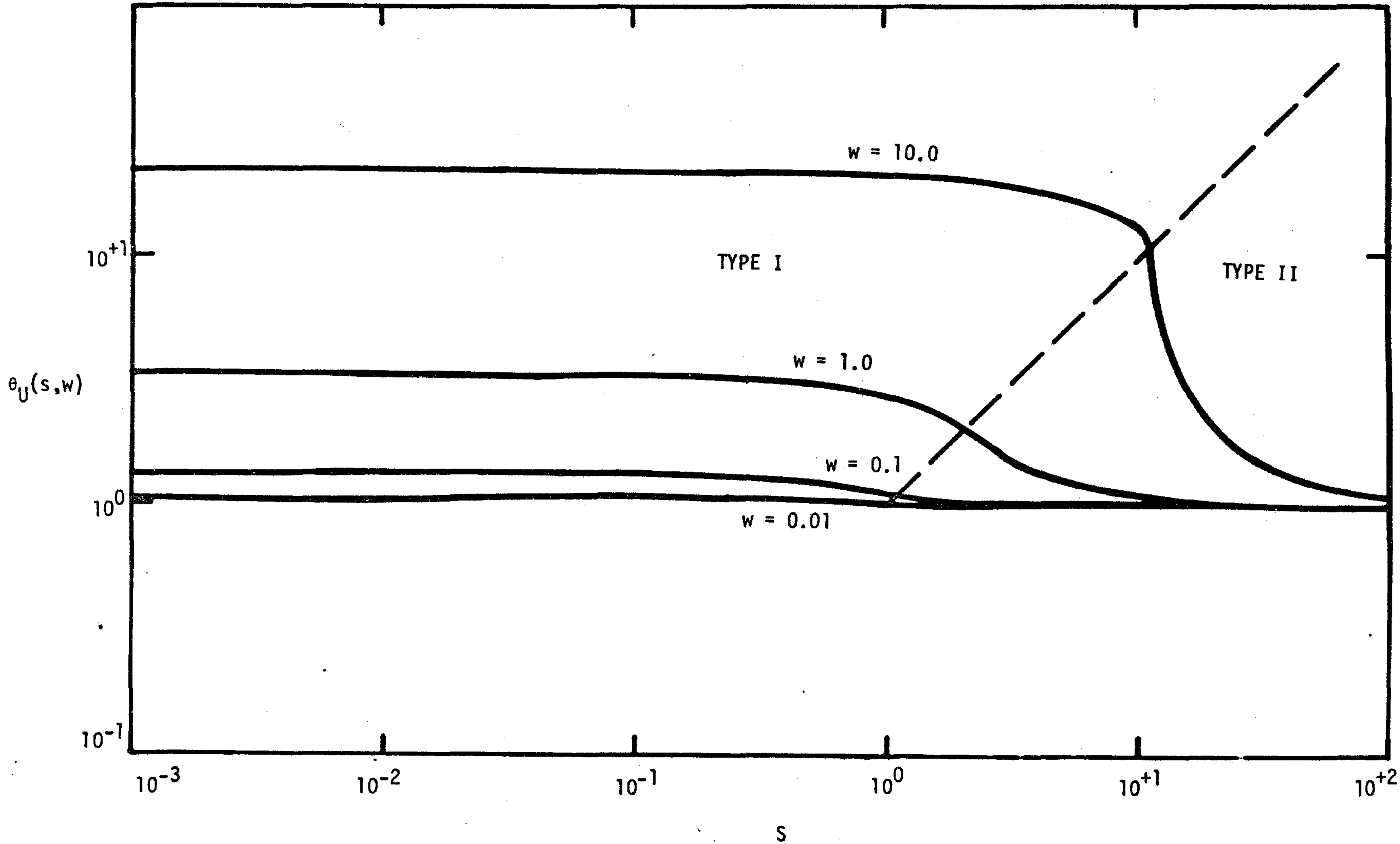

Figure 5. Threshold Function $\theta_U(s,w)$

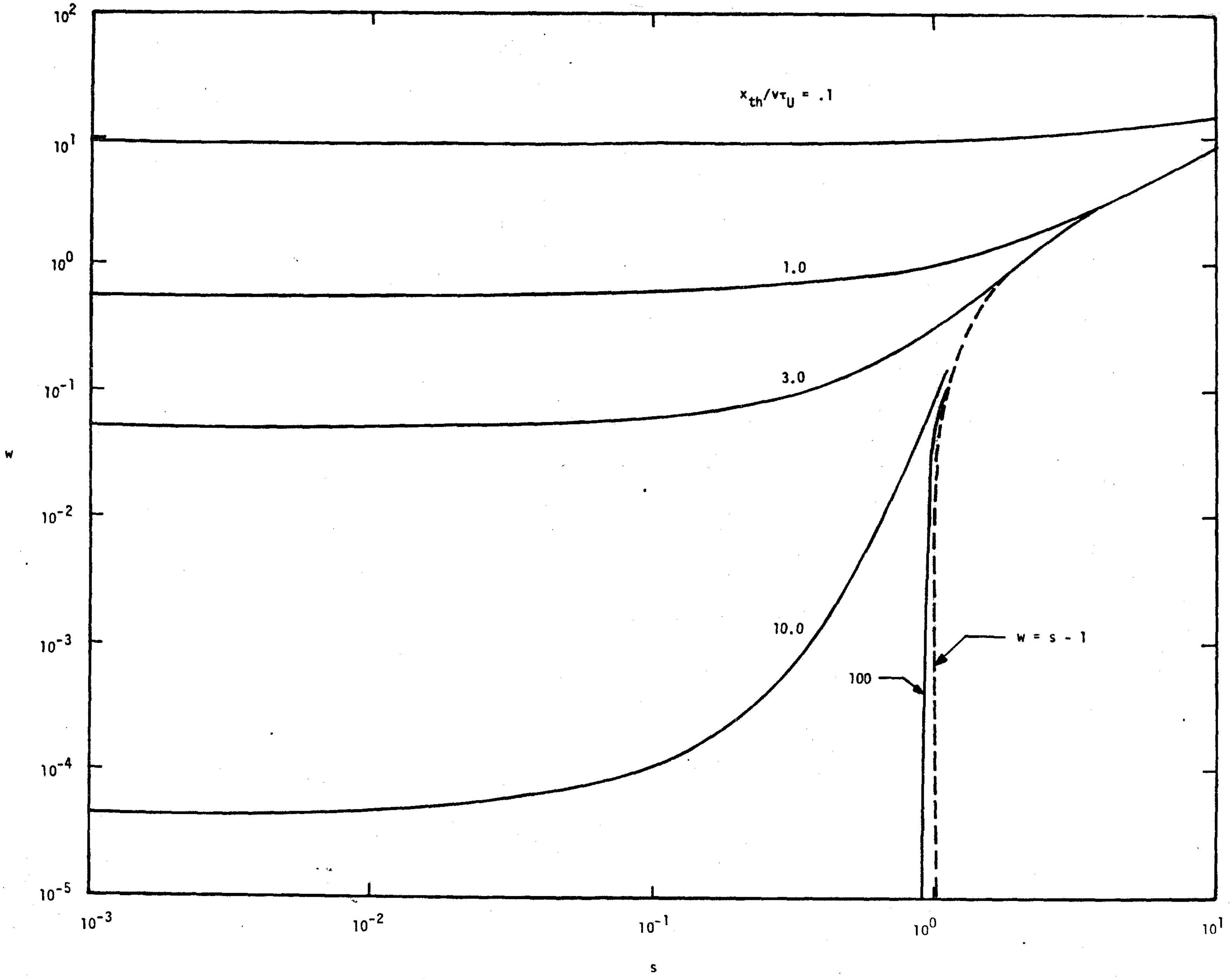

Figure 6. $x_{th}/v\tau_U$ versus s and w

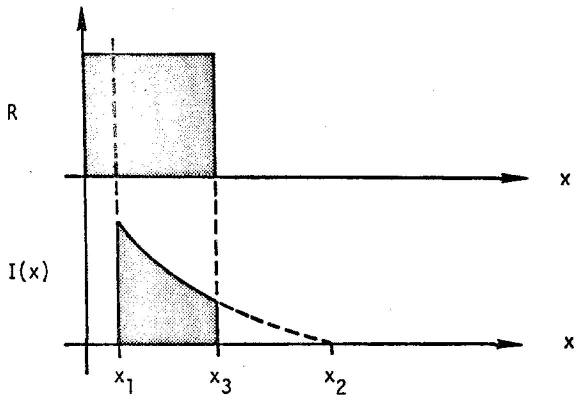

Figure 7a. Pumping Region Ends Before x_2.

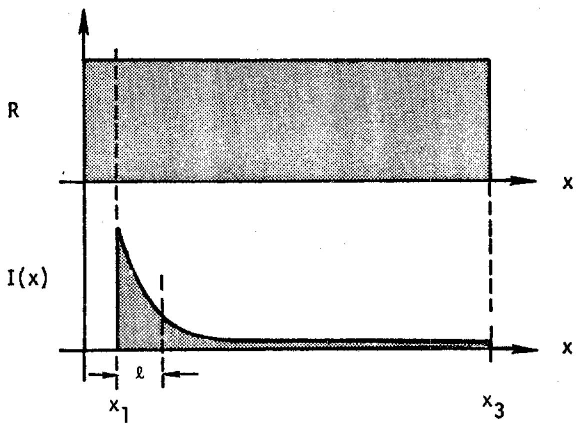

Figure 7b. Pumping Region Extending Downstream far Beyond ℓ ($n^*_\infty > 0$).

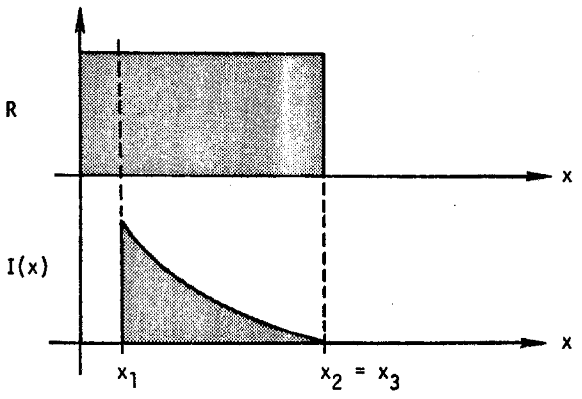

Figure 7c. Pumping Region Ends at the Lasing Region.

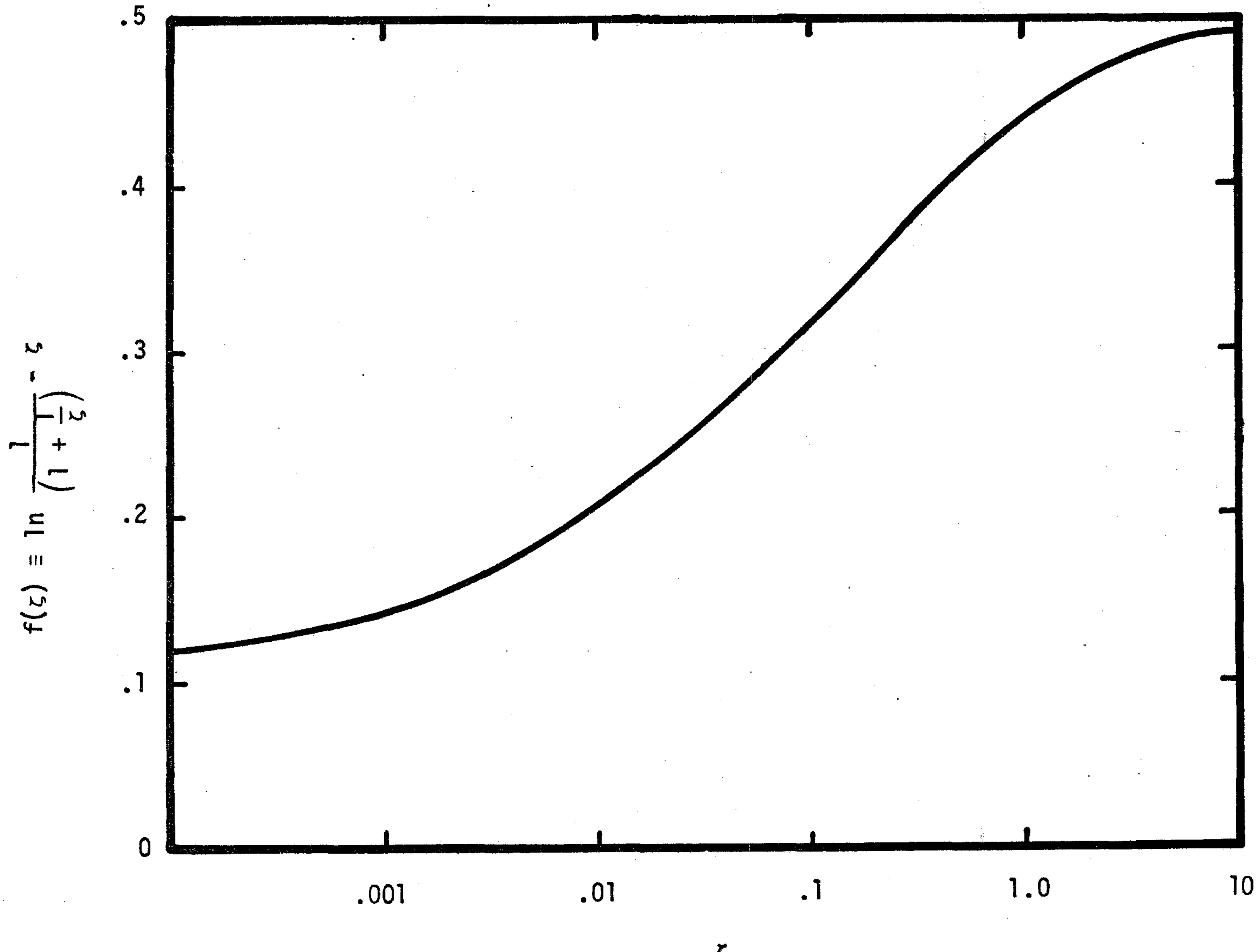

Figure 8. Graph of $f(\zeta)$

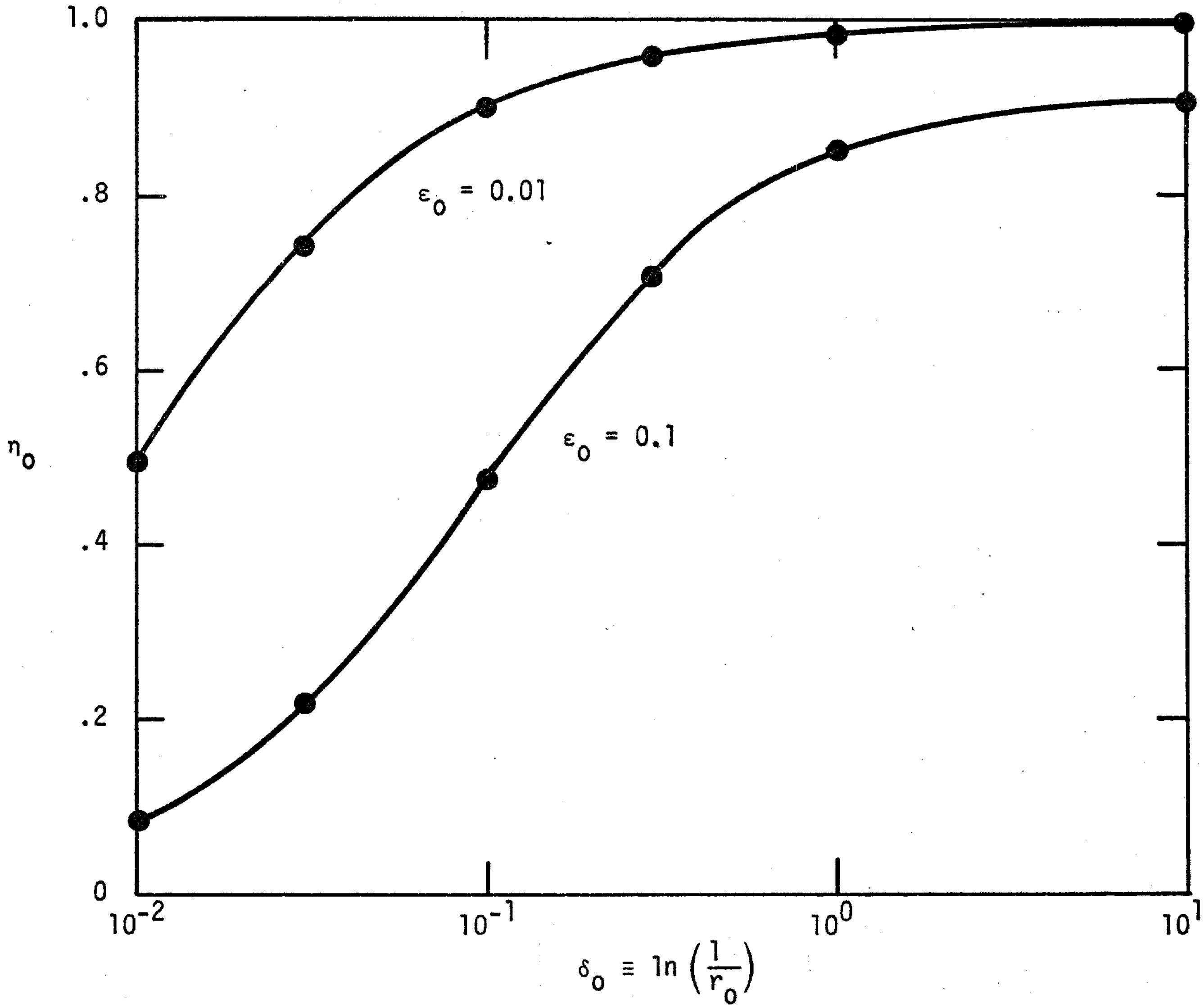

Figure 9. Optical Efficiency versus Reflective and Other Losses

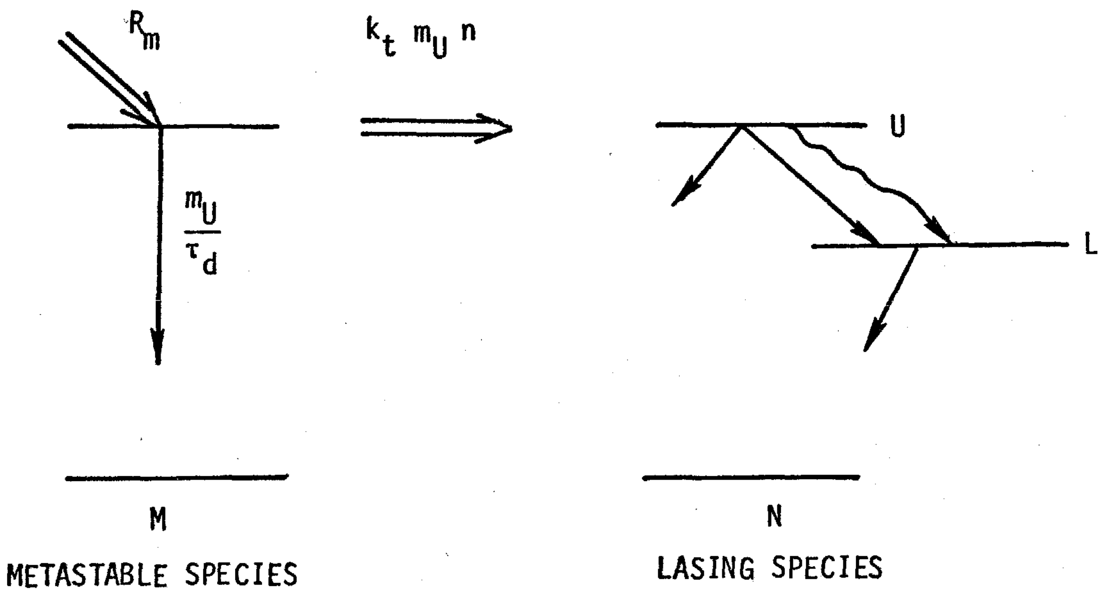

Figure 10. Energy Transfer from Metastable Species

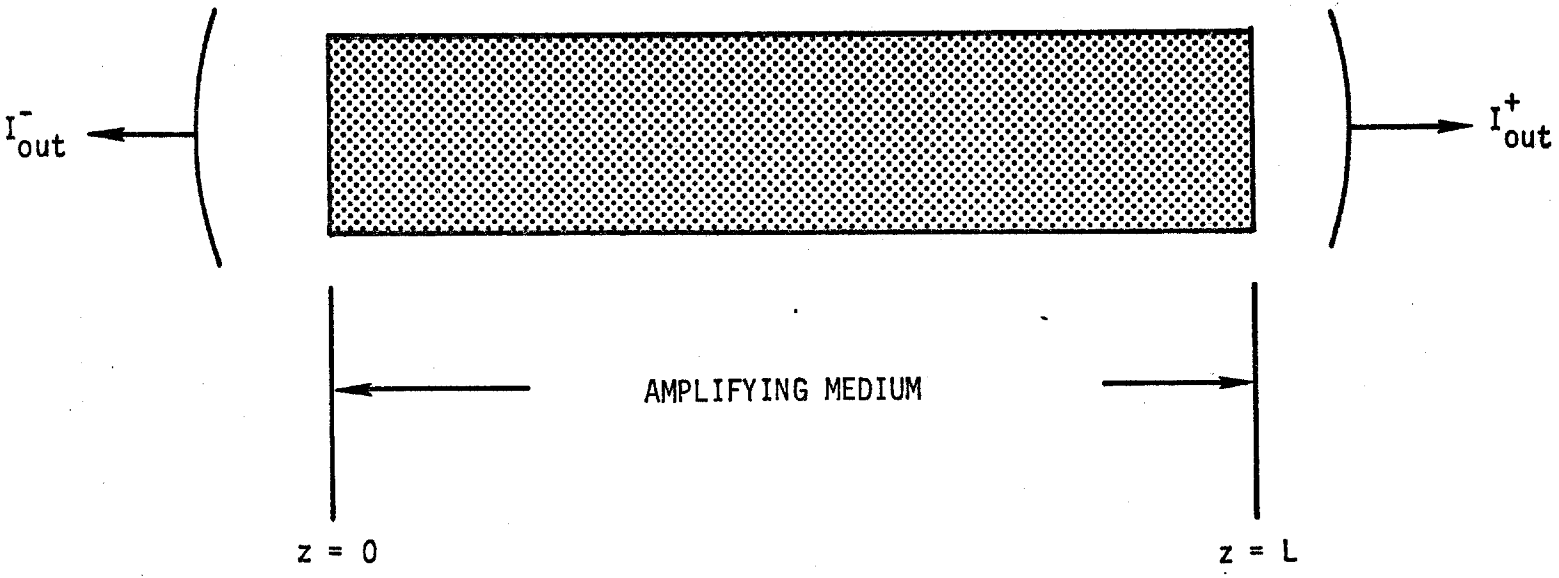

Figure A1.1. Optical Cavity

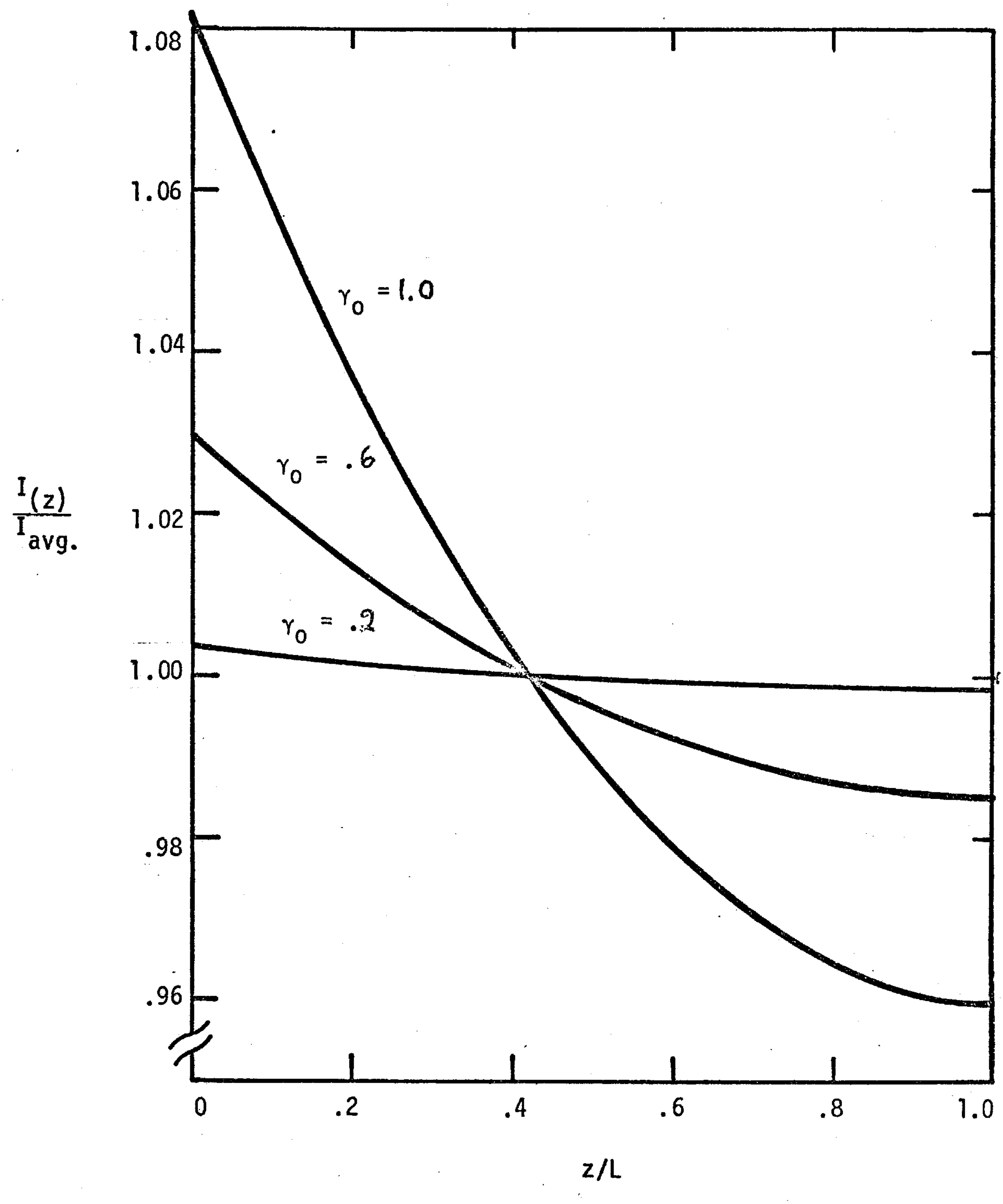

Figure A1.2. Intra-cavity Intensity as a Function of the Mirror Loss Parameter