V. EXPERIMENTAL RESULTS

To illustrate the good performance of the proposed controller, this section shows some experimental tests, performed at the Control Laboratory of the "Departamento de Automação e Sistemas" of the Federal University of Santa Catarina, Brazil. The control algorithm was implemented using real-time control structure based on the software SIMULINK [12] and a data acquisition system. In the experiments the proposed controller was used to control the flux of the air in a tube. The flux is controlled varying the velocity of a small fan allocated in one extreme of the tube. On the other side a small turbine is used to generate a voltage proportional to the flux. Also, the system allows to introduce some perturbations and to change the system dynamics by operating a manual mechanism in the middle of the tube. Although the system presents a nonlinear behavior, a simple model could be obtained using some experimental step-tests near the middle of the operating range (0–5 volts):

$$P(s) = \frac{1.02e^{-8.2s}}{(1 + 1.7s)}. \tag{13}$$

Because of the poor information about the uncertainties the proposed controller was tuned using $T_1 = T = 1.7$ seconds and $T_0 = L/2 = 4.1$ seconds.

The experimental results for some changes in the set-point and a perturbation introduced in $t = 70$ s are shown in Fig. 9 for two different positions of the manual mechanism.

As can be seen the controller performs well even when the flux is under the operating point 2.5 and the gain and dead-time of the process are different from the model. The tuning procedure is simple and, from the point of view of implementation, the 2DOF DTC real time blocks are simple PIs or filters usually used in industrial controllers.

VI. CONCLUSION

A unified approach for the robust tuning of a 2DOF DTC have been proposed. The tuning rules, defined using the two typical models of processes with delay that are found in the process industry, are simple and take into account the robust performance of the controller. A comparative analysis has shown that the 2DOF DTC is equivalent or superior to the recent modified SPs proposed in literature. Experimental results have demonstrated the performance of the controller and the simplicity of the tuning rule.

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Normal Forms for Underactuated Mechanical Systems With Symmetry

Reza Olfati-Saber

***Abstract*—In this note, we introduce cascade normal forms for underactuated mechanical systems that are convenient for control design. These normal forms include three classes of cascade systems, namely, nonlinear systems in strict feedback form, feedforward form, and nontriangular quadratic form (to be defined). In each case, the transformation to cascade systems is provided in closed-form. We apply our results to the Acrobot, the Rotating Pendulum, and the Cart-Pole system.**

***Index Terms*—Cascade systems, nonlinear control, normal forms, symmetry, underactuated systems.**

I. INTRODUCTION

Underactuated systems are mechanical control systems with fewer controls than configuration variables. In recent years, there has been extensive interest among the researchers in control of underactuated mechanical systems due to their broad applications (see [1], [2], and references therein). Many real-life control systems including aircraft, spacecraft, helicopters, underwater vehicles, surface vessels, mobile robots, walking robots, and flexible-link robots are examples of underactuated systems.

In this note, we introduce three classes of cascade normal forms for underactuated systems. Namely, cascade nonlinear systems in *strict feedback form*, *feedforward form*, and *nontriangular quadratic form*. The structure of these normal forms allows application of the existing control design methods like *backstepping* [3] and *forwarding* [4] to

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control of underactuated systems. In addition, the need for developing effective control design methods for nonlinear systems in nontriangular forms as in [5] and [6] becomes more evident.

The main contribution of this note is to provide diffeomorphisms in closed form that transform classes of underactuated systems with symmetry into cascade systems with structural properties. These transformations are physically meaningful and all of them are obtained from the Lagrangian functions of the original underactuated systems. Appropriate references are provided for control design methods associated with each obtained class of cascade normal forms.

The outline of the note is as follows. In Section II, the dynamics and partial feedback linearization methods for underactuated systems are presented. In Section III, we present our main results on cascade normal forms for low-order underactuated systems. In Section IV, applications of our theoretical results to three robotics benchmark examples are given. Finally, concluding remarks are made.

II. Dynamics of Underactuated Systems

In this note, we consider mechanical systems with configuration vector $q \in Q$ which is an n-dimensional vector and a (simple) Lagrangian

$$\mathcal{L}(q, \dot{q}) = K - V = \tfrac{1}{2}\,\dot{q}^T M(q)\dot{q} - V(q)$$

where K is the kinetic energy, $V(q)$ is the potential energy, and $M(q)$ is the *inertia matrix* of the system. Let us decompose the configuration vector of the system as $q = \mathrm{col}(q_1, q_2) \in Q_1 \times Q_2$ where the dimension of the manifold Q_i is denoted by $n_i = \dim(Q_i)$ for $i = 1, 2$ and $n_1 + n_2 = n$ ("col" means a column vector). The Euler–Lagrange equations of motion for this system can be written in the form

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} &= \tau_1 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} &= \tau_2 \end{aligned} \tag{1}$$

where τ_is are the control inputs satisfying the conditions of either of the following *actuation modes*:

A1) $\tau = \tau_2 \in \mathbb{R}^{n_2}$ is the control and $\tau_1 \equiv 0$;

A2) $\tau = \tau_1 \in \mathbb{R}^{n_1}$ is the control and $\tau_2 \equiv 0$.

Apparently, in both of the above cases system (1) is an underactuated system. Actuation modes A1) and A2) are important due to their applications in robotics. The Acrobot (Fig. 1) is actuated according to mode A1), while the Rotating Pendulum (Fig. 2) and the Cart-Pole system are actuated according to mode A2.

The equations in (1) can be simplified as

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q, \dot{q}) &= \tau_1 \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q, \dot{q}) &= \tau_2 \end{aligned} \tag{2}$$

where h_is contain Coriolis, centrifugal, and gravity terms. Due to Spong [2], [7], there exists an invertible change of control input

$$\tau_i = \alpha_i(q)u + \beta_i(q, \dot{q}), \qquad i = 1, 2 \tag{3}$$

which transforms the dynamics of (2) into

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f(q, p) + g(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \tag{4}$$

where for case A1), $f(q, p),\ g(q)$ are given by

$$\begin{aligned} f(q, p) &:= -m_{11}^{-1}(q)h_1(q, \dot{q}) \\ g(q) &:= -m_{11}^{-1}(q)m_{12}(q). \end{aligned} \tag{5}$$

In a similar way, expressions for f, g can be obtained for case A2). The corresponding partially linearizing change of control in (3) for A1) and A2) are called *collocated feedback* and *noncollocated feedback*, respectively [2], [7]. Notice that (4) is not a cascade nonlinear system. In [8], an algebraic sufficient condition is provided for the existence of a global change of coordinates that transforms (4) into a cascade nonlinear system in the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= F(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u. \end{aligned} \tag{6}$$

The main contribution of this work is to provide this transformation in closed-form for classes of underactuated systems with symmetry properties as defined in the following.

Definition 1 (Shape Variables): The set of configuration variables that appear in the inertia matrix $M(q)$ are called *shape variables*. In other words, if $M(q) = M(q_2)$, then q_2 is the vector of shape variables of the system. (See [9] for formal definitions of the notions of "shape space" and "symmetry" on manifolds).

III. Cascade Normal Forms

This section is devoted to normal forms with triangular and nontriangular structural properties for classes of underactuated systems with symmetry.

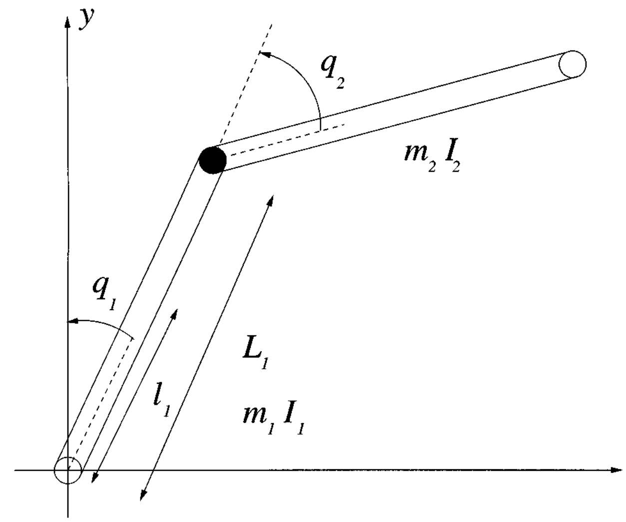

Fig. 1. The acrobot.

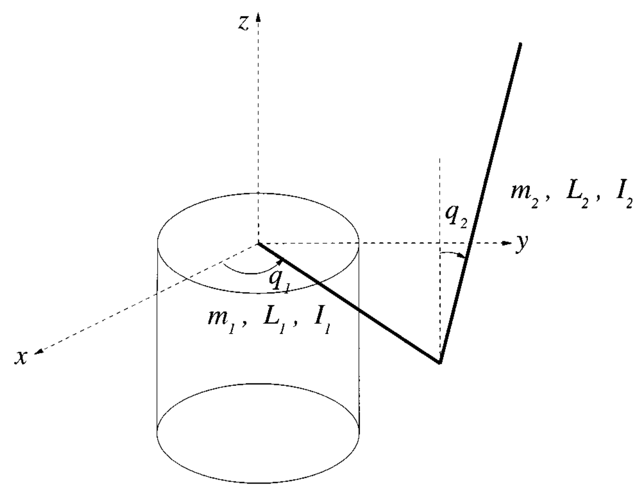

Fig. 2. The rotating pendulum.

Theorem 1: Consider the underactuated system in (2) with two degrees of freedom (q_1, q_2) and symmetry property $M(q) = M(q_2)$. Assume the shape variable q_2 is actuated [i.e., case A1)]. Then, the following global change of coordinates:

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{11}(q_2)p_1 + m_{12}(q_2)p_2 := \partial\mathcal{L}/\partial\dot{q}_1 \\ \xi_1 &= q_2 \\ \xi_2 &= p_2 \end{aligned} \tag{7}$$

transforms the dynamics of (2) into a cascade nonlinear system in strict feedback form

$$\begin{aligned} \dot{z}_1 &= m_{11}^{-1}(\xi_1)z_2 \\ \dot{z}_2 &= g(z_1, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{8}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)}\, ds \tag{9}$$

$$g(z_1, \xi_1) = -\left[D_{q_1}V(q_1, q_2)\right]_{q_1=z_1-\gamma(\xi_1),\, q_2=\xi_1}. \tag{10}$$

Proof: By definition of z_1 and z_2, $\dot{z}_1 = z_2/m_{11}(q_2)$. Since $M(q) = M(q_2)$, $\partial K/\partial q_1 = 0$ and we have

$$\dot{z}_2 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_1} = \frac{\partial\mathcal{L}}{\partial q_1} = \frac{\partial K}{\partial q_1} - \frac{\partial V(q)}{\partial q_1} = g(z_1, \xi_1) \tag{11}$$

and, thus, the result follows. □

Remark 1: Clearly, z_2 in (7) is the generalized momentum conjugate to q_1. In addition, z_1 can be viewed as a new generalized configuration variable replacing q_1. This means that the change of coordinates given in (7) is physically meaningful.

The control input for the z-subsystem of the normal form (8) is the shape variable $q_2 = \xi_1$. If there exists a state feedback $\xi_1 = k_1(z)$ that renders $z = 0$ globally asymptotically stable (GAS), then using backstepping procedure [3], [10] a state feedback $u = k_2(z, \xi)$ can be obtained that renders $(z, \xi) = 0$ GAS for the composite system in (8). Global asymptotic stabilization of the z-subsystem of (8) using a static state feedback is addressed in [11].

The following theorem gives the normal form for underactuated systems with 2 d.o.f. and actuation mode A2).

Theorem 2: Consider the underactuated system in (2) with two degrees of freedom (q_1, q_2) and symmetry property $M(q) = M(q_2)$. Assume the shape variable q_2 is unactuated [i.e., case A2)]. Then, the following global change of coordinates

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{21}(q_2)p_1 + m_{22}(q_2)p_2 := \partial\mathcal{L}/\partial\dot{q}_2 \\ \xi_1 &= q_2 \\ \xi_2 &= p_2 \end{aligned} \tag{12}$$

over the set $U = \{q_2 | m_{21}(q_2) \neq 0\}$ transforms the dynamics of (2) into a cascade nonlinear system in nontriangular quadratic normal form

$$\begin{aligned} \dot{z}_1 &= m_{21}^{-1}(\xi_1)z_2 \\ \dot{z}_2 &= g(z_1, \xi_1) + \Sigma(\xi_1, z_2, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{13}$$

where

$$\Sigma(\xi_1, z_2, \xi_2) = (z_2, \xi_2)\pi(\xi_1)(z_2, \xi_2)^T \tag{14}$$

is quadratic in (z_2, ξ_2) with a weight matrix $\pi(\xi_1) \in \mathbb{R}^{2\times 2}$ and

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{22}(s)}{m_{21}(s)}\, ds \tag{15}$$

$$g(z_1, \xi_1) = -\left[D_{q_2}V(q_1, q_2)\right]_{q_1=z_1-\gamma(\xi_1),\, q_2=\xi_1}. \tag{16}$$

Proof: The proof is rather similar to the proof of Theorem 1 with the difference that $\partial K/\partial q_2 =: \Sigma(\xi_1, z_2, \xi_2)$.

Remark 2: Stabilization of special classes of nonlinear systems in nontriangular normal form (13) is addressed in [5], [6]. In this case, backstepping/forwarding approaches are not applicable.

The following theorem shows that under further assumptions on the class of underactuated systems with an unactuated shape variable, the obtained normal form is a nonlinear system in feedforward form [12].

Theorem 3: Assume all the conditions in Theorem 2 hold. Denote $g(q_1, q_2) = -D_{q_2}V(q_1, q_2)$. Suppose the following properties hold:

i) m_{11} is constant;
ii) $g(q_1, q_2) = g(q_2)$, i.e., $D_{q_1}D_{q_2}V(q) \equiv 0$;
iii) $\psi(q_2) = g(q_2)/m_{21}(q_2)$ satisfies $\psi'(0) \neq 0$.

Then, applying the change of coordinates

$$y_1 = z_1, \qquad y_2 = z_2/m_{21}(q_2) \tag{17}$$

transforms system (13) into a cascade nonlinear system in strict feedforward form

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \psi(\xi_1) + \pi(\xi_1)\xi_2^2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{18}$$

with $\pi: \mathbb{R} \to \mathbb{R}$. Moreover, the origin for this feedforward system can be globally asymptotically stabilized using Teel's nested saturations (see [12]).

Proof: See [8, Prop. 4.1]. □

IV. EXAMPLES

In this section, we provide examples of underactuated systems with their corresponding normal forms.

Example 1: The *Acrobot* [13], as shown in Fig. 1, is a two-link planar robot with revolute joints and one actuator at the elbow. The inertia matrix for the Acrobot is given by

$$\begin{aligned} m_{11} &= m_1 l_1^2 + m_2(L_1^2 + l_2^2 + 2L_1 l_2 \cos(q_2)) \\ &\quad + I_1 + I_2 =: a + b \end{aligned}$$

$\cos(q_2)$

$$\begin{aligned} m_{12} &= m_{21}(q_2) = m_2(l_2^2 + L_1 l_2 \cos(q_2)) + I_2 \\ &=: c + (b/2) \end{aligned}$$

$\cos(q_2)$

$$m_{22} = m_2 l_2^2 + I_2 =: c.$$

Clearly, the symmetry condition $M(q) = M(q_2)$ holds and q_2 is an actuated shape variable for the Acrobot. Therefore, based on Theorem 1 after applying the global change of coordinates in (7), the dynamics of the Acrobot transforms into a cascade system in strict feedback form

$$\begin{aligned} \dot{z}_1 &= z_2/m_{11}(q_2) \\ \dot{z}_2 &= -(m_1 l_1 + m_2 L_1)g_0 \cos(z_1 - \gamma(q_2)) \\ &\quad - m_2 l_2 g_0 \cos(z_1 - \gamma(q_2) + q_2) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \tag{19}$$

where g_0 is the gravity constant and the function $\gamma(\cdot)$ is given by

$$\gamma(q_2) = \frac{q_2}{2} + \frac{(2c-a)}{\sqrt{a^2 - b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}} \tan\left(\frac{q_2}{2}\right)\right) \tag{20}$$

which is defined over $q_2 \in [-\pi, \pi)$ and $a, b, c > 0$ are three constants that determine $M(q)$. See [5] and [11] for further details on control of the Acrobot.

Example 2: The *Rotating Pendulum*, depicted in Fig. 2, consists of an inverted pendulum on a rotating arm [14]. The elements of the inertia matrix for the Rotating Pendulum are given by

$$\begin{aligned} m_{11} &= I_1 + m_1 l_1^2 + m_2(L_1^2 + l_2^2 \sin^2(q_2)) \\ &=: a + d\sin^2(q_2) \\ m_{12} &= m_{21} = m_2 L_1 l_2 \cos(q_2) =: b\cos(q_2) \\ m_{22} &= I_2 + m_2 l_2^2 =: c. \end{aligned}$$

The potential energy for this system is

$$V(q_1, q_2) = m_2 g l_2 \cos(q_2).$$

Apparently, $M(q) = M(q_2)$ and q_2 is an unactuated shape variable for the rotating pendulum. Based on Theorem 2, the dynamics of the rotating pendulum can be transformed into a nontriangular quadratic normal form using the change of coordinates (12) for $q_2 \in U = (-\pi/2, \pi/2)$. After setting $y_1 = z_1$, $y_2 = z_2/m_{21}(q_2)$, we obtain

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \frac{g}{L_1}\tan(q_2) + \frac{l_2}{L_1}\sin(q_2)\left(u_2 - \frac{m_{22}}{m_{21}(q_2)}p_2\right)^2 \\ &\quad + \frac{m_{22}}{m_{21}(q_2)}\tan(q_2)p_2^2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned}$$

where

$$\begin{aligned} y_1 &= q_1 - \gamma(q_2) \\ y_2 &= p_1 - m_{21}^{-1}(q_2)m_{22}(q_2)p_2 \end{aligned}$$

and

$$\gamma(q_2) = \frac{(m_2 l_2^2 + I_2)}{m_2 l_2 L_1} \log\left(\frac{1 + \tan(q_2/2)}{1 - \tan(q_2/2)}\right).$$

Example 3: The *Cart-Pole System* is an inverted pendulum on a cart. Let (q_1, q_2) denote the position of the cart and the angle of the pendulum, respectively. The inertia matrix for the cart-pole system is similar to the rotating pendulum with the difference that m_{11} is constant. In addition, the Cart-Pole system satisfies all other conditions of Theorem 3 over the upper half-plane [i.e., $q_2 \in (-\pi/2, \pi/2)$]. Thus, the origin $(q, p) = (0, 0)$ for the cart-pole system can be globally asymptotically stabilized over the upper half-plane using Teel's nested saturations [5], [8].

V. Conclusion

In this note, we introduced novel structured cascade normal forms for underactuated systems. Namely, cascade systems in strict feedback form, feedforward form, and nontriangular quadratic form. The corresponding control design methods and examples for each class are mentioned. These normal forms allow application of the exiting methods like backstepping and feedforwarding to control of complex underactuated systems. We also introduced fourth-order cascade normal forms for three robotics benchmark examples. Namely, the Acrobot, the Rotating Pendulum, and the Cart-Pole system.

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Stability Criteria of Sector- and Slope-Restricted Lur'e Systems

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***Abstract*—This note presents new stability criteria of sector- and slope-restricted Lur'e systems, in terms of linear matrix inequalities, by fully exploiting inherent properties of sector and slope restrictions in the time domain. Interpreting the time-domain criteria in the frequency domain, furthermore, supplies simpler expressions. Several examples show excellent performances of these criteria.**

***Index Terms*—LMIs, sector- and slope-restricted nonlinearities, stability.**

I. Introduction

The asymptotic stability analysis of Lur'e systems has been studied for several decades [1]–[19]. When the nonlinearity of the Lur'e system is only sector-restricted, the best stability condition is expressed with

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