Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

FrM01-2

Nonlinear Control and Reduction of Underactuated Systems with Symmetry II: Unactuated Shape Variables Case

Reza Olfati-Saber
California Institute of Technology
Control and Dynamical Systems 107-81
Pasadena, CA 91125
olfati@cds.caltech.edu

Abstract

In this paper, we address nonlinear control and reduction of two classes of high-order underactuated mechanical systems with kinetic symmetry called Class-II and Class-III systems. Class-II systems are underactuated systems with unactuated shape variables, decoupled inputs, and integrable normalized momentums (all to be defined). We show that all Class-II underactuated systems can be transformed into cascade systems in non-triangular quadratic form using an explicit change of coordinates and control. In addition, we characterize a subclass of Class-II systems, called Class-III systems, that can be explicitly transformed into cascade systems in strict feedforward form. This allows application of existing nonlinear control design methods like nested saturations and feedforwarding to control of underactuated systems

1 Introduction

Control design and analysis for underactuated mechanical systems is currently an active field of research. The importance of underactuated systems is due to their broad applications in robotics, aerospace vehicles, and marine vehicles. In addition, restriction of the control authority in underactuated systems offers challenging control problems from theoretical point of view (see [5, 9] for recent surveys).

This paper is part II of a series of articles that aim to address reduction and nonlinear control of broad classes of high-order underactuated systems. In [5], it is shown that underactuated systems can be essentially classified to eight main classes that overall cover the majority of the aforementioned real-life applications. In part II of this paper, we focus on reduction and control of underactuated systems with unactuated shape variables, integrable normalized momentums, and decoupled inputs (all defined in section 2). We refer to this particular class of underactuated systems as Class-II systems.

The reduction and control of underactuated systems with actuated shape variables and input coupling are addressed in parts I [7] and III of this paper, respectively.

The main contribution of part II of this paper is providing a systematic way for obtaining change of coordinates in closed-form that transform high-order Class-II underactuated systems into cascade systems in nontriangular quadratic form. In addition, we characterize a subclass of Class-II systems, called Class-III systems, that can be transformed into cascade systems in feedforward form. This allows application of Teel's nested saturations [13], nonlinear small-gain theorem [12], and feedforwarding methods due to Mazenc and Praly [3] to stabilization of Class-III underactuated systems. Stabilization of nonlinear systems in nontriangular forms is, in general, an open problem. In important special cases, this problem has been addressed in [5, 8].

The outline of the paper is as follows. In section 2, we provide some background on dynamics and symmetry properties of underactuated systems. In section 3, we present our main reduction and stabilization results. In section 4, we give a detailed example. Finally, we make concluding remarks.

2 Underactuated Systems with Symmetry

In this paper, we consider the class of simple Lagrangian systems with configuration vector $q = \mathrm{col}(q_x, q_s) \in Q_x \times Q_s$, configuration space $Q = Q_x \times Q_s$ of dimension n, and Lagrangian

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T M(q_s)\dot{q} - V(q_x, q_s)$$

where K is the kinetic energy, $V(q)$ is the potential energy, and $M(q) = M(q_s)$ is the *inertia matrix*. We say the system has *kinetic symmetry* w.r.t. q_x due to $\frac{\partial K}{\partial q_x} = 0$. We refer to q_x and q_s as the vectors of *external variables* and *shape variables*, respectively. The forced Euler-Lagrange equation for this system can be written as

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_x} - \frac{\partial \mathcal{L}}{\partial q} &= F_x(q)\tau \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_s} - \frac{\partial \mathcal{L}}{\partial q} &= F_s(q)\tau \end{aligned} \qquad (1)$$

where $\tau \in \mathbb{R}^m$ and $F(q) = \mathrm{col}[F_x(q), F_s(q)]$ is the *force matrix*. We say the mechanical system in (1) is an *underactuated system* if $m = \mathrm{rank}(F(q)) < n$. Since $F(q)$ has full column rank, there exists a permutation of q such that $F(q)$ can be partitioned as $\mathrm{col}[F_1(q), F_2(q)]$ where $F_1(q)$ is an invertible $m \times m$ matrix. If $F_2(q) \equiv 0$, we say (1) is a mechanical system with *decoupled inputs*. Otherwise, (1) is referred to as a mechanical system with *input coupling*. This paper is devoted to reduction and nonlinear control of high-order underactuated systems with *unactuated shape variables and decoupled inputs*. Without loss of generality, set $F_x(q) = I_m, F_s(q) = 0$ (otherwise, replace τ by $F_x^{-1}(q)\tau$). For the sake of simplicity of notation, we assume $\dim(Q_x) = \dim(Q_s)$ (the general case can be found in [5](chapter 4)).

By *reduction*, we mean transformation of the mechanical control system in (1) into a cascade nonlinear system in the form

$$\begin{aligned} \dot{x} &= f(x,\xi) \\ \dot{\xi} &= g(\xi, u) \end{aligned} \qquad (2)$$

using a smooth invertible change of coordinates and control (i.e. diffeomorphism)

$$(x,\xi) = \Phi_1(q,\dot{q}),\ u = \Phi_2(q,\dot{q},\tau) \qquad (3)$$

Remark 1. Notice that, in general, a system with Lagrangian that is considered here does not possess symmetry properties in the classical sense according to [2], unless $V(q_x, q_s) = V(q_x)$. As a result the generalized momentum $p_s = \partial\mathcal{L}/\partial\dot{q}_x$ conjugate to q_s is not anymore a conserved quantity for the unforced Lagrangian system. This is a fundamental difference between kinetic symmetry and classical symmetry.

The Lagrangian equations of motion in (1) with $F(q) = [I_m, 0]^T$ can be rewritten as

$$\begin{array}{rcl} m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q,\dot{q}) & = & \tau \\ m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q,\dot{q}) & = & 0 \end{array} \quad (4)$$

where h_x, h_s contain the Coriolis, centrifugal, and gravity terms. In a similar approach taken in [11], the dynamics of an underactuated system in the form (4) can be partially linearized using an invertible change of control in the form $\tau = \alpha(q)u + \beta(q,\dot{q})$ over the following set

$$U = \{q_s \mid \det(m_{sx}(q_s)) \neq 0\} \quad (5)$$

This feedback is called a *noncollocated partially-linearizing feedback* [10] and it reduces the dynamics of q_s to a set of double-integrators as $\ddot{q}_s = u$.

The key tools in reduction of high-order underactuated systems with kinetic symmetry are generalized momentums, normalized momentums, and their integrals which are defined in the sequel. Let p_s be the *generalized momentum* conjugate to q_s, then for system (4)

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = m_{sx}(q_s)\dot{q}_x + m_{ss}(q_s)\dot{q}_s \quad (6)$$

We define the *normalized momentum* conjugate to q_s over U as

$$\pi_s = m_{sx}^{-1}(q_s)\frac{\partial \mathcal{L}}{\partial \dot{q}_s} = \dot{q}_x + m_{sx}^{-1}(q_s)m_{ss}(q_s)\dot{q}_s \quad (7)$$

We say the normalized momentum π_s is *integrable*, if there exists a smooth function $h(q) = (h_1(q), \dots, h_m(q))^T$ such that $\pi_s = \dot{h}$ where $\dot{h} := Dh(q)\dot{q}$ and $Dh(q) = (\nabla h_1(q), \dots, \nabla h_m(q))$. Whenever π_s is integrable, we call $h(q)$ the *integral* of π_s. Now, we are ready to define Class-II underactuated systems.

Definition 1. (Class-II systems) We refer to the class of underactuated mechanical systems with unactuated shape variables, decoupled inputs, and integrable normalized momentum π_s as *Class-II underactuated systems.*

We find the following quadratic forms convenient for representation of the normal forms of Class-II underactuated systems.

Definition 2. (Vector Quadratic Forms) Consider a mapping $\Sigma : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Sigma(x,v) = v^T\Pi(x)v := (v^T\pi_1 v, \dots, v^T\pi_n v)^T \quad (8)$$

where $\Pi(x) : \mathbb{R}^p \to \mathbb{R}^{n\times n\times n}$ is a cubic matrix with layers $\pi_i(x) : \mathbb{R}^p \to \mathbb{R}^{n\times n}, i = 1, \dots, n$ which are square matrices. We call $\Sigma(x,v)$ a *vector quadratic form* in v.

3 Main Results

By definition, any Class-II underactuated system satisfies the following assumption.

Assumption 1. (integrability condition) Assume all the elements of

$$\omega = m_{sx}^{-1}(q_s)m_{ss}(q_s)dq_s\ , \ \forall q_s \in U \quad (9)$$

are exact one-forms and let $\omega = d\gamma(q_s)$. In other words, denoting $\mu(q_s) := m_{sx}^{-1}(q_s)m_{ss}(q_s)$, assume all the one-forms

$$\omega_i = \Sigma_{j=1}^m \mu_{ij}(q_s)dq_s^j$$

are exact for $i = 1, \dots, m$ (μ_{ij} and q_s^j are elements of $\mu(q_s)$ and q_s, respectively). Let $\omega_i = d\gamma_i(q_s)$, then $\gamma(q_s) := (\gamma_1(q_s), \dots, \gamma_m(q_s))^T$.

Here is our first main result on reduction of Class-II underactuated systems:

Theorem 1. *Consider the underactuated system in (1) and suppose Assumption 1 holds. Then, there exists a change of coordinates (i.e. diffeomorphism) obtained from the Lagrangian of the system*

$$\begin{aligned} q_r &= q_x + \gamma(q_s) \\ p_r &= m_{sx}(q_s)\dot{q}_x + m_{ss}(q_s)\dot{q}_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_s} \end{aligned} \tag{10}$$

that transforms the dynamics of the underactuated system (4) into a cascade nonlinear system in nontriangular quadratic form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{11}$$

where $\Sigma = \nabla_{q_s} K$ *(K is the kinetic energy) is a vector quadratic form in* (p_r, p_s)

$$\Sigma(q_s, p_r, p_s) = \begin{bmatrix} p_r \\ p_s \end{bmatrix}^T \Pi(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

with a cubic weight matrix $\Pi(q_s)$ *and*

$$\begin{aligned} m_r(q_s) &:= m_{sx}(q_s) \\ g_r(q_r, q_s) &:= -\left[\nabla_{q_s} V(q_x, q_s)\right]_{q_x = q_r - \gamma(q_s)} \end{aligned}$$

Proof. See page 66 in [5]. □

Remark 2. The normal form for Class-II underactuated systems in (11) possesses a nontriangular structure which does not allow application of backstepping [4, 1] or forwarding methods [3, 13]. Stabilization of different classes of nonlinear systems in nontriangular forms has been addressed in [5, 8].

The following corollary states a physical property of the (q_r, p_r)-subsystem in (11).

Corollary 1. *In theorem 1, assume* $V(q) = V(q_x)$. *Then, the* (q_r, p_r)*-subsystem is a Lagrangian system with configuration vector* q_r *and reduced Lagrangian*

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2}\dot{q}_r^T m_r(q_s)\dot{q}_r \tag{12}$$

that satisfies the forced Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = \Sigma(q_s, p_r, p_s) \tag{13}$$

We need the following definition, before presenting our next result.

Definition 3. (Differentially Symmetric Rows) We say a square matrix function $m(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ has *differentially symmetric rows*, if the i^{th} row of the matrix $m_{i*}(x)$ satisfies

$$\frac{\partial m(x)}{\partial x_i} = \frac{\partial m_{i*}(x)}{\partial x} \tag{14}$$

for $i = 1, \ldots, n$. (the derivative $\partial/\partial x_i$ on the LHS of (14) is taken element-wise and the RHS of (14) is a Jacobian matrix).

Definition 4. (Class-III systems) The subclass of Class-II underactuated systems that satisfy conditions i), ii), and iii) of Theorem 2 are called *Class-III underactuated systems.*

The following theorem provides sufficient conditions such that Class-II underactuated systems can be transformed into feedforward nonlinear systems using a change of coordinates in explicit form.

Theorem 2. *Assume all the conditions in theorem 1 hold. In addition, the underactuated system (4) satisfies the following conditions*

i) $m_{xx}(q_s)$ *is constant.*

ii) $m_{sx}(q_s)$ *has differentially symmetric rows.*

iii) $V(q) = V_x(q_x) + V_s(q_s),\ V_i : Q_i \to \mathbb{R}$ *for* $i \in \{x, s\}$.

Then, applying the change of coordinates

$$z_1 = q_r,\ z_2 = m_r^{-1}(q_s)p_r$$

(where (q_r, p_r) *are defined in (10)) transforms the original system (4) into a cascade system in feedforward form as the following*

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& \psi(q_s) + p_s^T \Pi(q_s) p_s \\ \dot{q}_s &=& p_s \\ \dot{p}_s &=& u \end{array} \tag{15}$$

where $\Pi(q_s)$ *is a cubic matrix and* $\psi : Q_s \to \mathbb{R}^m$ *is defined as*

$$\psi(q_s) = -m_r^{-1}(q_s)\nabla_{q_s} V_s(q_s)$$

Moreover, if $\psi(0) = 0$ *and* $\psi(q_s)$ *has an invertible Jacobian* $\nabla_{q_s}\psi(q_s)$ *at* $q_s = 0$*, then the origin for (15) (and (4)) can be globally asymptotically and locally exponentially stabilized over* U *using a state feedback in explicit form as nested saturations.*

Proof. See pages 69–70 in [5]. □

Remark 3. The feedforward structure of the normal form of Class-III underactuated systems in (15) makes it possible to apply the existing control design methods for feedforward nonlinear systems in [13, 12, 3] to stabilization of a broad class of underactuated systems.

4 Example: The Pendubot

In [5], it is shown that the pendubot, the rotating-pendulum, the planar cart-pole system, the beam-and-ball system, and the inertia wheel pendulum [6], are all examples of Class-II underactuated systems. Among all of them, only the cart-pole system and the inertia wheel pendulum are Class-III systems. Here, we provide the details for reduction of the pendubot.

Example 1. The *pendubot* is a two-link planar robot with revolute joints and one actuator at the shoulder, as shown in Fig. 1. The

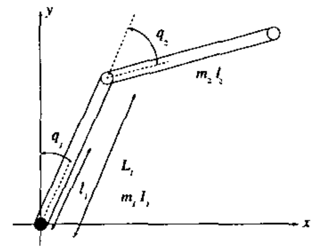

Figure 1: The Pendubot

inertia matrix of the pendubot is given by

$$\begin{array}{rcl} m_{11}(q_2) &=& a + 2b\cos(q_2) \\ m_{12}(q_2) &=& m_{21}(q_2) = c + b\cos(q_2) \\ m_{22}(q_2) &=& c \end{array}$$

where $a, b, c > 0$ are given by

$$\begin{array}{rcl} a &=& m_1 l_1^2 + m_2(L_1^2 + l_2^2) + I_1 + I_2 \\ b &=& m_2 l_2 L_1 \\ c &=& m_2 l_2^2 + I_2 \end{array}$$

Apparently, $q_s = q_2$ is an unactuated shape variable for the pendubot. Thus, the pendubot is a Class-II underactuated system. After a noncollocated partial feedback linearization process, one obtains $\ddot{q}_2 = u$. Now, applying the change of coordinates

$$\begin{array}{rcl} q_r &=& q_1 + \gamma(q_2) \\ p_r &=& m_{11}(q_2)p_1 + m_{22}p_2 \end{array}$$

with

$$\begin{array}{rcl} \gamma(q_2) &=& \displaystyle\int_0^{q_2} \frac{m_{22}}{m_{21}(\theta)} d\theta \\ &=& \displaystyle\frac{2c}{\sqrt{c^2 - b^2}} \arctan\left(\sqrt{\frac{c-b}{c+b}} \tan(\frac{q_2}{2})\right) \end{array}$$

and $c > b, q_2 \in [-\pi, \pi)$, we get

$$\begin{array}{rcl} \dot{q}_r &=& p_r / m_{21}(q_s) \\ \dot{p}_r &=& m_2 l_2 g \sin(q_r - \gamma(q_s) + q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &=& p_s \\ \dot{p}_s &=& u \end{array}$$

where

$$\Sigma = -\frac{b\sin(q_s)(p_r - m_{22}p_s)(p_r - (m_{11}+m_{22})p_s)}{m_{11}^2}$$

Clearly, the equation of $\dot{p}_r$ consists of a quadratic term in (p_r, p_s) and a reduced gravity term $g_r(q_r, q_s) = m_2 l_2 g \sin(q_r - \gamma(q_s) + q_s)$ as in Theorem 1.

5 Conclusion

In this paper, we presented explicit change of coordinates for reduction of high-order underactuated systems with unactuated shape variables, decoupled inputs, and integrable normalized momentums (i.e. Class-II systems). Under further conditions, we showed that a subclass of Class-II systems called Class-III underactuated systems can be transformed into nonlinear cascade systems in feedforward form. We provided several examples of underactuated systems with unactuated shape variables, and applied our results to reduction of the pendubot.

References

[1] A. Isidori. *Nonlinear Control Systems.* Springer, 1995.

[2] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry.* Springer, 1999.

[3] F. Mazenc and L. Praly. "Adding integrations, saturated controls, and stabilization for feedforward Systems". *IEEE Trans. on Automatic Control*, 40:1559–1578, 1996.

[4] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design.* John Wiley & Sons, 1995.

[5] R. Olfati-Saber. *"Nonlinear Control of Underactuated Mechanical Systems with Application to Robotics and Aerospace Vehicles".* PhD thesis, Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, February 2001.

[6] R. Olfati-Saber. "Global Stabilization of a Flat Underactuated System: the Inertia Wheel Pendulum". *Proc. of the 40th Conf. on Decision and Control*, Orlando, Fl, Dec. 2001.

[7] R. Olfati-Saber. "Nonlinear control and reduction of underactuated systems with symmetry I: actuated shape variables case". *Proc. of the 40th Conf. on Decision and Control*, Orlando, Fl, Dec. 2001.

[8] R. Olfati-Saber. "Fixed point controllers and stabilization of the cart-pole system and the rotating pendulum". *Proc. of the 38th Conf. on Decision and Control*, 2:1174–1181, Phoenix, AZ, Dec. 1999.

[9] M. Reyhanoglu, A. van der Schaft, N. H. McClamroch, and I. Kolmanovsky. "Dynamics and control of a class of underactuated mechanical systems". *IEEE Trans. on Automatic Control*, 44(9):1663–1671, 1999.

[10] M. W. Spong. "Underactuated mechanical systems". In B. Siciliano and K. P. Valavanis, editor, *Control Problems in Robotics and Automation*, Springer-Verlag, London, UK, 1997.

[11] M. W. Spong. "Energy based control of a class of underactuated mechanical systems". *1996 IFAC World Congress*, July 1996.

[12] A. R. Teel. "A nonlinear small gain theorem for the analysis of control systems with saturation". *IEEE Trans. on Automatic Control*, 40:1256–1270, 1996.

[13] A. R. Teel. "Using Saturation to stabilize a class of single-input partially linear composite systems". *IFAC NOLCOS'92 Symposium*, pages 369–374, June 1992.