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On the evaluation of instantaneous fluid-dynamic forces
on a bluff body

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On the evaluation of instantaneous fluid-dynamic forces on a bluff body*

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In this article, we present an expression (first derived in 1952) for the evaluation of instantaneous forces on a bluff body in a cross-flow, when only the velocity field (and, therefore, the vorticity field as well) is known in its vicinity. This expression is particularly useful for experimental methods such as DPIV which do not provide any information about the pressure field, but do yield the velocity and vorticity fields in a finite domain.

Several interesting features of this expression are noteworthy:

- It does not require the knowledge of the pressure field.
- It is valid for incompressible, viscous, rotational, and time dependent flows.
- It does not require a knowledge of the velocity field over the whole wake; the control volume can be chosen arbitrarily (it must include the body though).

Before we give a derivation of this expression, we will remind the reader of some classical expressions for the evaluation of instantaneous forces and the assumptions underlying these expressions, and we will show how these expressions contain the pressure explicitly.

We will then present a generalized Green's transformation, the Burgatti identity, which will be the key to the removal of the pressure terms from the force equations. As a starting point, the Burgatti identity will be first applied to the derivation of the force equation in vortex methods. In particular, it will be shown how the pressure terms drop out naturally from the equation, without any *ad hoc* assumptions about the pressure field.

Then, the general expression will be derived. Again, stress will be placed on the absence of the pressure terms in the resulting equation. As a check, we will show that the force equations used in vortex methods as well as in inviscid, rotational flows do follow from the general equation.

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1 Instantaneous force measurements in general flows

1.1 General expression for fluid dynamic forces

The starting point is the integral formulation for the momentum law in continuum mechanics. Consider a material control volume $V_m(t)$ bounded by a material surface $S_m(t)$. The momentum law in continuum mechanics can then be written as¹:

$$\frac{d}{dt} \int_{V_m(t)} \rho \mathbf{u} dV = \oint_{S_m(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS, \quad (1)$$

where ρ is the medium density, \mathbf{u} its velocity, and $\boldsymbol{\Sigma}$ the stress tensor. Here, $V_m(t)$ is a material control volume enclosed by the surface $S_m(t)$ with unit outward normal $\hat{\mathbf{n}}$. For a Newtonian fluid, we have:

$$\boldsymbol{\Sigma} = -p\mathbf{I} + \mathbf{T}, \quad (2)$$

where p is the pressure and \mathbf{T} is the viscous stress tensor:

$$\mathbf{T} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \quad (3)$$

As seen in Figure 1., the surface $S_m(t)$ can be decomposed into the surface of the body $S_b(t)$ (with outward normal pointing *into* the body), the exterior surface $S_m^*(t)$, and the surface of the umbilicus (or “branch cut”) $S_u(t)$ which joins the exterior surface to the body surface, that is:

$$S_m(t) = S_m^*(t) \oplus S_b(t) \oplus S_u(t), \quad (4)$$

such that:

$$\oint_{S_m^*(t) \oplus S_b(t) \oplus S_u(t)} \Phi dS = \int_{S_m^*(t)} \Phi dS + \int_{S_b(t)} \Phi dS + \int_{S_u(t)} \Phi dS. \quad (5)$$

¹The equation should have another term on the right hand side to include volume forces:

$$\int_{V_m(t)} \rho \mathbf{f} dV.$$

Here, \mathbf{f} is the volume force per unit mass (of gravitational or electromagnetic origin, for example). An *a priori* knowledge of it is needed to carry on the algebra. Each case needs to be dealt with separately. In electromagnetics, the coupling between fluid flow and electromagnetic forces is quite intricate. Even in the case of a gravitational field, only a few cases lend themselves to further simplifications, as in the case of a fluid of uniform and constant density. For the rest of the article, *it will be assumed that the volume force $\rho \mathbf{f}$ is independent of the flow*. In other words, it can be measured in advance when the flow is turned off, and can be subtracted from the forces measured when the flow is turned on (in constant and uniform density flows, this procedure would be equivalent to removing buoyancy forces). Therefore, without too much loss in generality, the term in volume force is excluded from the rest of the discussion.

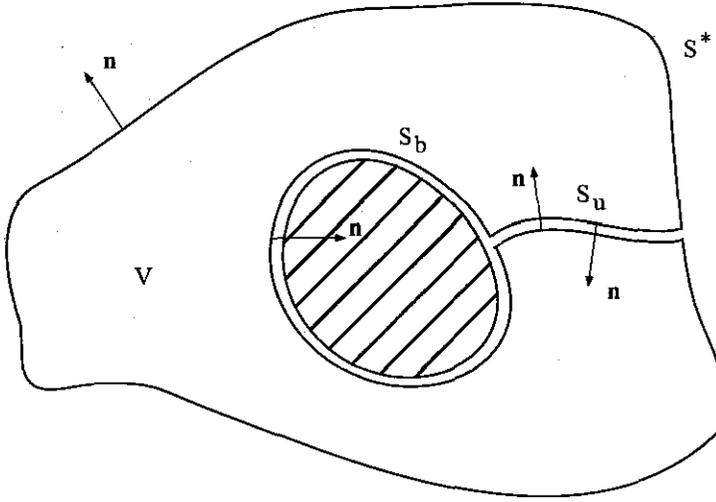


Figure 1: Control volume analysis.

Let us look closely at the surface integral over the umbilicus. If the umbilicus is chosen to be infinitesimal cross section, then the integrand varies only with the longitudinal coordinate of the umbilicus. In other words, it can be approximated to be constant along the perimeter of the umbilicus. By slicing the umbilicus into rings of infinitesimal area $\delta S_u(t)$, and taking the surface integral over one of these rings, one obtains:

$$\int_{\delta S_u(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS \approx \boldsymbol{\Sigma}^T[\delta S_u(t)] \cdot \int_{\delta S_u(t)} \hat{\mathbf{n}} dS, \quad (6)$$

where $\boldsymbol{\Sigma}^T[\delta S_u(t)]$ is the value of the integrand $\boldsymbol{\Sigma}^T$ at the longitudinal position of the ring $\delta S_u(t)$. Since:

$$\int_{\delta S_u(t)} \hat{\mathbf{n}} dS = 0, \quad (7)$$

the integration over the umbilicus does not contribute to the total surface integral, and we can write:

$$\oint_{S_m^*(t) \oplus S_b(t) \oplus S_u(t)} \Phi dS = \oint_{S_m^*(t)} \Phi dS + \oint_{S_b(t)} \Phi dS, \quad (8)$$

where it is to be remembered that the unit normal $\hat{\mathbf{n}}$ over $S_b(t)$ is *into* the body.

Now, let us look at the surface integral over the body. This integral is just the force exerted by the body on the fluid. For the force of the fluid on the body, we then have:

$$\mathbf{F} = - \oint_{S_b(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \quad (9)$$

Finally, we obtain:

$$\mathbf{F} = -\frac{d}{dt} \int_{V_m(t)} \rho \mathbf{u} dV + \oint_{S_m^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \quad (10)$$

In practical applications, one prefers to use an arbitrary volume instead of a material volume. Assume that at time t , the material volume $V_m(t)$ coincides with an arbitrary volume $V(t)$. Then, we can use the identity²:

$$\frac{d}{dt} \int_{V_m(t)} \Phi dV = \frac{d}{dt} \int_{V(t)} \Phi dV + \oint_{S(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \Phi dS, \quad (11)$$

where $S(t) = S_m(t)$ and \mathbf{u}_S is the velocity of the surface $S(t)$. As a result, the force exerted by the fluid on the body is:

$$\begin{aligned} \mathbf{F} = & -\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV - \oint_{S(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \rho \mathbf{u} dS \\ & + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \end{aligned} \quad (12)$$

This is the most general expression for the hydrodynamic force acting on a body. Note that the second integral in Equation 12 is taken over the whole surface $S(t)$ whereas the third one is taken over only the exterior surface $S^*(t)$. At this point, several options are possible, depending on how one wishes to use this expression for drag measurements. The possibilities will be reviewed in the following sections.

1.2 Boundary conditions

An important feature of bluff body flows is the motion of the body itself as well as the behavior of the flow velocity at the body surface. A myriad of boundary conditions are possible, and here, only the main ones will be mentioned.

1. Fluid flow conditions at the body surface

- No through flow condition:

$$(\mathbf{u} - \mathbf{u}_S) \cdot \hat{\mathbf{n}}|_b = 0. \quad (13)$$

- No slip condition:

$$(\mathbf{u} - \mathbf{u}_S)|_b = 0. \quad (14)$$

2. Wall velocity conditions:

- Rigid body motion:

$$\mathbf{u}_S|_b = \mathbf{f}(t) + \mathbf{x}_b \wedge \mathbf{g}(t), \quad (15)$$

where $\mathbf{f}(t)$ is the *translational* velocity of the body and $\mathbf{g}(t)$ is the *angular* velocity.

²Candel S.: *Mécanique des fluides*, Dunod, 1990.

- Body motion with sliding walls:

$$\mathbf{u}_S \wedge \hat{\mathbf{n}}|_b = \mathbf{f}[\mathbf{x}_b(t), t], \quad (16)$$

$$\mathbf{u}_S \cdot \hat{\mathbf{n}}|_b = \mathbf{g}(t). \quad (17)$$

1.3 Application to wake surveys for time-averaged flows

In some drag measurement procedures, it is desired to measure the average drag of a body by measuring quantities on the surface of the control volume only. It is customary (and practical) in wake surveys to have the exterior surface of the control volume fixed. Decomposing as before the surface $S(t)$ into the exterior surface S^* (now assumed to be independent of time), the body surface $S_b(t)$ (still a function of time if the body moves), and the umbilicus surface $S_u(t)$, and making the surface integral over the umbilicus to vanish, we obtain:

$$\begin{aligned} \mathbf{F} = & -\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV - \oint_{S_b(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \rho \mathbf{u} dS \\ & + \oint_{S^*} \hat{\mathbf{n}} \cdot (-\rho \mathbf{u} \mathbf{u} + \boldsymbol{\Sigma}) dS. \end{aligned} \quad (18)$$

Note that the volume $V(t)$ is still a function of time despite of the fact that the exterior surface is fixed. The reason is that the the body can be deformable or be moving around as a solid body³.

We will now assume that the body motion coupled with the flow is stochastically periodic with average period T . By this statement, we mean that:

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=N} \int_{t+nT}^{t+(n+1)T} \left(\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV \right) dt \equiv 0, \quad (19)$$

or, in abbreviated notation:

$$\left\langle \frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV \right\rangle \equiv 0. \quad (20)$$

In other words, wake surveys allow drag measurements of time-averaged flows only. Note that the time averaging was necessary to remove the integral over

³One way to see this would be to expand the time derivative of the integral as follows:

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \mathbf{u} dV &= \int_{V(t)} \frac{\partial \mathbf{u}}{\partial t} dV \\ &+ \oint_{S_b(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S \mathbf{u} dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S \mathbf{u} dS, \end{aligned}$$

where the surface integral over S^* vanishes if the surface is fixed. If the volume $V(t)$ had not been a function of time, the surface integral over $S_b(t)$ would not have been there. A sufficient (although not necessary) condition for this surface integral to vanish is for $\mathbf{u}_S \cdot \hat{\mathbf{n}}$ to be equal to zero.

the volume $V(t)$. Finally:

$$\begin{aligned} \langle \mathbf{F} \rangle &= - \left\langle \oint_{S_b(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \rho \mathbf{u} dS \right\rangle \\ &+ \oint_{S^*} \langle \hat{\mathbf{n}} \cdot (-\rho \mathbf{u} \mathbf{u} + \boldsymbol{\Sigma}) \rangle dS. \end{aligned} \quad (21)$$

Moreover, if it is assumed that the no-through flow condition at the body surface is satisfied, then we have, by construction:

$$\hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S)|_b \equiv 0, \quad (22)$$

and the surface integral over the body vanishes⁴.

Finally:

$$\langle \mathbf{F} \rangle = \oint_{S^*} \hat{\mathbf{n}} \cdot \langle -\rho \mathbf{u} \mathbf{u} + \boldsymbol{\Sigma} \rangle dS. \quad (23)$$

Equation 23 cannot be used as yet without further assumptions regarding the value of the stress tensor $\boldsymbol{\Sigma}$ on the surface S^* . If the surface lies some distance away from the body, the viscous stress tensor \mathbf{T} becomes negligible, and only the pressure term remains.

$$\langle \mathbf{F} \rangle = \oint_{S^*} \hat{\mathbf{n}} \cdot \langle -\rho \mathbf{u} \mathbf{u} - p \mathbf{l} \rangle dS \quad (24)$$

Since wake surveys involve only flow velocity measurements on the surface S^* , the pressure has to be inferred. In general, it is assumed that the pressure in wake of the body is equal to the pressure outside the wake of the body, where we know Bernoulli's equation can be effectively used⁵ (similar to a boundary layer assumption). This assumption is in general valid only far downstream of the body. Other possibilities exist for inferring this pressure with more precision⁶.

To conclude, wake surveys can be used only for time-averaged forces, and have to be made several body diameters away from the body, where further assumptions about pressure can be made.

2 Instantaneous force measurements in incompressible flows

2.1 Expression for fluid dynamic forces

In this section, it will be assumed that the density is uniform and constant, and we will set $\rho = 1$. The starting point is again Equation 12, which we rewrite for

⁴This term would be present in the case of flows with surface injection, but it could be evaluated in a straightforward manner.

⁵Schlichting H., *Boundary Layer Theory*, McGraw Hill, 1987.

⁶Dimotakis P., *Laser Doppler velocimetry momentum defect measurements of cable drag at low to moderate Reynolds numbers*, NCBC Report R541, 1977.

the sake of clarity:

$$\mathbf{F} = -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV - \oint_{S(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \quad (25)$$

This equation will be the starting point for the derivation of our equation.

Some preliminary comments can be made about this equation. If the no-through flow condition still applies, and the outer surface S^* is taken as being fixed, then it is straightforward to show that:

$$\mathbf{F} = -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV + \oint_{S^*} \hat{\mathbf{n}} \cdot (-\mathbf{u}\mathbf{u} + \boldsymbol{\Sigma}) dS, \quad (26)$$

or in terms of the pressure and the viscous stress tensor:

$$\mathbf{F} = -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV + \oint_{S^*} \hat{\mathbf{n}} \cdot (-\mathbf{u}\mathbf{u} - p\mathbf{l} + \boldsymbol{\tau}) dS. \quad (27)$$

Except for the pressure, all other quantities are computable from the velocity field or its derivatives. In theory, the pressure can be evaluated at the surface of the body from the relation⁷:

$$\hat{\mathbf{n}} \wedge \nabla p = -\hat{\mathbf{n}} \wedge \mathbf{a}_b + \nu \hat{\mathbf{n}} \cdot \nabla \boldsymbol{\omega} - \nu (\hat{\mathbf{n}} \wedge \nabla) \cdot [(\boldsymbol{\omega} - 2\boldsymbol{\Omega}_b) \wedge \hat{\mathbf{n}}], \quad (28)$$

where \mathbf{a}_b and $\boldsymbol{\Omega}_b$ are respectively the acceleration and the angular velocity of the body surface. This equation can be integrated for the pressure p up to an additive constant. This additive constant can be inferred if, for example, the front part of the body is in contact with irrotational flow⁸. If the surface S^* is then made to coincide with the body surface, the force on the body can be computed. Note, however, that this method requires the evaluation of vorticity derivatives on the body surface. Experimentally, it may be very difficult to measure these quantities, not withstanding the fact that the additive constant has to be evaluated.

Alternatively, if the surface S^* is chosen arbitrarily, the Poisson equation:

$$\nabla^2 p = \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \quad (29)$$

could be integrated in the domain $V(t)$ knowing the pressure at infinity and on the surface of the body. However, the dilemma is still present since the pressure at infinity is not known, and has to be inferred from additional assumptions.

⁷Wu J. Z., *Acta Aerodyn. Sinica*, 4 (1986) 168.

⁸Spalart P.R., *Von Karman Lectures*, March 1988.

2.2 The Burgatti identity

Most of the transformations that will be used in the next sections rely on the following identity:

$$\int_V \mathbf{x} \wedge \nabla \wedge \mathbf{a} dV = (\mathcal{N} - 1) \int_V \mathbf{a} dV + \oint_S \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a}) dS, \quad (30)$$

where \mathcal{N} is the dimension of the space you are working in ($\mathcal{N} = 3$ in 3D and $\mathcal{N} = 2$ in 2D for example). This identity was already known to Lamb⁹, but only through integration by parts. Burgatti¹⁰ gave a generalized form of this identity, and for this reason, we will refer to the derived form in Equation 30 as *the Burgatti identity*. It is used extensively by Saffman¹¹.

Note that the form of the equation depends strongly on the dimension of the space, \mathcal{N} . It is instructive to see how the space dimension arises in this identity. Consider the generalized form of this identity as given by Burgatti:

$$\int_V [(\nabla \wedge \mathbf{a}) \wedge \mathbf{b} + (\nabla \wedge \mathbf{b}) \wedge \mathbf{a} + \mathbf{a}(\nabla \cdot \mathbf{b}) + \mathbf{b}(\nabla \cdot \mathbf{a})] dV = \oint_S [\hat{\mathbf{n}} \cdot (\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})\hat{\mathbf{n}}] dS. \quad (31)$$

Now let $\mathbf{b} = \mathbf{x}$, and rewrite the above equation, keeping in mind that:

$$\nabla \wedge \mathbf{x} = 0, \quad (32)$$

and

$$\nabla \cdot \mathbf{x} = \mathcal{N}, \quad (33)$$

which yields:

$$\int_V [(\nabla \wedge \mathbf{a}) \wedge \mathbf{x} + \mathcal{N}\mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a})] dV = \oint_S [\hat{\mathbf{n}} \cdot (\mathbf{a}\mathbf{x} + \mathbf{x}\mathbf{a}) - (\mathbf{a} \cdot \mathbf{x})\hat{\mathbf{n}}] dS. \quad (34)$$

Also, using the vector identities:

$$\mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a}) = (\mathbf{a} \cdot \mathbf{x})\hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot (\mathbf{x}\mathbf{a}), \quad (35)$$

and

$$\nabla \cdot (\mathbf{a}\mathbf{x}) = \mathbf{x}(\nabla \cdot \mathbf{a}) + \mathbf{a}, \quad (36)$$

one obtains:

$$\begin{aligned} \int_V \mathbf{x} \wedge (\nabla \wedge \mathbf{a}) dV = & \\ & (\mathcal{N} - 1) \int_V \mathbf{a} dV + \oint_S \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a}) dS \\ & + \int_V \nabla \cdot (\mathbf{a}\mathbf{x}) dV - \oint_S \hat{\mathbf{n}} \cdot (\mathbf{a}\mathbf{x}) dS. \end{aligned} \quad (37)$$

⁹Lamb H., *Hydrodynamics*, §152, Cambridge University Press, 1924.

¹⁰Burgatti P., *Bolletino della Unione Matematica Italiana*, 10 (1931) 1-5; see also Truesdell C., *Kinematics of vorticity*, Equation 7.4, Indiana University Press, 1954.

¹¹Saffman P.E., *Vortex dynamics*, Cambridge University Press, 1993.

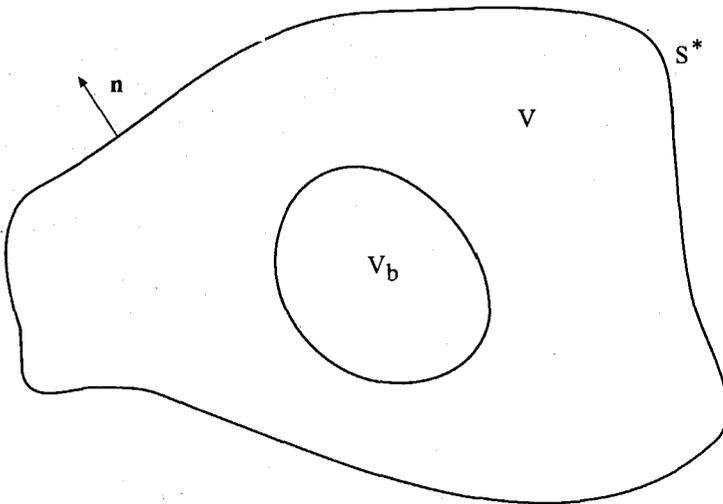


Figure 2: The fluidic body.

By Green's theorem, the last two integrals cancel, and one finally obtains Burgatti identity as given in Equation 30.

Throughout the derivations, the space dimension \mathcal{N} will be kept in the formulae. However, most of the equations involve the vorticity vector which can be defined rigorously only in three dimensions, that is $\mathcal{N} = 3$. Nevertheless, a vorticity vector for planar flows can be defined by extending the vortex lines to infinity in a direction normal to the plane of the flow. In this case, the flow is effectively two dimensional, and we can set $\mathcal{N} = 2$.

2.3 The “fluidic” body

To carry on the algebra, it is of practical interest to replace the body by a mass of fluid of unit density (as the surrounding fluid). The picture that now emerges is the one shown in Figure 2, in which a no-through flow envelope separates the fluidic body of volume V_b from the external fluid of volume V . The exterior surface S^* bounds the volumes V and V_b .

The viscosity of the “fluidic” body is left arbitrary. Many cases of interest arise, depending on the viscosity of the body:

1. The body has the same viscosity as the surrounding fluid. This is a case of slip flow for either a viscous or inviscid flow.
2. The body has a different viscosity than the surrounding fluid. Two cases of interest arise:
 - The body has infinite viscosity and the external fluid is inviscid. This is a case of slip flow over a solid (not necessarily rigid) body. Note that a vortex sheet is present at the surface of the body.

- The body has infinite viscosity and the external fluid is viscous. This is the general case of no-slip viscous flow past a solid body. In this case, the flow velocity equals the body surface velocity (boundary layer flow), and no vortex sheet is present (which just means that the flow velocity is continuous across the “fluidic” body surface).

In the discussion that follows, it will be assumed that there is no surface injection, and as such, that the volume of the body remains constant. The body will be, therefore, a material volume.

3 “Vortex method” equation

3.1 Boundary conditions at infinity

The topic of boundary conditions at infinity is not easy to tackle. In this section, only boundary conditions proper to vortex methods will be considered.

In vortex methods, all the vorticity is produced at the surface of the body and is generally confined in a finite region of space. As such, some deductions can be made regarding vorticity and velocity at infinity.

When the vorticity decays exponentially at large distances, then the velocity obeys the following relations¹²:

$$r \rightarrow \infty, \quad |\boldsymbol{\omega}| \sim e^{-\alpha r}, \quad |\mathbf{u}| \sim \begin{cases} r^{-3} & \text{3D} \\ r^{-1} & \text{2D} \end{cases} \quad (38)$$

Note, however, that no condition can be imposed on the pressure at infinity.

3.2 Preliminary assumptions

As a result of the boundary conditions at infinity, the integral of the velocity terms over S^* in Equation 27 vanishes, that is:

$$\oint_{S^*} \hat{\mathbf{n}} \cdot (-\mathbf{u}\mathbf{u} + \mathbf{T}) dS \sim 0. \quad (39)$$

A priori, nothing can be said about the pressure integral. In vortex methods, it is tacitly assumed that this pressure integral is null:

$$\oint_{S^*} p \hat{\mathbf{n}} dS \sim 0. \quad (40)$$

However, we will see that this assumption is not necessary, notwithstanding the fact that it is incorrect as well. As a result, we will depart slightly from a typical vortex method derivation so as to include this pressure integral. The resulting equation which is of use in vortex methods:

$$\mathbf{F} = -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV - \oint_{S^*} p \hat{\mathbf{n}} dS. \quad (41)$$

¹²Batchelor G. K., *An introduction to fluid dynamics*, Cambridge University Press, 1967; Ting L., *J. Fluid Mech.*, **127** (1983) 497.

3.3 Vortex method equation

What follows is a derivation similar to one given by Leonard¹³. First, Equation 41 is modified by considering the body to be of unit density, and as such, to be part of the whole fluid. The volume under consideration is now the volume $V(t) + V_b(t)$ bounded by the surface S^* , where $V_b(t)$ is the volume of the “fluidic” body. Then:

$$\mathbf{F} = -\frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{u} dV + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV - \oint_{S^*} p \hat{\mathbf{n}} dS. \quad (42)$$

By using the Burgatti identity with $\mathbf{a} = \mathbf{u}$ on the first integral, one obtains:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV + \frac{1}{\mathcal{N}-1} \frac{d}{dt} \oint_{S^*} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS - \oint_{S^*} p \hat{\mathbf{n}} dS. \end{aligned} \quad (43)$$

Let us look at the time derivative of the surface integral over S^* . The time derivative can be taken inside the integral:

$$\frac{1}{\mathcal{N}-1} \frac{d}{dt} \oint_{S^*} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS = \frac{1}{\mathcal{N}-1} \oint_{S^*} \mathbf{x} \wedge \left(\hat{\mathbf{n}} \wedge \frac{\partial \mathbf{u}}{\partial t} \right) dS. \quad (44)$$

The integrand can be evaluated through the use of the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p - \nabla \cdot (\mathbf{u}\mathbf{u}) + \nabla \cdot \mathbf{T}. \quad (45)$$

With the proper behavior of the velocity at infinity (as indicated above), the integral of the inertia and viscous terms over S^* vanishes, that is:

$$\oint_{S^*} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (-\nabla \cdot (\mathbf{u}\mathbf{u}) + \nabla \cdot \mathbf{T})] dS = 0. \quad (46)$$

The pressure term can be given a different form by using the Burgatti identity (Equation 30), and this yields:

$$-\frac{1}{\mathcal{N}-1} \oint_{S^*} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \nabla p) dS = \int_{V(t)+V_b(t)} \nabla p dV, \quad (47)$$

where the integration is over the closed surface S^* which encloses both the fluid and the “fluidic” body. The right hand side can be converted to a surface integral by Green’s theorem:

$$\int_{V(t)+V_b(t)} \nabla p dV = \oint_{S^*} p \hat{\mathbf{n}} dS. \quad (48)$$

¹³see for example Koumoutsakos P. & Leonard A., *Direct numerical simulations of unsteady separated flows using vortex methods*, PhD Thesis, California Institute of Technology, 1993.

Leonard again tacitly assumes that this integral vanishes, but we see that this assumption is not necessary. As a matter of fact, this integral cancels the one in Equation 43, and the resulting equation becomes:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV - \frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV. \end{aligned} \quad (49)$$

This is the most general equation for the instantaneous force on a body with the following assumptions:

- The flow is incompressible.
- The fluid is of unit density, and so is the “fluidic” body.
- The no-through flow condition applies on the “body” boundary.
- The volume $V(t)$ extends to infinity and encloses *all* the vorticity.
- The surface integrals of the viscous and convective terms vanish at infinity.
- No assumption is made about pressure at infinity.

4 Instantaneous force equation for incompressible, viscous, rotational flows: the Moreau equation

4.1 Derivation

We are now ready to derive a general equation for the force on a bluff body in cross flow, in terms of the vorticity field in its vicinity. The expression for the force was first obtained in 1952 by Moreau¹⁴. The key step in the derivation lies in the use of the Burgatti identity.

Let us look back at Equation 25, which we reproduce:

$$\begin{aligned} \mathbf{F} = & -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV - \oint_{S(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS \\ & + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \end{aligned} \quad (50)$$

The surface integral over $S(t)$ can be decomposed, as before, into three surface integrals, and by making the surface integral over the umbilicus to vanish, we obtain:

$$\oint_{S(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS = \oint_{S_b(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS. \quad (51)$$

¹⁴Moreau J. J., *J. Math. Pures Appl.* **31** (1952) 355-375; **32** (1953) 1-78.

If the no through flow condition applies at the body surface, the surface integral over $S_b(t)$ vanishes. Now, the surface integral over $S^*(t)$ can be decomposed into two terms as follows:

$$\oint_{S^*(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) \mathbf{u} dS = \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u} \mathbf{u} dS - \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S \mathbf{u} dS. \quad (52)$$

The first integral can be transformed into a volume integral using Green's theorem:

$$\oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u} \mathbf{u} dS = \int_{V(t)+V_b(t)} \nabla \cdot (\mathbf{u} \mathbf{u}) dV, \quad (53)$$

where the volume integration includes the fluidic body as well. We then use the identity:

$$\nabla \cdot (\mathbf{u} \mathbf{u}) = (\nabla \cdot \mathbf{u}) \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}. \quad (54)$$

Since the flow is incompressible:

$$\nabla \cdot \mathbf{u} = 0, \quad (55)$$

and as a consequence:

$$\nabla \cdot (\mathbf{u} \mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u}. \quad (56)$$

We then use the vector identity:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left(\frac{1}{2} u^2 \right) - \mathbf{u} \wedge \boldsymbol{\omega}, \quad (57)$$

where $\boldsymbol{\omega}$ is the vorticity:

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u}. \quad (58)$$

Equation 53 becomes:

$$\int_{V(t)+V_b(t)} \nabla \cdot (\mathbf{u} \mathbf{u}) dV = \int_{V(t)+V_b(t)} \nabla \left(\frac{1}{2} u^2 \right) dV - \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV. \quad (59)$$

Finally, we convert the first integral back into a surface integral using Green's theorem:

$$\int_{V(t)+V_b(t)} \nabla \left(\frac{1}{2} u^2 \right) dV = \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \frac{1}{2} u^2 \mathbf{l} dS, \quad (60)$$

and Equation 50 becomes:

$$\begin{aligned} \mathbf{F} = & -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV + \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\ & + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \left(-\frac{1}{2} u^2 \mathbf{l} + \boldsymbol{\Sigma} + \mathbf{u}_S \mathbf{u} \right) dS. \end{aligned} \quad (61)$$

Now, the stress tensor can be expanded into the pressure tensor and the viscous stress tensor such that:

$$\begin{aligned} \mathbf{F} = & -\frac{d}{dt} \int_{V(t)} \mathbf{u} dV + \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\ & + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \left[\left(-p - \frac{1}{2} u^2 \right) \mathbf{l} + \mathbf{T} + \mathbf{u}_S \mathbf{u} \right] dS. \end{aligned} \quad (62)$$

The integral involving the time derivative can be transformed using Burgatti identity such that:

$$\begin{aligned}
\mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV + \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\
& + \frac{1}{\mathcal{N}-1} \frac{d}{dt} \oint_{S^*(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV \\
& + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot [(-p - \frac{1}{2}u^2)\mathbf{I} + \mathbf{T} + \mathbf{u}_S \mathbf{u}] dS.
\end{aligned} \tag{63}$$

The time derivative of the surface integral can be manipulated through the use of a kinematic identity¹⁵:

$$\frac{d}{dt} \oint_{S(t)} \hat{\mathbf{n}} \cdot \Phi dS = \oint_{S(t)} \hat{\mathbf{n}} \cdot \left[\frac{\partial \Phi}{\partial t} + \mathbf{u}_S (\nabla \cdot \Phi) \right] dS. \tag{64}$$

We can write the integrand of interest as follows:

$$\mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) = \hat{\mathbf{n}} \cdot [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}], \tag{65}$$

so that the Aris identity in Equation 64 can be used with:

$$\Phi = (\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}, \tag{66}$$

which yields:

$$\frac{d}{dt} \oint_{S(t)} \hat{\mathbf{n}} \cdot [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] dS = \tag{67}$$

$$\oint_{S(t)} \hat{\mathbf{n}} \cdot \left\{ \frac{\partial}{\partial t} [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] + \mathbf{u}_S \nabla \cdot [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] \right\} dS. \tag{68}$$

The time derivative can be transformed back to:

$$\frac{\partial}{\partial t} [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] = \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \frac{\partial \mathbf{u}}{\partial t}), \tag{69}$$

where the \mathbf{u} time derivative can be obtained from the Navier Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla(p + \frac{1}{2}u^2) + \mathbf{u} \wedge \boldsymbol{\omega} + \nabla \cdot \mathbf{T}. \tag{70}$$

The divergence term becomes:

$$\nabla \cdot [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] = \nabla(\mathbf{x} \cdot \mathbf{u}) - \nabla \cdot (\mathbf{x}\mathbf{u}). \tag{71}$$

Each term can be evaluated separately:

$$\nabla(\mathbf{x} \cdot \mathbf{u}) = (\mathbf{x} \cdot \nabla)\mathbf{u} + \mathbf{u} + \mathbf{x} \wedge \boldsymbol{\omega}, \tag{72}$$

¹⁵Aris R., *Vectors, tensors, and the basic equations of fluid mechanics*, Dover, 1962; Aris only gives the relation for a vector, but is easy to generalize it to a tensor.

and:

$$\nabla \cdot (\mathbf{x}\mathbf{u}) = \mathcal{N}\mathbf{u} + (\mathbf{x} \cdot \nabla)\mathbf{u}, \quad (73)$$

therefore:

$$\nabla \cdot [(\mathbf{x} \cdot \mathbf{u})\mathbf{I} - \mathbf{x}\mathbf{u}] = -(\mathcal{N} - 1)\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\omega}. \quad (74)$$

Finally, we obtain:

$$\begin{aligned} \frac{d}{dt} \oint_{S^*(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS = \\ \oint_{S^*(t)} \mathbf{x} \wedge \left\{ \hat{\mathbf{n}} \wedge \left[-\nabla \left(p + \frac{1}{2}u^2 \right) + \mathbf{u} \wedge \boldsymbol{\omega} + \nabla \cdot \mathbf{T} \right] \right\} dS \\ + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S [-(\mathcal{N} - 1)\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\omega}] dS. \end{aligned} \quad (75)$$

Now, the integral involving the pressure tensor can be transformed using Burgatti identity:

$$\begin{aligned} \oint_{S^*(t)} \mathbf{x} \wedge \left\{ \hat{\mathbf{n}} \wedge \left[-\nabla \left(p + \frac{1}{2}u^2 \right) \right] \right\} dS = \\ (\mathcal{N} - 1) \oint_{V(t)+V_b(t)} \nabla \left(p + \frac{1}{2}u^2 \right) dV, \end{aligned} \quad (76)$$

which can be converted back to a surface integral using Green's theorem:

$$\oint_{S^*(t)} \mathbf{x} \wedge \left\{ \hat{\mathbf{n}} \wedge \left[-\nabla \left(p + \frac{1}{2}u^2 \right) \right] \right\} dS = (\mathcal{N} - 1) \oint_{S^*(t)} \left(p + \frac{1}{2}u^2 \right) \hat{\mathbf{n}} dS. \quad (77)$$

Finally:

$$\begin{aligned} \frac{d}{dt} \oint_{S^*(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS = \\ \oint_{S^*(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\mathbf{u} \wedge \boldsymbol{\omega})] dS \\ + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S [-(\mathcal{N} - 1)\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\omega}] dS \\ + (\mathcal{N} - 1) \oint_{S^*(t)} \left(p + \frac{1}{2}u^2 \right) \hat{\mathbf{n}} dS + \oint_{S^*(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\nabla \cdot \mathbf{T})] dS. \end{aligned} \quad (78)$$

The first integral on the right hand side can be modified further:

$$\begin{aligned} \oint_{S^*(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\mathbf{u} \wedge \boldsymbol{\omega})] dS = \\ - \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u} (\mathbf{x} \wedge \boldsymbol{\omega}) dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\omega} (\mathbf{x} \wedge \mathbf{u}) dS. \end{aligned} \quad (79)$$

Then, Equation 78 becomes:

$$\begin{aligned}
\frac{d}{dt} \oint_{S^*(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS = & \\
- \oint_{S^*(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) (\mathbf{x} \wedge \boldsymbol{\omega}) dS & \quad (80) \\
- (\mathcal{N} - 1) \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \mathbf{u}_S \mathbf{u} dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\omega} (\mathbf{x} \wedge \mathbf{u}) dS & \\
+ (\mathcal{N} - 1) \oint_{S^*(t)} (p + \frac{1}{2}u^2) \hat{\mathbf{n}} dS + \oint_{S^*(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\nabla \cdot \boldsymbol{\tau})] dS. &
\end{aligned}$$

Inserting Equation 80 back into Equation 63, one finally obtains the desired equation:

$$\begin{aligned}
\mathbf{F} = & - \frac{1}{\mathcal{N} - 1} \frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV + \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\
& - \frac{1}{\mathcal{N} - 1} \oint_{S^*(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S) (\mathbf{x} \wedge \boldsymbol{\omega}) dS \quad (81) \\
& + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV + \frac{1}{\mathcal{N} - 1} \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\omega} (\mathbf{x} \wedge \mathbf{u}) dS \\
& + \frac{1}{\mathcal{N} - 1} \oint_{S^*(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\nabla \cdot \boldsymbol{\tau})] dS + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\tau} dS.
\end{aligned}$$

This is the most general equation for the force on a body with a no-through flow condition at the surface of the body. It is identical to the equation obtained by Moreau¹⁶. The body is not required to be neither rigid nor solid, but its density has to be the same as the one of the fluid and its volume has to remain constant. It is valid for rotational and viscous flows. The control volume which encloses the body is arbitrary, and can be chosen at will. The surface of the control volume can lie anywhere in the flow as long as it englobes the body.

4.2 Alternative forms

This expression can take two alternative forms. First, we may note the identity:

$$\int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV = \oint_{S^*(t)} [\frac{1}{2}u^2 \hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \mathbf{u}) \mathbf{u}] dS, \quad (82)$$

and, therefore, Equation 81 becomes:

$$\begin{aligned}
\mathbf{F} = & - \frac{1}{\mathcal{N} - 1} \frac{d}{dt} \int_{V(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV + \oint_{S^*(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\tau} dS \\
& + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV - \frac{1}{\mathcal{N} - 1} \frac{d}{dt} \int_{V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV, \quad (83)
\end{aligned}$$

¹⁶Moreau J. J., *J. Math. Pures Appl.*, Equation 14.5, **32** (1953) 41.

where the tensor \mathbf{T} is:

$$\begin{aligned} \mathbf{T} = & \frac{1}{2}u^2\mathbf{I} - \mathbf{u}\mathbf{u} - \frac{1}{\mathcal{N}-1}(\mathbf{u} - \mathbf{u}_S)(\mathbf{x} \wedge \boldsymbol{\omega}) + \frac{1}{\mathcal{N}-1}\boldsymbol{\omega}(\mathbf{x} \wedge \mathbf{u}) \\ & + \frac{1}{\mathcal{N}-1}[\mathbf{x} \cdot (\nabla \cdot \mathbf{T})\mathbf{I} - \mathbf{x}(\nabla \cdot \mathbf{T})] + \mathbf{T}. \end{aligned} \quad (84)$$

Yet another form can be obtained by considering the volume $V(t)$ to be material such that on $S^*(t)$, $\mathbf{u}_S = \mathbf{u}$. Then:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1}\frac{d}{dt}\int_{V_m(t)+V_b(t)}\mathbf{x} \wedge \boldsymbol{\omega} dV + \int_{V_m(t)+V_b(t)}\mathbf{u} \wedge \boldsymbol{\omega} dV \\ & + \oint_{S_m^*(t)}\hat{\mathbf{n}} \cdot \mathbf{T}_m dS \\ & + \frac{d}{dt}\int_{V_b(t)}\mathbf{u} dV, \end{aligned} \quad (85)$$

where:

$$\mathbf{T}_m = \frac{1}{\mathcal{N}-1}\boldsymbol{\omega}(\mathbf{x} \wedge \mathbf{u}) + \frac{1}{\mathcal{N}-1}[\mathbf{x} \cdot (\nabla \cdot \mathbf{T})\mathbf{I} - \mathbf{x}(\nabla \cdot \mathbf{T})] + \mathbf{T}. \quad (86)$$

4.3 Alternative derivation

For the sake of completeness, we will derive this equation by an alternative procedure which makes use of material volumes. The starting point is Equation 10:

$$\mathbf{F} = -\frac{d}{dt}\int_{V_m(t)}\rho\mathbf{u} dV + \oint_{S_m^*(t)}\hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS. \quad (87)$$

Now, using Burgatti identity, the first integral can be transformed to give:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1}\frac{d}{dt}\int_{V_m(t)+V_b(t)}\mathbf{x} \wedge \boldsymbol{\omega} dV \\ & + \frac{1}{\mathcal{N}-1}\frac{d}{dt}\oint_{S_m^*(t)}\mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS + \oint_{S_m^*(t)}\hat{\mathbf{n}} \cdot \boldsymbol{\Sigma} dS \\ & + \frac{d}{dt}\int_{V_b(t)}\mathbf{u} dV. \end{aligned} \quad (88)$$

The time derivative of the surface integral can be manipulated through the use of Zorawski kinematic identity¹⁷:

$$\frac{d}{dt}\oint_{S_m(t)}\hat{\mathbf{n}} \cdot \Phi dS = \oint_{S_m(t)}\hat{\mathbf{n}} \cdot \left[\frac{D\Phi}{Dt} + \Phi(\nabla \cdot \mathbf{u}) - (\nabla \mathbf{u})^T \cdot \Phi \right] dS, \quad (89)$$

where Φ is a tensor and \mathbf{u} the flow velocity. The integrand can be transformed by noting that:

$$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + (\mathbf{u} \cdot \nabla)\Phi, \quad (90)$$

¹⁷Truesdell C., *Kinematics of vorticity*, Indiana University Press, 1954.

and by using the following identity:

$$\nabla \wedge (\mathbf{u} \wedge \Phi) = \mathbf{u}(\nabla \cdot \Phi) + (\nabla \mathbf{u})^T \cdot \Phi - \Phi(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\Phi, \quad (91)$$

where \mathbf{u} is an arbitrary vector. Then, Zorawski identity can be written as:

$$\frac{d}{dt} \oint_{S_m(t)} \hat{\mathbf{n}} \cdot \Phi dS = \oint_{S_m(t)} \hat{\mathbf{n}} \cdot \left[\frac{\partial \Phi}{\partial t} - \nabla \wedge (\mathbf{u} \wedge \Phi) + \mathbf{u}(\nabla \cdot \Phi) \right] dS. \quad (92)$$

Now, since the integration is over the complete surface, then:

$$\oint_{S_m(t)} \hat{\mathbf{n}} \cdot [\nabla \wedge (\mathbf{u} \wedge \Phi)] dS \equiv 0, \quad (93)$$

and Equation 92 becomes:

$$\frac{d}{dt} \oint_{S_m(t)} \hat{\mathbf{n}} \cdot \Phi dS = \oint_{S_m(t)} \hat{\mathbf{n}} \cdot \left[\frac{\partial \Phi}{\partial t} + (\nabla \cdot \Phi)\mathbf{u} \right] dS, \quad (94)$$

which is just Aris identity given by Equation 64 with $\mathbf{u}_S = \mathbf{u}$. Actually, we could have used Aris identity from the start and set $\mathbf{u}_S = \mathbf{u}$ since the surface is material. The procedure is then identical to the one following Equation 64, and the result is Equation 85.

4.4 Force equation for inviscid, rotational flows

This equation is fully derived in Saffman monograph¹⁸. This equation can be reproduced with the two following assumptions:

$$\boldsymbol{\omega} \cdot \hat{\mathbf{n}}|_{S^*(t)} = 0, \quad (95)$$

and:

$$\mathbf{T}|_{S^*(t)} = 0. \quad (96)$$

Then, Equation 81 takes the form:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV + \int_{V(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\ & - \frac{1}{\mathcal{N}-1} \oint_{S^*(t)} \hat{\mathbf{n}} \cdot (\mathbf{u} - \mathbf{u}_S)(\mathbf{x} \wedge \boldsymbol{\omega}) dS \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV, \end{aligned} \quad (97)$$

or, in terms of material volumes:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V_m(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV + \int_{V_m(t)+V_b(t)} \mathbf{u} \wedge \boldsymbol{\omega} dV \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV, \end{aligned} \quad (98)$$

¹⁸Saffman P.E., *Vortex dynamics*, Chapter 3, 4 & 7, Cambridge University Press, 1993.

where the body is assumed to be a material volume. We have, thus, recovered Saffman equation¹⁹. We see that the assumption of inviscid flow can be relaxed by the more precise statement that the viscous stresses have to be null on the surface $S^*(t)$. The assumptions underlying this equation are therefore:

- The flow is incompressible and viscous.
- The fluid is of unit density, and so is the “fluidic” body.
- The no-through flow condition applies on the “body” boundary.
- On the surface $S^*(t)$, the flow is irrotational and the viscous stresses are null.
- No assumption is made about pressure.

If the surface $S^*(t)$ lies in rotational flow, then Equation 81 has to be used.

4.5 Vortex method equation

With boundary conditions given at infinity by Equation 38, we see that the surface integral over S^* (which is now assumed to be fixed) in Equation 83 cancels since:

$$r \rightarrow \infty, \quad |\boldsymbol{\Upsilon}| \sim \begin{cases} r^{-6} & \text{3D} \\ r^{-2} & \text{2D} \end{cases} \quad (99)$$

and therefore:

$$\oint_{S^*} \hat{\mathbf{n}} \cdot \boldsymbol{\Upsilon} dS \sim 0, \quad (100)$$

which yields:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV - \frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV. \end{aligned} \quad (101)$$

We have thus recovered the vortex method force equation, which is subject to the following assumptions:

- The flow is incompressible.
- The fluid is of unit density, and so is the “fluidic” body.
- The no-through flow condition applies on the “body” boundary.
- The volume $V(t)$ extends to infinity and encloses *all* the vorticity, which decays exponentially at infinity.
- No assumption is made about pressure at infinity.

¹⁹Saffman P.E., *Vortex dynamics*, §4.2 & §7.6, Cambridge University Press, 1993.

4.6 Added mass

To check the validity of the equation, the concept of added mass has to be recovered. Let us assume that the external flow is irrotational. Moreover, let the surface S^* extend to infinity such that the velocity decays as dictated by Equation 38. Then, the force on the body is just given by:

$$\mathbf{F} = \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV - \frac{1}{\mathcal{N} - 1} \frac{d}{dt} \int_{V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV \quad (102)$$

Even if the flow is irrotational *outside* the body, it has to be rotational (most of the time) *inside* the body to preserve the no-through flow condition on the surface of the body. For a sphere or a cylinder, this is equivalent to there being a doublet inside the body.

This equation can be modified by the Burgatti identity to yield:

$$\mathbf{F} = -\frac{1}{\mathcal{N} - 1} \frac{d}{dt} \oint_{S_b(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS \quad (103)$$

and this is exactly the *virtual momentum* of the body²⁰. This expression for the virtual momentum is valid whether the body is liquid (finite viscosity) or solid (infinite viscosity).

If the body is assumed to be solid (“fluidic” body of infinite viscosity), and the external flow is kept irrotational, then the virtual momentum of the body can be recovered if enough care is exercised. As a matter of fact, straight use of Equation 102 to evaluate the virtual momentum leads to problems. Since the body is rigid, the integrals over the body can be evaluated explicitly:

$$\mathbf{F} = V_b \frac{d\mathbf{u}_b}{dt} - 2V_b \frac{d}{dt} (\mathbf{x}_b \wedge \boldsymbol{\omega}_b) \quad (104)$$

For a non-rotating body and for irrotational flow, the force on the body would reduce to the first term $V_b \dot{\mathbf{u}}_b$. This, however, is not the virtual momentum of the body. A flat plate of zero volume placed in a cross flow would have no virtual momentum by this formula! Let us see where we went wrong. We have to remember that the “fluidic” body (even with infinite viscosity) is part of the flow. To clarify this statement, let us rewrite the equation used in vortex methods, Equation 101, as follows:

$$\begin{aligned} \mathbf{F} = & -\frac{1}{\mathcal{N} - 1} \frac{d}{dt} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV \\ & + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV, \end{aligned} \quad (105)$$

where now the volume integral encloses the whole fluid (including the “fluidic” body). Since there is slip flow on the body, and the body is solid, there results

²⁰Saffman P.E., *Vortex dynamics*, Chapter 4, Cambridge University Press, 1993.

a *vortex sheet* on the surface of the body, and this is the missing ingredient that led us astray before. The flow is thus wholly irrotational except for this vortex sheet. The volume integral is therefore not null. It yields a term which, when subtracted from $V_b \dot{\mathbf{u}}_b$, gives the right virtual momentum. As a matter of fact, for a flat plate in cross flow, it is the vortex sheet that is wholly responsible for the added mass (since the plate has no net volume).

Let the vortex sheet be given by:

$$\boldsymbol{\omega} = [\hat{\mathbf{n}} \wedge (\mathbf{u} - \mathbf{u}_b)] \delta(\mathbf{x} - \mathbf{x}_b), \quad (106)$$

where now \mathbf{u} is the flow velocity and \mathbf{u}_b is the body velocity. Then, the volume integration becomes:

$$\begin{aligned} \int_{V(t)+V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega} dV &= \int_{V(t)+V_b(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\mathbf{u} - \mathbf{u}_b)] \delta(\mathbf{x} - \mathbf{x}_b) dV \\ &= \oint_{S_b(t)} \mathbf{x} \wedge [\hat{\mathbf{n}} \wedge (\mathbf{u} - \mathbf{u}_b)] dS. \end{aligned} \quad (107)$$

Then, Equation 105 becomes:

$$\begin{aligned} \mathbf{F} &= -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{S_b(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS \\ &\quad + \frac{d}{dt} \int_{V_b(t)} \mathbf{u} dV + \frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{S_b(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}_b) dS. \end{aligned} \quad (108)$$

The last two terms can be transformed with Burgatti identity, so that:

$$\begin{aligned} \mathbf{F} &= -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{S_b(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS \\ &\quad + \frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{V_b(t)} \mathbf{x} \wedge \boldsymbol{\omega}_b dV, \end{aligned} \quad (109)$$

but since the body was assumed not to rotate, the last integral vanishes, and we recover the expression for the virtual momentum of the solid:

$$\mathbf{F} = -\frac{1}{\mathcal{N}-1} \frac{d}{dt} \int_{S_b(t)} \mathbf{x} \wedge (\hat{\mathbf{n}} \wedge \mathbf{u}) dS \quad (110)$$

where now \mathbf{u} is the *flow* velocity at the surface of the bluff body.