A SIMPLE COLLOCATION METHOD FOR FITTING VISCOELASTIC MODELS TO EXPERIMENTAL DATA

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ABSTRACT

An easily applied collocation method is discussed for fitting the response of finite-element viscoelastic models to experimental stress-strain curves. It can be used with creep, relaxation, and steady-state oscillation data. The method is illustrated by means of two examples. As the first one, a model is obtained utilizing the dynamic shear compliance of polyisobutylene. In the second example we calculate a model from the tensile relaxation modulus of polymethyl methacrylate. With each case the model's response agreed with the experimental data within graphical accuracy over the entire frequency (or time) scale.

1. Introduction

It is well known that mechanical models consisting of springs and dashpots can be used to represent linear viscoelastic stress-strain behavior[1]. Furthermore it is not necessary to use a general array of springs and dashpots, but only the simple arrangement of Voigt elements in series (Kelvin model) or Maxwell elements in parallel (Wiechert model)[2]. We shall not discuss the mechanics and mathematics of such models since reference (2) contains a relatively complete catalog of this information. Our objective here is only to present a simple method of calculating the parameters (spring and dashpot constants) in finite element models by fitting their response to experimental stress-strain data.

Other investigators have suggested various graphical and numerical schemes for obtaining model parameters. For example, Bland and Lee[3] give a simple graphical scheme for fitting the response of four-element models to a narrow frequency band of steady-state sinusoidal ("dynamic") data. Welch[4] also develops a graphical method for obtaining model parameters using dynamic data, however it is not limited to models with only four elements. Elder[5] presents a least-squares technique, also for use with dynamic data, which leads to a set of non-linear algebraic equations for the model parameters.

In this paper we discuss a simple collocation method which can be used to closely fit the entire viscoelastic response curves

obtained in the standard tests of creep, relaxation, and steady-state oscillation. The method is presented by applying it to dynamic shear data obtained on glass-filled polyisobutylene and to the relaxation tensile modulus of polymethyl methacrylate. The stress-strain equations corresponding to the two models calculated in this paper are those used in reference (6) in two examples of visco-elastic stress analysis.

2. Property Representation of Polyisobutylene

Consider the problem of fitting the response of a model to the stress-strain data shown in Figure 1. This data was obtained on glass-filled polyisobutylene under constant frequency sinusoidal loading in shear.[7] The stress, $\sigma_o e^{i\omega t}$, and strain, $\epsilon_o e^{i\omega t}$, ($i = \sqrt{-1}$, ω = frequency given in radians per unit time) in such a test can be related by means of the complex compliance $J^*(\omega)$; thus

$$\frac{\epsilon_o e^{i\omega t}}{\sigma_o e^{i\omega t}} = \frac{\epsilon_o}{\sigma_o} = J^*(\omega) = J'(\omega) - i\,J''(\omega) \qquad (1)$$

where $J'(\omega)$ and $J''(\omega)$ are real functions which are defined as the real and imaginary components of $J^*(\omega)$, respectively.

Let us now fit a Kelvin model to the data by collocating the model's response, $J_m^*(\omega) = J_m'(\omega) - i\,J_m''(\omega)$, with the experimental values. The response of the model is given by [2]

$$J_m'(\omega) = J_g + \sum_{i=1}^{n} \frac{J_i}{\omega^2 \tau_i^2 + 1} \qquad (2a)$$

$$J_m''(\omega) = \sum_{i=1}^{n} \frac{\omega \tau_i J_i}{\omega^2 \tau_i^2 + 1} \qquad (2b)$$

where Jg is the glassy (high frequency) compliance, J_i and τ_i are model constants which are to be evaluated by equating J_m' and J' at n values of frequency, ω_i, and choosing the τ_i such that $\omega_i \tau_i = 1$ (i = 1, 2, ---n)*. This procedure leads to a set of n linear algebraic equations for the constants J_i **. The way in which the τ_i are chosen is somewhat arbitrary; however by setting

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* It may often be desirable to add another term to the series (2) in order to satisfy the condition

$$J_g + \sum_{i=1}^{n+1} J_i = J(\omega)\Big|_{\omega=0}$$

However, for the present example it turned out that this condition was closely satisfied without making it an explicit requirement.

** Of course, one could collocate at 2n points, say, in order to evaluate all 2n constants, J_i and τ_i. However, the resulting set of equations would be nonlinear and consequently quite difficult to solve.

$\tau_i = 1/\omega_i$, each point of collocation corresponds to a frequency at which one of the terms in the series (2a) attains a value which is one half of its maximum ($\omega = 0$) value. In the present example this choice will be seen to produce a set of equations which are easily solved.

It should be observed that once the constants J_i and τ_i are known, the imaginary component J_m'' can be calculated from (2b) and compared with the data in Figure 2. If J_m' and J' are in good agreement then, apart from measurement errors, any significant difference between J_m'' and J'' can be attributed to nonlinearities or inertia effects.

By referring to Figure 1, it is seen that a good fit might be obtained if

$$\omega_i = 10^{(i-2)} \quad ; (i = 1, 2, \cdots 10) \tag{3}$$

so that the "retardation times" τ_i are chosen to be

$$\tau_i = 10^{(2-i)} \quad ; \quad (i = 1, 2, \cdots 10) \tag{4}$$

Collocation at the values (3) yields the following system of linear equations for evaluation of the J_i :

$$
\begin{bmatrix}
\frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \\ J_7 \\ J_8 \\ J_9 \\ J_{10} \end{bmatrix}
=
\begin{bmatrix} J'(10^1) - J_9 \\ J'(1) - J_8 \\ J'(10) - J_7 \\ J'(10^2) - J_6 \\ J'(10^3) - J_5 \\ J'(10^4) - J_4 \\ J'(10^5) - J_3 \\ J'(10^6) - J_2 \\ J'(10^7) - J_1 \\ J'(10^8) - J_0 \end{bmatrix}
\tag{5}
$$

where $J_g = 3.16 \times 10^{-11}$. The elements of the matrix which are shown as unity or zero are not exactly these values, but are close enough to permit the approximation. Since the matrix is nearly triangular the equations can be easily solved by iteration as follows: Let the first estimation of the J_i be denoted as $J_i^{(1)}$. Solve for $J_i^{(1)}$ by neglecting all elements 1/101 and $1/10^4 + 1$. This provides a set of 1 x 1 equations by starting with $J_{10}^{(1)}$, e.g.

$$
J_{10}^{(1)} = 2 \left[J'(10^8) - J_8 \right] \tag{6}
$$

The second approximation $J_i^{(2)}$ is then obtained by retaining all

elements in (5), and letting the first solution $J_1^{(1)}$ multiply the elements which were initially neglected. For example, the $J_1^{(2)}$ are calculated from:

$$\begin{aligned}
&\frac{J_1^{(2)}}{2} + \frac{J_2^{(2)}}{1.01} + J_3^{(2)} + J_4^{(2)} + \cdots + J_{10}^{(2)} = J'(10^{-1}) - J_g \\
&\frac{J_1^{(1)}}{101} + \frac{J_2^{(2)}}{2} + \frac{J_3^{(2)}}{1.01} + J_4^{(2)} + \cdots + J_{10}^{(2)} = J'(1) - J_g \\
&\frac{J_1^{(1)}}{10^4+1} + \frac{J_2^{(1)}}{101} + \frac{J_3^{(2)}}{2} + \frac{J_4^{(2)}}{1.01} + \cdots + J_{10}^{(2)} = J'(10) - J_g \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned} \tag{7}$$

etc.

which can be solved as a set of 1 x 1 equations by starting with $J_{10}^{(2)}$. This iteration procedure is to be repeated until the desired accuracy is obtained. For the problem at hand it was found that J_m' and J' agreed within graphical accuracy by using the third approximation, which is

$$\begin{aligned}
J_1 &= 357 \times 10^{-11} & J_6 &= 250 \times 10^{-11} \\
J_2 &= 533 \times 10^{-11} & J_7 &= 80.8 \times 10^{-11} \\
J_3 &= 3960 \times 10^{-11} & J_8 &= 22.2 \times 10^{-11} \\
J_4 &= 3580 \times 10^{-11} & J_9 &= 4.00 \times 10^{-11} \\
J_5 &= 1210 \times 10^{-11} & J_{10} &= 2.22 \times 10^{-11}
\end{aligned} \tag{8}$$

It was mentioned above that (2b) can be used to predict the imaginary component of J^*, and hence serve as a check on the linearity assumption and the assumption that inertia is negligible. The predicted and experimental curves which are shown in Figure 2 agree well except at high frequencies.

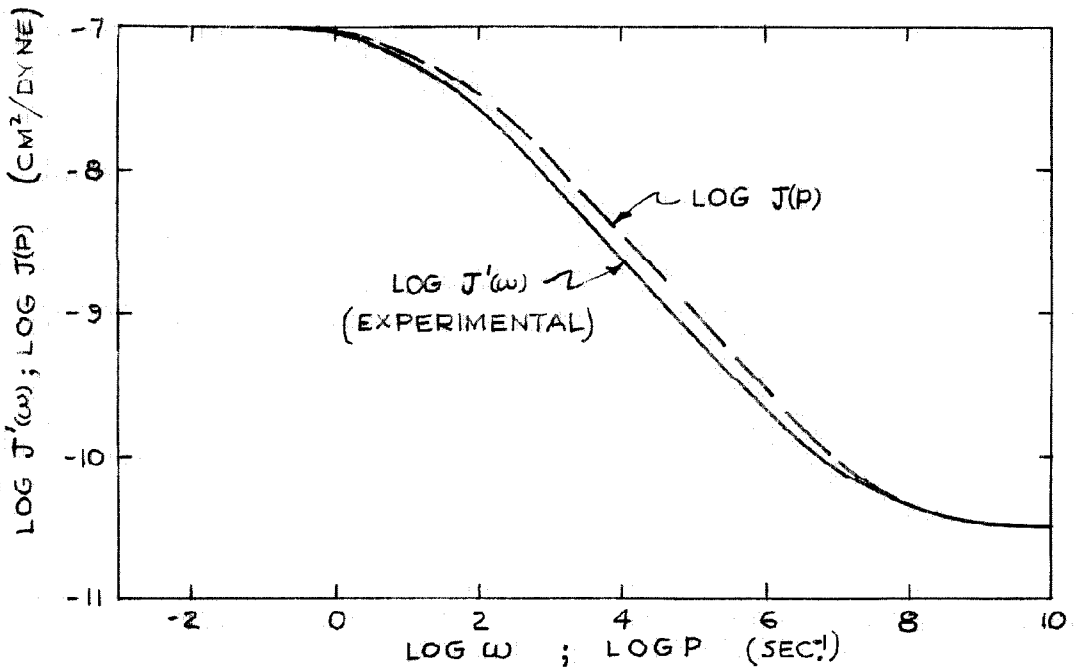

FIGURE 1. REAL PART OF COMPLEX COMPLIANCE AND OPERATIONAL COMPLIANCE FOR POLYISOBUTYLENE AT 12.5°C (IN SHEAR)

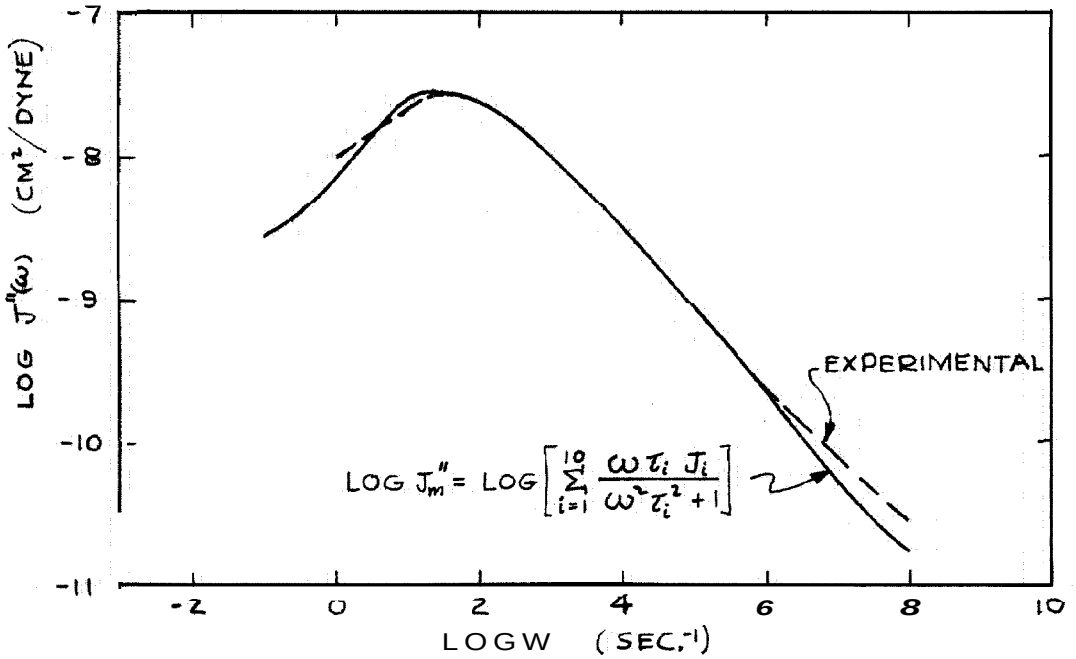

FIGURE 2. IMAGINARY PART OF COMPLEX COMPLIANCE FOR POLYISOBUTYLENE AT 12.5°C (IN SWEAR). COMPARISON OF EXPERIMENTAL AND MODEL RESPONSE.

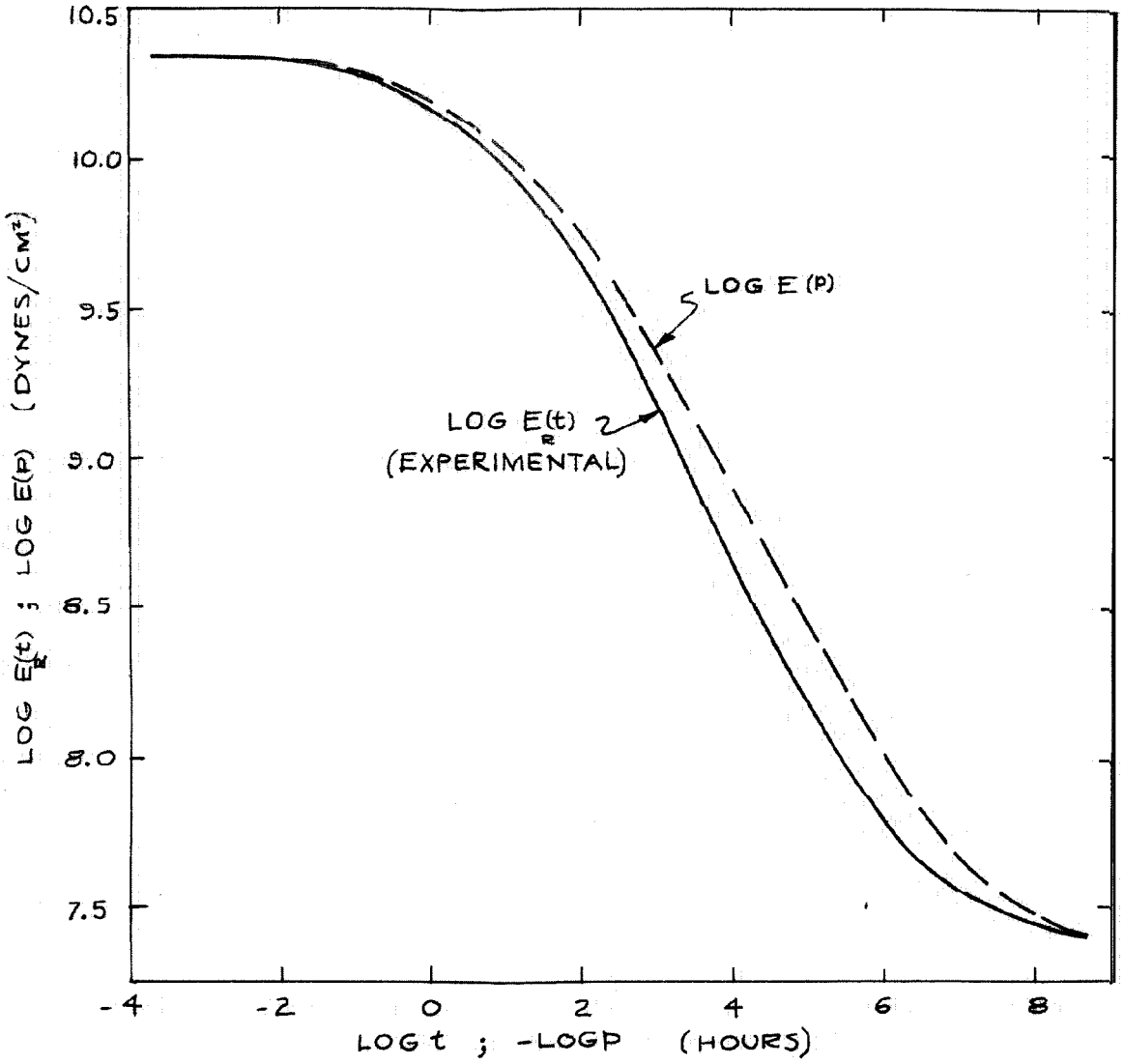

FIGURE 3. RELAXATION MODULUS AND OPERATIONAL MODULUS FOR POLYMETHYL METHACRYLATE AT 80°C (IN SIMPLE TENSION)