Velocity dispersion due to anelasticity; implications for seismology and mantle composition

Hsi-Ping Liu *Seismological Laboratory, and Division of Engineering and Applied Sciences, California Institute of Technology, Pasadena, California 91125, USA*

Don L. Anderson and Hiroo Kanamori *Seismological Laboratory, Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, California 91125, USA*

Summary. The concept of a relaxation spectrum is used to compute the absorption and dispersion of a linear anelastic solid. The Boltzmann after-effect equation is solved for a solid having a linear relationship between stress and strain and their first time derivatives, the 'standard linear solid', and having a distribution of relaxation times. The distribution function is chosen to give a nearly constant Q over the seismic frequency range. Both discrete and continuous relaxation spectra are considered. The resulting linear solid has a broad absorption band which can be interpreted in terms of a super-position of absorption peaks of individual relaxation mechanisms.

The accompanying phase and group velocity dispersion imply that one cannot directly compare body wave, surface wave, and free oscillation data or laboratory and seismic data without correcting for absorption. The necessary formalism for making these corrections is given. In the constant Q regions the correction is the same as that implied in the theories of Futterman, Lomnitz, Strick and Kolsky.

1 Introduction

It is usually assumed in seismological studies that elastic moduli and seismic wave velocities are independent of frequency. This is equivalent to assuming that the Earth is perfectly elastic or that seismic frequencies are far away from any absorption bands. However, the fact that seismic waves attenuate with distance and that free oscillations decay with time indicate that the Earth is not an ideal elastic body. The observation that Q^{-1}, the specific dissipation function or seismic quality factor, is roughly constant from the body wave to the free oscillation band indiates that Q^{-1} cannot satisfy the ω or ω^{-1}-dependence required by physical mechanisms of attenuation at frequencies remote from the characteristic frequency of the attenuation mechanism. For example, the observation that ScS waves of periods of 10–30 s have approximately the same Q as toroidal oscillations of periods in excess of 1000 s (e.g.

Anderson 1967a; Anderson, Ben-Menahem & Archambeau 1965; Anderson & Kovach 1964), provides a strong constraint on the dissipation spectrum and the associated dispersion, even if Q is not precisely independent of frequency. The consequences of dispersions are that one cannot directly compare body wave, surface wave, and free oscillation data or compare laboratory ultrasonic and shock wave data with seismic data, unless corrections are made for phase velocity dispersion arising from anelasticity. The importance of phase velocity dispersion due to anelasticity has been discussed by Lomnitz (1957), Jeffreys (1965), Carpenter & Davies (1966), Futterman (1962), Strick (1967), Lamb (1962) and Randall (1976). Although body wave data are often corrected for dispersion (e.g. Carpenter 1967; Helmberger 1973) using Futterman's theory (Futterman 1962), the effect of physical dispersion is usually ignored in surface wave and free oscillation interpretations. The reason, possibly, is that there are arbitrary assumptions and physical shortcomings in most of the theories that have been put forward to explain the near constancy of Q and the resulting required phase velocity dispersion. In this paper we show from general physical considerations, that the phase velocity dispersion due to anelasticity is important. This affects inversion of surface wave and free oscillation data, inferences from body wave spectra, comparison of body wave and normal mode earth models, discussions of the base line problems in travel times and oceanic—continental differences and use of laboratory high pressure, high temperature data to infer the composition of the Earth.

2 Background

The widely-adopted assumption that seismic velocity dispersion due to anelasticity can be ignored, or is a second order effect, is perhaps due to the fact that a large amount of body wave, surface wave and free oscillation data can be satisfied by a purely elastic Earth with frequency-independent moduli. Also, the velocity dispersion due to anelasticity is a second order effect in special circumstances, for example, at frequencies far away from the characteristic frequency of a standard linear solid, the low frequency limit of a Kelvin—Voigt solid and the high frequency limit of a Maxwellian solid. However, in these regions the Q itself varies directly or inversely with frequency.

The Futterman type theories (Futterman 1962; Lamb 1962; Strick 1967; Azimi, Kalinin & Pivovarov 1968; Randall 1976) and the Lomnitz model (Lomnitz 1957) give frequency-dependent seismic velocities which depend to first order on anelasticity. The Futterman theory is based on Kramers—Krönig relations, which connect dispersion and absorption, and on cut-off frequencies outside of the seismic band. The Lomnitz theory is based on a logarithmic creep function and Boltzmann's after-effect equation (e.g. Gross 1968). The Lomnitz theory does not pass in the limit to the static case (Kogan 1966), which predicts that primary creep would continue indefinitely. This is in contrast with the satellite observation of gravitational potential which shows that stress differences on the order of 100 b must exist in the upper mantle on the geological time scale. The Futterman type theory is empirical and does not have its basis on physical mechanisms of attenuation.

We use the concept of a spectrum of relaxation mechanisms to determine the absorption and dispersion of a linear viscoelastic solid. We solve the Boltzmann after-effect equation for a standard linear solid with both a finite number and a continuous distribution of relaxation times. It is shown that in the frequency range where the coefficient of internal friction is approximately constant, the phase velocity depends to first order on Q. In the high and low frequency limits Q varies directly as frequency or period and velocity varies only to the second order of frequency or period. The derivation of these results and their significance in seismology and mantle composition investigations will be given in the following sections.

3 Theory

Anderson (1967b), Gordon & Nelson (1966), and Jackson & Anderson (1970) reviewed the possible mechanisms for seismic wave attenuation. Two possible non-linear attenuation mechanisms reviewed by Jackson & Anderson (1970) are the static hysteresis mechanisms of friction across microcracks and unpinning of dislocations. Another possible nonlinear mechanism is the kink model of dislocation damping proposed by Mason (1969). Under modest pressures (10 kb), microcracks in rocks would be closed and the frictional mechanism would not operate. The effects of dislocation unpinning and the kink model of dislocation attenuation may be small compared with other attenuation mechanisms because the high temperature in the mantle and the long geological time is likely to have annealed out most dislocations and to have unpinned the rest. The other mechanisms of attenuation considered by these authors can all be modelled by linear mechanisms. We, therefore, restrict our investigation to linear viscoelastic theory of sound absorption and dispersion.

Formally, the theory of linear viscoelasticity is embodied in Boltzmann's after-effect equation (e.g. Gross 1968) which can be written as

$$\epsilon(t) = \int_{-\infty}^{t} \dot{\sigma}(\tau) \, \phi(t - \tau) \, d\tau. \tag{3}$$

Boltzmann's equation assumes that the strain at time t, $\epsilon(t)$, is caused linearly by the total history of stress $\sigma(\tau)$ up to the time t and therefore incorporates both the superposition principle and the causality principle. The function $\phi(t)$ is the creep function and is determined by the mechanism of attenuation.

For a plane wave propagating in a linear viscoelastic solid,

$$\sigma(t) = A \exp\left[i(\omega t - Kx)\right] \tag{4}$$

K is the wave number defined by

$$K(\omega) = \frac{\omega}{v_p(\omega)} - i\alpha(\omega) \tag{5}$$

$v_p(\omega)$ is the phase velocity and $\alpha(\omega)$ is the attenuation factor. $v_p(\omega)$ and $\alpha(\omega)$ are determined by the physical mechanism of attenuation (or rheology of the material) through the creep function.

A formal integral relation exists which connects $v_p(\omega)$ and $\alpha(\omega)$. In terms of index of refraction $n(\omega)$, defined by $n(\omega) = K(\omega)/K_0(\omega)$, where $K_0(\omega)$ is the 'non-dispersive' behaviour at the same frequency,

$$\text{Re}\left[n(\omega_1) - n(\infty)\right] = \frac{2}{\pi} P \int_0^{\infty} d\omega \, \frac{\text{Im}\{n(\omega)\} \, \omega}{\omega^2 - \omega_1^2} \tag{6a}$$

or

$$\text{Re}\left[n(\omega_1) - n(0)\right] = \frac{2\omega_1^2}{\pi} P \int_0^{\infty} d\omega \, \frac{\text{Im}\{n(\omega)\}}{\omega(\omega^2 - \omega_1^2)} \tag{6b}$$

where P denotes Cauchy principal value.

Equation (6) and the inverse relation expressing $\text{Im}\left[n(\omega)\right]$ in terms of $\text{Re}\left[n(\omega)\right]$ are the Kramers–Krönig relations. These relations can be derived from Boltzmann's after-effect equation as shown in the theory of generalized susceptibility (e.g. Landau & Lifshitz 1968).

Since they do not contain any information not already in the Boltzmann's after-effect equation, and because of the ambiguity in the definition of $K_0(\omega)$, we will not use these relations in our theoretical considerations.

The creep function in the Boltzmann's after-effect equation is the strain response when $\sigma(t)$ is the unit step function $H(t)$. For $\sigma(t) = H(t)$, $\dot{\sigma}(t) = \delta(t)$, delta function, and Boltzmann's after-effect equation (3) becomes

$$\epsilon(t) = \int_{-\infty}^{t} \delta(\tau) \, \phi(t - \tau) \, d\tau = \phi(t). \tag{7}$$

For a Kelvin–Voigt solid,

$$\sigma = 2\mu\epsilon + 2\mu' \frac{\partial \epsilon}{\partial t} \tag{8}$$

where σ is the shear stress and ϵ is the shear strain. Under a unit step function loading, $\sigma(t) = H(t)$, equation (8) gives

$$\phi(t) = \epsilon(t) = \frac{1}{2\mu} \left[1 - \exp\left(-t\mu/\mu'\right) \right] H(t). \tag{9}$$

The Boltzmann's after-effect equation for the Kelvin–Voigt solid is therefore

$$\epsilon(t) = \int_{-\infty}^{t} \dot{\sigma}(\tau) \frac{1}{2\mu} \left[1 - \exp\left\{-\left(\frac{\mu}{\mu'}\right)(t - \tau)\right\} \right] d\tau. \tag{10}$$

For a suddenly applied shear stress there is no instantaneous or initial strain; the strain gradually approaches an asymptotic value. Rocks, however, have instantaneous strain upon sudden application of stress and an instantaneous recovery upon removal of stress. A more appropriate mechanical model is the 'standard linear solid' model represented by the stress–strain relation

$$\sigma + \tau_\sigma \dot{\sigma} = M_R (\epsilon + \tau_\epsilon \dot{\epsilon}) \tag{11}$$

where M_R is a deformation modulus and τ_σ is the stress relaxation time under constant strain and τ_ϵ is the strain relaxation time under constant stress. Such a solid is called a 'standard linear solid' because it is the most general linear equation in stress, strain and their first time derivatives. The strain for a unit step function loading is

$$\epsilon(t) = \frac{1}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\epsilon} \right) \exp\left(-t/\tau_\epsilon \right) \right] H(t) \tag{12}$$

at

$$t = 0, \qquad \epsilon(t) = \frac{1}{M_R} \frac{\tau_\sigma}{\tau_\epsilon} H(t) = \frac{1}{M_R} \frac{\tau_\sigma}{\tau_\epsilon} \sigma(t) \tag{13}$$

at

$$t = \infty, \qquad \epsilon(t) = \frac{1}{M_R} H(t) = \frac{1}{M_R} \sigma(t). \tag{14}$$

Since the final value of the ratio of stress to strain, the value after relaxation has taken place, is M_R, this quantity is known as the 'relaxed elastic modulus'. The initial value of the ratio

of stress to strain, before relaxation has time to occur, is $M_R(\tau_\epsilon/\tau_\sigma)$. This quantity is known as the 'unrelaxed elastic modulus', and is denoted by M_U.

$$\frac{M_U}{M_R} = \frac{\tau_\epsilon}{\tau_\sigma}. \tag{15}$$

The Boltzmann after-effect equation for a standard linear solid is

$$\epsilon(t) = \int_{-\infty}^{t} \dot{\sigma}(\tau) \frac{1}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right) \exp\{-(t-\tau)/\tau_\epsilon\} \right] d\tau.$$

With $\sigma(-\infty) = 0$ and $t - \tau = \theta$

$$\epsilon(t) = \frac{\sigma(t)}{M_R} - \frac{1}{M_R} \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right) \int_0^\infty \exp(-\theta/\tau_\epsilon) \dot{\sigma}(t-\theta) d\theta. \tag{16}$$

For sinusoidal disturbance

$$\epsilon(t) = \frac{\sigma(t)}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right) \int_0^\infty i\omega \exp(-i\omega\theta) \exp(-\theta/\tau_\epsilon) d\theta \right]$$

$$= \frac{\sigma(t)}{M_R} \left[1 - \frac{\omega^2 \tau_\epsilon^2}{1 + \omega^2 \tau_\epsilon^2} \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right) - i \frac{\omega(\tau_\epsilon - \tau_\sigma)}{1 + \omega^2 \tau_\epsilon^2} \right] = \frac{\sigma(t)}{M_R} (A - iB) \tag{17}$$

where

$$A = 1 - \frac{\omega^2 \tau_\epsilon^2}{1 + \omega^2 \tau_\epsilon^2} \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right)$$

$$B = \omega(\tau_\epsilon - \tau_\sigma)/(1 + \omega^2 \tau_\epsilon^2).$$

The complex modulus M_c is defined as

$$M_c \equiv \frac{M_R}{A - iB}. \tag{18}$$

We have therefore

$$\frac{K(\omega)}{\omega/v_e} = \left[\frac{M_R}{M_c(\omega)} \right]^{1/2} \tag{19}$$

where $v_e = \sqrt{M_R/\rho}$, is the phase velocity associated with the relaxed elastic modulus, and ρ is density. Combine equation (19) with equation (5).

$$v_p^2 = \frac{2v_e^2}{B^2} \Omega(\omega)$$

$$\alpha^2 = \frac{\omega^2}{2v_e^2} \Omega(\omega) \tag{20}$$

where

$$\Omega(\omega) = A \left[\left(1 + \frac{B^2}{A^2}\right)^{1/2} - 1 \right].$$

The internal friction coefficient, $\tan \delta$, is

$$\tan \delta = \frac{\mathrm{Im}(M_c)}{\mathrm{Re}(M_c)} = \frac{\omega(\tau_\epsilon - \tau_\sigma)}{1 + \omega^2 \tau_\epsilon \tau_\sigma} = \left[\sqrt{\frac{M_U}{M_R}} - \sqrt{\frac{M_R}{M_U}} \right] \frac{\omega\tau}{1 + \omega^2 \tau^2} \tag{21}$$

where

$$\tau \equiv \sqrt{\tau_\epsilon \tau_\sigma}$$

$\tan \delta = Q^{-1} = (1/2\pi)(\Delta W/W)$ for small attenuation where W is the elastic strain energy per unit cycle per unit volume, and ΔW is the energy dissipated per unit cycle per unit volume. These are well-known results. The phase velocity dispersion and internal friction coefficient are plotted as a function of frequency in Fig. 1. The internal friction has a ω-dependence at low frequencies and $(1/\omega)$-dependence at high frequencies. The phase velocity variation with frequency has zero slope at low and high frequencies ($\omega \ll 1/\sqrt{\tau_\epsilon \tau_\sigma}$, $\omega \gg \sqrt{\tau_\epsilon \tau_\sigma}$).
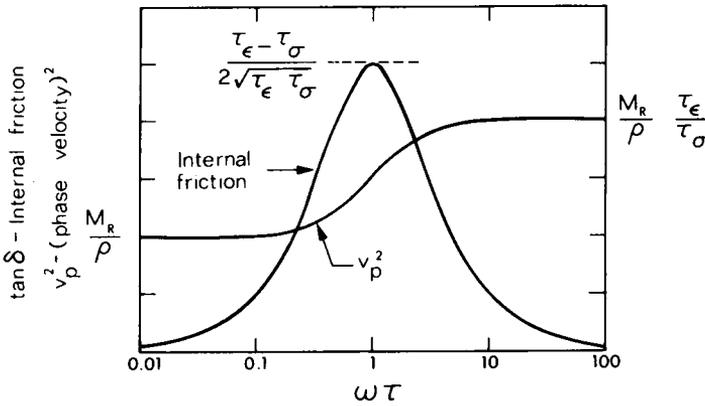
Figure 1. Phase velocity dispersion and internal friction coefficient as a function of frequency in a solid with a single relaxation mechanism.

Many processes such as thermoelasticity, diffusional motion of dislocations and point defects, stress-induced ordering, phase changes and twinning—detwinning of crystals interact with an external strain wave. Energy is absorbed in half of the stress cycle and given up in the next half cycle. A finite time is required for the energy exchange. This relaxation time makes strain out of phase with stress. The standard linear solid models very well these processes (Zener 1948; Walsh 1968). A typical relaxation spectrum for metals is shown in Fig. 2. Many peaks are seen in the internal friction coefficient versus frequency curve, and each peak can be attributed to some relaxation process. Some relaxation mechanisms such as interstitial atom relaxation can be closely approximated by the standard linear solid with a single relaxation time. Other processes, such as grain boundary relaxation, have $\tan \delta$ curve that are much broader than a single relaxation curve and involve a distribution of relaxation times. For Earth materials, τ for system undergoing phase changes is a kinetic rate constant of the order of 10 s for partial melting and 10^4 s for solid phase transformation (Vaisnys 1968). τ for intergranular thermal diffusion in polycrystalline solids is of the order of 0.02–200 s for grain sizes of 1 mm–10 cm (Savage 1965). Relaxation of interstitial solute atoms also satisfies the above equations and have relaxation times in the seismic range (Anderson 1967b). There are, therefore, several relaxation mechanisms which have peak attenuation in the seismic frequency band and it cannot be assumed that seismic phase
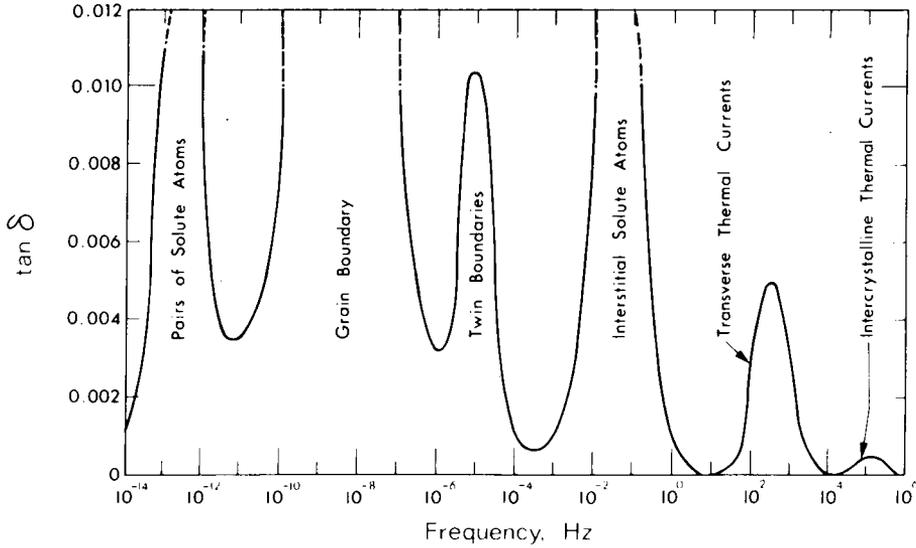
Figure 2. A typical relaxation spectrum (after Zener 1948).

velocities are frequency independent. In general, the relaxation time depends on temperature and pressure. Absorption bands in the Earth can therefore be expected to be broadened since seismic waves which of necessity traverse temperature and pressure gradients. This broadening is in addition to the intrinsic breadth due to the spectrum of atomic relaxation times, grain sizes, etc., existing in natural rocks. It seems natural therefore to attempt to explain the relative constancy of Q in the Earth by a distribution of relaxation times.

Generalizing equation (16) to a spectrum of relaxation mechanisms

$$\epsilon(t) = \frac{\sigma(t)}{M_R} - \frac{1}{M_R} \sum_{k=1}^{N} \left(1 - \frac{\tau_{\sigma k}}{\tau_{\epsilon k}}\right) \int_0^\infty \exp(-\theta/\tau_{\epsilon k}) \, \dot{o}(t - \theta) \, d\theta \tag{22}$$

for a finite number of relaxation mechanisms and

$$\epsilon(t) = \frac{\sigma(t)}{M_R} - \frac{1}{M_R} \int_0^\infty d\theta \int_0^\infty d\tau_\epsilon \int_0^\infty d\tau_\sigma D(\tau_\epsilon, \tau_\sigma) \cdot \left(1 - \frac{\tau_\sigma}{\tau_\epsilon}\right) \exp(-\theta/\tau_\epsilon) \, \dot{o}(t - \theta) \tag{23}$$

for a continuous distribution of relaxation mechanisms, $D(\tau_\epsilon, \tau_\sigma)$ is the distribution function. For sinusoidal disturbance,

$$\epsilon(t) = \frac{\sigma(t)}{M_R} (A - iB) \tag{24}$$

where

$$A = 1 - \sum_{k=1}^{N} \frac{\omega^2 \tau_{\epsilon k} (\tau_{\epsilon k} - \tau_{\sigma k})}{1 + \omega^2 \tau_{\epsilon k}^2}$$

$$\tag{25}$$

$$B = \sum_{k=1}^{N} \frac{\omega(\tau_{\epsilon k} - \tau_{\sigma k})}{1 + \omega^2 \tau_{\epsilon k}^2}$$

for discrete spectrum and

$$A = 1 - \int_0^\infty d\tau_\sigma \int_0^\infty d\tau_\epsilon D(\tau_\sigma, \tau_\epsilon) \frac{\omega^2 \tau_\epsilon (\tau_\epsilon - \tau_\sigma)}{1 + \omega^2 \tau_\epsilon^2}$$

$$B = \int_0^\infty d\tau_\sigma \int_0^\infty d\tau_\epsilon D(\tau_\sigma, \tau_\epsilon) \frac{\omega(\tau_\epsilon - \tau_\sigma)}{1 + \omega^2 \tau_\epsilon^2} \tag{26}$$

for a continuous spectrum.

The phase velocity v_p and attenuation factor are given by equation (20) with the newly-defined A and B — equation (25) or (26). The internal friction coefficient $\tan \delta$ is given by

$$\tan \delta = \frac{B}{A}. \tag{27}$$

For $\tan \delta \ll 1, B/A \ll 1$

$$\alpha = \frac{\omega}{2v_e} \frac{B}{\sqrt{A}}$$

$$v_p = \frac{v_e}{\sqrt{A}}. \tag{28}$$

The group velocity is given by

$$v_g = \frac{1}{(d/d\omega)(\omega/v_p)} = (2\Omega)^{3/2} v_e \bigg/ \left(2\Omega B + 2\Omega\omega \frac{dB}{d\omega} - B\omega \frac{d\Omega}{d\omega}\right). \tag{29}$$

For $\tan \delta \ll 1$

$$v_g = v_p \bigg/ \left(1 + \frac{\omega}{B} \frac{dB}{d\omega} - \frac{A}{B^2} \omega \frac{d\Omega}{d\omega}\right). \tag{30}$$

4 Results

4.1 VELOCITY DISPERSION AND ATTENUATION CALCULATED FROM A FINITE NUMBER OF RELAXATION MECHANISMS

Fig. 3(a) shows the internal friction, Fig. 3(b) shows the phase and group velocity dispersion and Fig. 3(c) shows the attenuation factor as a function of frequency for a relaxation spectrum of twelve mechanisms. The parameters of these twelve mechanisms are shown in Table 1. The composite Q is ~ 100 and constant to ± 1.25 per cent from 10 to 3×10^{-4} Hz. Two sets of parameters of twelve mechanisms which make the Q equal to ~ 500 and ~ 1000 within ± 1.25 per cent over the same frequency range are also listed in Table 1. The phase velocity varies monotonically between $v_p(\omega = 0)$ and $v_p(\omega = \infty)$. These are the lower and upper limit of the phase velocities for a linear viscoelastic solid constructed from a spectrum of relaxation mechanisms. These properties can be proved from equations (25) and (28). Equation (25) shows that A is a monotonic decreasing function of frequency, decreasing from 1 to

$$1 - \sum_{k=1}^N \left(\frac{\tau_{\epsilon k} - \tau_{\sigma k}}{\tau_{\epsilon k}}\right).$$
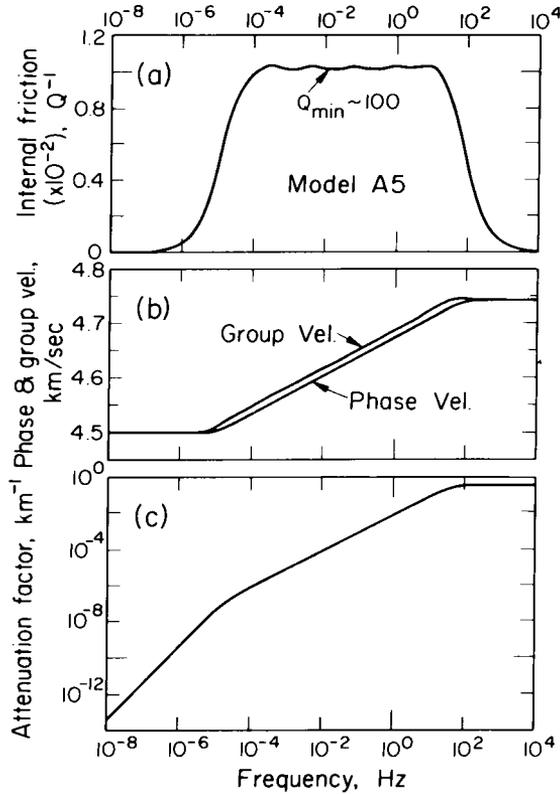
Figure 3. (a) Internal friction coefficient as a function of frequency (b) phase and group velocity dispersion, (c) attenuation factor as a function of frequency, for a solid with twelve relaxation mechanisms. The relaxation mechanism parameters are listed in Table 1.

The phase velocity at zero frequency corresponds to the relaxed elastic modulus M_R and the phase velocity at infinite frequency corresponds to the unrelaxed modulus M_U defined by

$$M_U = M_R \bigg/ \left[1 - \sum_{k=1}^{N} \left(\frac{\tau_{\epsilon k} - \tau_{\sigma k}}{\tau_{\epsilon k}} \right) \right].$$

4.2 VELOCITY DISPERSION AND ATTENUATION CALCULATED FROM A CONTINUOUS DISTRIBUTION OF RELAXATION MECHANISMS

We first go back to equation (17) for a single relaxation mechanism,

$$\epsilon(t) = \frac{\sigma(t)}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\epsilon} \right) \int_0^\infty i\omega \exp(-i\omega\theta) \exp(-\theta/\tau_\epsilon) \, d\theta \right]. \tag{17a}$$

Choose the particular case such that

$$1 - \frac{\tau_\sigma}{\tau_\epsilon} = C = \text{constant}, \qquad C \ll 1, \tag{31}$$

Hsi-Ping Liu, D. L. Anderson and Hiroo Kanamori

Table 1. Relaxation time spectrum parameters

Model designation	Minimum Q	$\tau_{\epsilon k}$ (s)	$\tau_{\sigma k}$ (s)
A5	~ 100	0.727567×10^4	0.720983×10^4
		0.194894×10^4	0.193243×10^4
		0.522413×10^3	0.517598×10^3
		0.139847×10^3	0.138822×10^3
		0.375044×10^2	0.371650×10^2
		0.100403×10^2	0.996725×10^1
		0.269236×10^1	0.266864×10^1
		0.720840	0.715635
		0.193279	0.191624
		0.517539×10^{-1}	0.513802×10^{-1}
		0.138813×10^{-1}	0.137534×10^{-1}
		0.371759×10^{-2}	0.368711×10^{-2}
B5	~500	0.723915×10^4	0.723619×10^4
		0.194230×10^4	0.193904×10^4
		0.520475×10^3	0.519525×10^3
		0.139436×10^3	0.139231×10^3
		0.373684×10^2	0.373002×10^2
		0.100111×10^2	0.999625×10^1
		0.268290×10^1	0.267806×10^1
		0.718767	0.717698
		0.192621	0.192279
		0.516056×10^{-1}	0.515278×10^{-1}
		0.138307×10^{-1}	0.138038×10^{-1}
		0.370559×10^{-2}	0.369904×10^{-2}
C3	~1000	0.724591×10^4	0.723943×10^4
		0.194148×10^4	0.193985×10^4
		0.520238×10^3	0.519762×10^3
		0.139385×10^3	0.139282×10^3
		0.373513×10^2	0.373172×10^2
		0.100074×10^2	0.999997×10^1
		0.268169×10^1	0.267927×10^1
		0.718500	0.717964
		0.192536	0.192364
		0.515862×10^{-1}	0.515472×10^{-1}
		0.138240×10^{-1}	0.138104×10^{-1}
		0.370395×10^{-2}	0.370068×10^{-2}

i.e. τ_σ depends on τ_ϵ through a simple linear relation. Generalizing to a continuous distribution in τ_ϵ, equation (17a) becomes

$$\epsilon(t) = \frac{o(t)}{M_R} \left[1 - C \int_0^\infty d\tau_\epsilon \int_0^\infty i\omega \exp\left(- i\omega\theta\right) \exp\left(- \theta/\tau_\epsilon\right) D(\tau_\epsilon) \, d\theta \right]. \tag{32}$$

Equation (32) is a special case of equation (23). Choose the particular distribution function $D(\tau_\epsilon)$:

$$D(\tau_\epsilon) = \begin{cases} 1/\tau_\epsilon & \tau_1 > \tau_\epsilon > \tau_2 \\ \\ 0 & \tau_\epsilon \geqslant \tau_1, \tau_\epsilon \leqslant \tau_2. \end{cases} \tag{33}$$

Equation (32) integrates to

$$\epsilon(t) = \frac{\sigma(t)}{M_R} \left[1 - \frac{C}{2} \ln \frac{1 + \omega^2 \tau_1^2}{1 + \omega^2 \tau_2^2} + iC \tan^{-1} \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2 \tau_1 \tau_2} \right], \tag{34}$$

i.e.

$$A = 1 - \frac{C}{2} \ln \frac{1 + \omega^2 \tau_1^2}{1 + \omega^2 \tau_2^2}$$

$$B = C \tan^{-1} \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2 \tau_1 \tau_2} \tag{35}$$

in equation (20) and (28). The internal friction, $\tan\delta$, and the phase velocity $v_p(\omega)$, are

$$\left. \begin{aligned} \tan\delta &= \frac{B}{A} \cong C \tan^{-1} \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2 \tau_1 \tau_2} \\ v_p(\omega) &= \sqrt{\frac{M_R}{\rho}} \left(1 - \frac{C}{2} \ln \frac{1 + \omega^2 \tau_1^2}{1 + \omega^2 \tau_2^2} \right)^{-1/2} = \sqrt{\frac{M_R}{\rho}} \left(1 + \frac{C}{4} \ln \frac{1 + \omega^2 \tau_1^2}{1 + \omega^2 \tau_2^2} \right) \end{aligned} \right\}. \tag{36}$$

If, in the seismic frequencies, τ_1 is very large and τ_2 is very small, for example, $\tau_1 = 2 \times 10^4$ s, $\tau_2 = 0.6$ s, then for $\tau_1^{-1} \ll \omega \ll \tau_2^{-1}$

$$\tan\delta \cong C \tan^{-1} \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2 \tau_1 \tau_2} \sim \frac{C\pi}{2} = 1/Q_m = \text{constant.} \tag{37}$$

Equation (37) is the result obtained by Savage (1965) for an intergranular thermoelastic relaxation model. The phase velocity in the range $\tau_1^{-1} \ll \omega \ll \tau_2^{-1}$ is

$$v_p(\omega) \cong v_e \left(1 + \frac{1}{\pi Q_m} \ln\omega + \frac{1}{\pi Q_m} \ln\tau_1 \right)$$

or for $\tau_1^{-1} \ll \omega_1, \omega_2 \ll \tau_2^{-1}$

$$\frac{v_p(\omega_2)}{v_p(\omega_1)} \cong 1 + \frac{1}{\pi Q_m} \ln \frac{\omega_2}{\omega_1}. \tag{38}$$

Equation (38) was derived by Kolsky (1956) for constant Q material based on the experimental observation that the shape of an input pulse remains similar after travelling different

distances through the material. Identical results can be derived from the theories of Futterman (1962) and Lomnitz (1957).

4.3 EFFECT OF DISPERSION DUE TO ANELASTICITY ON SURFACE WAVE PHASE VELOCITIES AND ON THE PERIODS OF FREE OSCILLATION

The effect of velocity dispersion due to anelasticity on Rayleigh and Love wave phase velocities and on the spheroidal and toroidal free oscillation periods can be computed in general by a perturbation approach (Liu & Archambeau 1975, 1976). However, if the sound velocity dispersion due to attenuation is given by a relation like equation (38), a simpler scheme is possible. If we consider an earth model derived from short period ($\sim$1 Hz) body wave data as the reference model, the shear wave velocity at the Love wave angular frequency, ω, for example, is given by equation (38) to be:

$$\frac{\beta_l(\omega)}{\beta_l(2\pi)} = 1 + \frac{1}{\pi Q_{\beta l}} \ln \frac{\omega}{2\pi}; \quad \frac{\Delta \beta_l}{\beta_l(2\pi)} = \frac{\beta_l(\omega) - \beta_l(2\pi)}{\beta_l(2\pi)} = \frac{1}{\pi Q_{\beta l}} \ln \frac{\omega}{2\pi} \tag{39}$$

where l is the layer index.

It follows that the phase velocity of Love waves computed for a structure defined at a body wave frequency should be corrected at each frequency by

$$\Delta c_{\mathrm{L}} = \sum_{l=1}^{N} \frac{\partial c_{\mathrm{L}}}{\partial \beta_l} \Delta \beta_l = c_{\mathrm{L}} \left(\frac{1}{\pi} \ln \frac{\omega}{2\pi} \right) \sum_{l=1}^{N} \frac{\beta_l}{c_{\mathrm{L}}} \frac{\partial c_{\mathrm{L}}}{\partial \beta_l} Q_{\beta l}^{-1}$$

or

$$\Delta c_{\mathrm{L}}/c_{\mathrm{L}} = \frac{Q_{\mathrm{L}}^{-1}}{\pi} \ln \frac{\omega}{2\pi} \tag{40}$$

where Q_{L}^{-1} is the quality factor of the Love wave at angular frequency ω (Anderson *et al.* 1965), before it is compared with the observed phase velocity of Love waves at the same frequency. Similar arguments lead to

$$\Delta c_{\mathrm{R}}/c_{\mathrm{R}} = \frac{Q_{\mathrm{R}}^{-1}}{\pi} \ln \frac{\omega}{2\pi} \tag{41}$$

for Rayleigh waves.

By using available data of Q_{R} and Q_{L} — Fig. 4(a) — from Smith (1972), $\Delta c_{\mathrm{L}}/c_{\mathrm{L}}$ and $\Delta c_{\mathrm{R}}/c_{\mathrm{R}}$ are computed. The results are shown in Fig. 4(b). The longer period data correspond to free oscillations. Corrections of 0.5–1.5 per cent are required to make the observed phase velocity and normal mode data compatible with short-period ($\sim$1 Hz) body wave data. These corrections are very significant and should be considered in future inversion studies. This point has been made by Carpenter & Davies (1966) and Randall (1976). A reinversion of seismic data, corrected for attenuation, is in progress at the Seismological Laboratory.

4.4 EFFECT OF SLIGHT DEPARTURE FROM THE CONSTANT Q ASSUMPTION ON THE IMPULSE RESPONSE OF A LINEAR VISCOELASTIC SOLID

The pulse shape of a delta function input after travelling through an attenuative medium (impulse response) is important in the construction of synthetic seismograms (e.g. Carpenter 1967; Helmberger 1973). The impulse response of the linear viscoelastic solid model shown
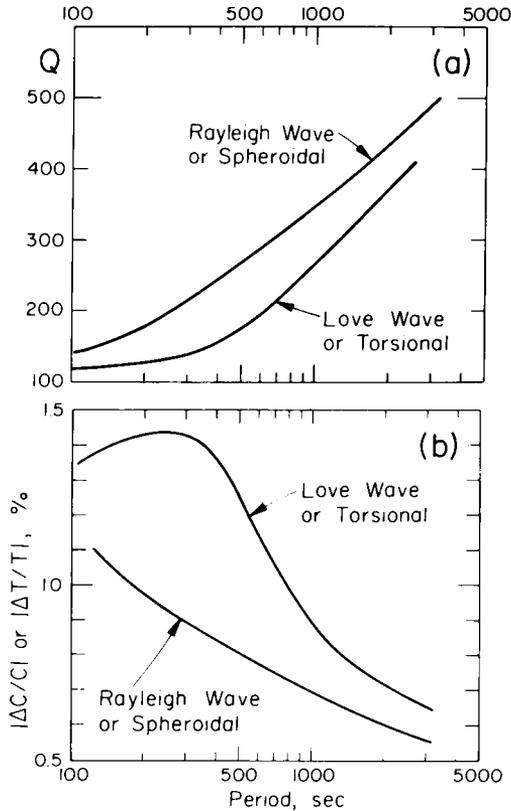
Figure 4. (a) Rayleigh and Love wave Q values (after Smith 1972). (b) Rayleigh and Love wave fractional phase velocity shift due to anelasticity.

in Fig. 3 is computed for distance $d = 4v_e Q_m$, $Q_m = 98.413$. Q_m^{-1} is the average Q^{-1} value of the superposition model in the approximately constant Q region. The result is compared with the impulse response computed from Futterman's attenuation—dispersion model D1 (Futterman 1962). Since the internal friction of the model in Fig. 3 is constant to ±1.25 per cent and the reduced quality factor Q_0 of Futterman's model D1 is precisely constant over the seismic band ($Q_0 = 98.413$), there is little difference in the two impulse responses (Fig. 5). The present observations of seismic attenuation, however, show that the seismic quality factor, Q^{-1}, may vary by a factor of 2 or 3 over the body wave spectrum and do not require a constant Q^{-1} over the seismic frequencies. An attenuation—dispersion pair with ±8 per cent variation in internal friction — Fig. 6(a) and (b) — is constructed from a superposition of relaxation mechanisms and the impulses response of a Futterman D1 model whose constant Q_0^{-1} value is selected to be the average Q_0^{-1} value of the model shown in Fig. 6(a). (Average over the seismic frequencies only). The pulse widths of these two impulse responses differ by ~12 per cent.

5 Discussion

5.1 COMPARISON WITH OTHER THEORIES

Attenuation—dispersion pairs over the seismic frequencies for a linear viscoelastic medium have been considered previously (Lomnitz 1957; Futterman 1962; Strick 1967; Azimi *et al.*
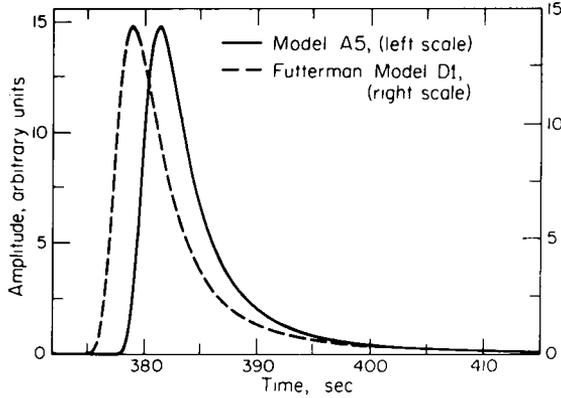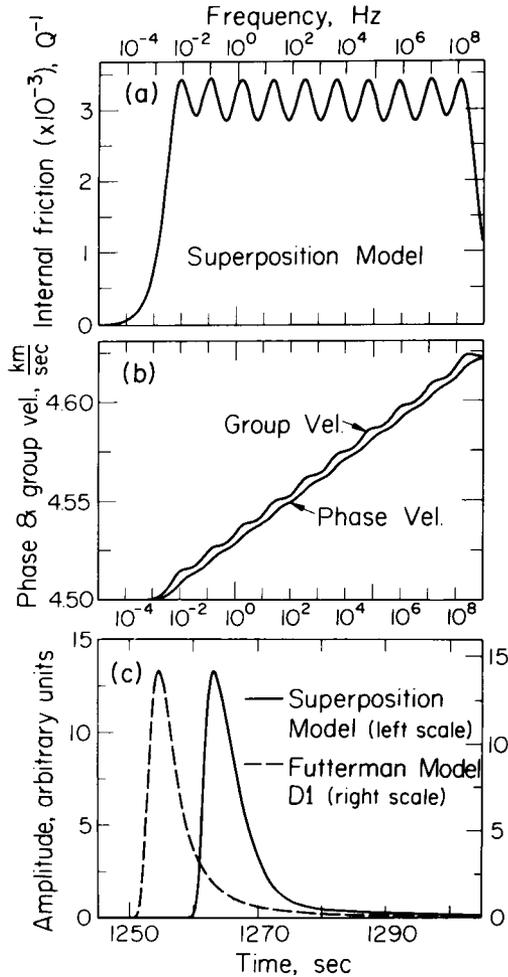
Figure 5. Comparison of impulse responses between relaxation model shown in Fig. 3 and Futterman model D1. Both models have constant Q over the seismic frequencies, and there is little difference in the pulse shapes. (Impulse response calculated by discrete Fourier transform. Arrival time corresponds closely to phase velocity at the upper cut-off frequency of the discrete Fourier transform window.)

1968; Randall 1976). Lomnitz used an empirical logarithmic creep function in the Boltzmann's after-effect equation to obtain an internal friction function that is approximately constant over a wide frequency range. The logarithmic creep function fails to pass at the limit to the static case (Kogan 1966). Most of the other considerations of attenuation— dispersion can be included in one category as the Futterman-type theory. This type of theory lacks a physical basis. An empirical $\text{Im}\{n(\omega)\}$ that is constant with frequency (or almost constant with frequency) over the seismic frequency range is substituted either into equation (6a) or equation (6b) to obtain the phase velocity dispersion. Both the manner and the choice of the cut-off frequencies of the constant behaviour of $\text{Im}\{n(\omega)\}$ is quite arbitrary. For Futterman's model D1 ($\text{Im}\{n(\omega)\} = 1/2Q_0$, $\omega > \omega_0$; $\text{Im}\{n(\omega)\} = 0$, $\omega < \omega_0$), the resulting velocity has a logarithmic singularity at $\omega = \omega_0$ and for the empirical attenuation adopted by Randall (1976), $\text{Im}\{n(\omega)\} = (\exp[-\tau_1\omega] - \exp[-\tau_2\omega])/2Q_0$, the maximum phase velocity exceeds $v_p(\infty)$. These models can not be represented by a linear viscoelastic solid with a spectrum of relaxation mechanisms, the phase velocity of which varies monotonically between $v_p(0)$ and $v_p(\infty)$. Futterman chose his 'non-dispersive' behaviour at zero frequency and therefore his impulse response arrives earlier than the non-dispersive signal. Strick (1967), Azimi *et al.* (1968) and Randall (1976), on the other hand, chose their 'non-dispersive' behaviour at infinite frequency, and their impulse responses arrive later than the non-dispersive signal. The difference in choice is whether to use the relaxed or the unrelaxed elastic modulus of the solid as the 'non-dispersive' modulus. This difference does not imply that Futterman's theory is acausal, as is generally asserted in the literature (e.g. Stacey *et al.* 1975). The present method of constructing the absorption—dispersion pair is based on observed relaxation behaviour in solids.

The phase velocity dispersion, equation (38),

$$\frac{v_p(\omega_2)}{v_p(\omega_1)} \cong 1 + \frac{1}{\pi Q} \ln\left(\frac{\omega_2}{\omega_1}\right)$$

holds for all the different theories (Lomnitz, Futterman type, Kolsky, superposition of relaxation mechanisms) in the frequency range where the internal friction function is assumed to be constant.

Figure 6. (a) Internal friction coefficient of a relaxation model which has ± 8 per cent over the seismic frequencies, (b) phase and group velocity dispersion of the same model, (c) comparison of impulse responses between this relaxation model and Futterman model D1. The pulse widths differ by ~ 12 per cent.

5.2 IMPLICATIONS IN SEISMOLOGY

Equation (38) implies a 1 per cent phase velocity dispersion between 1 and 1000 s (3 decades) for $Q \cong 220$. For $Q = 60$, as is the case for shear waves travelling in the Earth's low velocity, low Q zone, the phase velocity dispersion over 3 decades amounts to ~ 4 per cent. This large dispersion, together with the shifts in free oscillation period computed in Section 4.3, suggests the necessity of joint inversion of elastic and anelastic parameters in construction of an earth model from body wave travel-time and free oscillation period data.

The impulse response pulse width computed from an attenuation–dispersion model with ± 8 per cent variation in Q^{-1} over the seismic frequency band differs by ~ 12 per cent from that computed from a precisely constant Q^{-1} model. This does not have a serious effect on the synthetic seismograms constructed for natural earthquakes since such a difference is smaller in magnitude than the uncertainty in the time history of the fault motion. The precise dependence of Q^{-1} on frequency is clearly an important unknown. Even a ± 8 per

cent variation in Q^{-1} could affect the construction of synthetic seismogram for explosions whose focal time history can be determined more precisely than natural earthquakes.

Discussions of the base line problem in travel times and oceanic-continental travel-time differences should also take into account the effects of different regional attenuation on the phase velocities.

5.3 IMPLICATIONS IN MANTLE COMPOSITION STUDIES

The dispersion of phase velocity, i.e. elastic modulus, due to absorption affects the application of high pressure and high temperature laboratory measurements to interpretation of seismic data. Extrapolation using lattice dynamics (e.g. Sammis 1972), finite strain theories (e.g. Davies, 1973), and averaging schemes to determine elastic properties of aggregates from component minerals (e.g. Domany, Gubernatis & Krumhansl 1975) are necessary in using laboratory measurements of limited pressure and temperature range to infer mantle composition by comparison of laboratory data with seismic data. Dynamic shock wave data (Hugoniots), when used in mantle composition studies, are reduced to equivalent adabats and isotherms with time-independent equation of state (e.g. Duvall & Fowles 1963). The Earth and the minerals are assumed to be perfectly elastic in these extrapolation, averaging, and reduction schemes. Ultrasonic elastic properties, static compression data, shock wave data, and seismic earth models are assumed to correspond to the same frequency-independent elastic properties. This simplifying assumption is not justified considering the wide frequency range covered by shock wave data ($\sim 10^9$ Hz), ultrasonic data ($> 10^7$ Hz), seismic frequency band (~ 10 Hz to $\sim 3 \times 10^{-4}$ Hz), and static compression experiments ($\sim 10^{-5}$ Hz) and a lack of detailed absorption measurements across these frequencies. Under this assumption, a single crystal and a polycrystalline aggregate of the same chemical composition are assumed to have the same frequency-independent elastic properties. However, an isotropic single crystal of very high Q would not have the same elastic properties as a polycrystal of the same composition. The difference, aside from scattering, is determined by the additional absorption mechanisms present at grain boundaries of the polycrystalline material. One would have to measure and correct for scattering, the elastic modulus dispersion due to absorption and for the difference in absorption between single crystal and polycrystalline aggregates in order to use high pressure, high temperature ultrasonic, shock wave and static compression data to investigation of the composition of the mantle.

6 Conclusion

We have shown that a distribution of relaxation mechanisms can give a Q which satisfies seismic observations. A necessary consequence is a significant velocity dispersion. One cannot compare directly body wave, surface wave, and free oscillation data. This point has been made by Jeffreys (1967). One should take into account regional attenuation properties in comparing travel-time differences. Laboratory data should be corrected for dispersion and scattering before being used to infer mantle composition by comparing them with seismic data.

Although we have considered only absorption effects that can be attributed to linear relaxation mechanisms, other nonlinear effects such as static friction, nonlinear motion of dislocations, and scattering, distort the wave form and affect both the travel time of body waves and the apparent phase velocity of the surface waves. The magnitude of dispersion due to these effects can be estimated by physical considerations and by perturbation calculations.

Acknowledgments

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