Cubic interactions in *pp*-wave light-cone string field theory

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We use the supergravity modes to clarify the role of the prefactor in the light-cone superstring field theory on a *pp*-wave background. We verify some of the proposals of Constable *et al.* and give further evidence for the correspondence between $\mathcal{N}=4$ Super Yang-Mills gauge theory and string theory on a *pp*-wave. We also consider energy-preserving processes and find that they give a vanishing cubic interaction Hamiltonian matrix.

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I. INTRODUCTION

Recently, Berenstein, Maldacena, and Nastase (BMN) [1] came up with a very exciting proposal where we can test anti–de Sitter (AdS)/conformal field theory (CFT) correspondence beyond the supergravity approximation. It is based on the discovery [2,3] that Green-Schwarz strings on a pp-wave background are exactly solvable in the light-cone gauge, and the observation [4,5] that the pp-wave background can be obtained from $AdS_5\times S^5$ in the Penrose limit. Via the AdS/CFT dictionary, the authors of [1] have identified the corresponding limit in $\mathcal{N}=4$ super Yang-Mills (SYM) theory and argued that type IIB string theory on the pp-wave background is dual to a sector of operators with a large R charge $J\sim\sqrt{N}$ and finite $\Delta-J$ in the limit $N\to\infty$ while keeping g_{YM} fixed. In this limit, although the usual 't Hooft coupling g_{YM}^2N goes to infinity, perturbative SYM theory is well defined due to the near Bogomol'nyi-Prasad-Sommerfield (BPS) property of the operators under consideration. In particular, the duality allows one to compute the *free* string spectrum from a perturbative SYM calculation [1,6–8].

It is a very important and fascinating question whether this success can be extended to the *interacting* string theory. However, the holographic idea of AdS/CFT [9–11] does not seem to be directly applicable here and the first principle is not yet available.[1] Without fully understanding it, the natural framework for the interacting string theory on a pp-wave background is believed to be the light-cone string field theory, and the authors of [16] constructed the cubic interaction Hamiltonian following the light-cone string field theory formalism of [17–19]. The cubic interaction Hamiltonian was, roughly speaking, a three-string delta functional with a prefactor which was argued to be the same as that in flat spacetime.

Shortly after this development, the corresponding SYM objects were proposed to leading order in $g_{YM}^2N/J^2=1/(\mu\alpha' p^+)^2$ to be [8]

$$\langle 123|H_3\rangle=\mu(\Delta_3-\Delta_1-\Delta_2)C_{123}, \tag{1}$$

where Δ_i is the conformal dimension of the corresponding gauge theory operator $\mathcal{O}_i$ and C_{123} is the coefficient of the three-point function in the planar and free theory limit. It should be emphasized that the proposal is limited to processes where light-cone energy is preserved to leading order in g_{YM}^2N/J^2 so that it can be captured in perturbative SYM theory.[2] In this article, we perform an explicit check of this proposal for supergravity modes. Note that, for supergravity modes, if light-cone energy is preserved to leading order in $g_{YM}^2N/J^2=1/(\mu\alpha' p^+)^2$, it is preserved exactly since there are no corrections. In [8], it is further conjectured that C_{123} corresponds to the three-string interaction vertex and the dressing factor $\mu(\Delta_3-\Delta_1-\Delta_2)$ is reproduced by the prefactor. We explicitly check that this is the case for the particular bosonic excitations considered in [8] and extend the proposal to other excitations.

This paper is organized as follows. In Sec. II, we briefly review relevant materials and fix our convention. In Sec. III, we discuss the prefactor, and in Sec. IV, we compute the bosonic three-string Hamiltonian matrix elements and compare them to three-point functions on the gauge theory side. In Sec. V we consider general cubic interactions and in Sec. VI we end with a discussion.

While this manuscript was being prepared, some related articles [20–22] appeared in the archive. The authors of [20] fix the normalization of the cubic Hamiltonian constructed by [16], and then compute the matrix elements of chiral primaries and find agreement with SYM calculations as in Eq. (1). Some parts of Sec. IV were first computed in [21] by dropping the prefactor. Our work clarifies this point and considers general supergravity matrix elements.

II. REVIEW

A. *pp*-wave string/SYM correspondence

In this section, we briefly review pp-wave/SYM correspondence and fix our notation and convention. The pp-wave background is obtained by taking the Penrose limit of $AdS_5\times S^5$ and is given as

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[1] For some recent progress in this direction, see [12–15].

[2] It is argued in [8] that the process where light-cone energy is not preserved in leading order in g_{YM}^2N/J^2 corresponds to nonperturbative effects on the gauge theory side.

$$ds^2 = -4dx^+dx^- - \mu^2\left(\sum_{i=1}^{8} x^i x^i\right)dx^{+2} + \sum_{i=1}^{8} dx^{i\,2} \tag{2}$$

with Ramond-Ramond (RR) flux

$$F_{+1234} = F_{+5678} = 2\mu. \tag{3}$$

The metric has SO(8) rotational symmetry of the x^i's but it is explicitly broken to SO(4)×SO(4) by the RR flux. The Green-Schwarz string in this background is exactly solvable in the light-cone gauge [2,3] and the spectrum is a tower of free massive harmonic oscillators:

$$H_2 = \frac{1}{\alpha' p^+}\sum_{n=-\infty}^{\infty}\omega_n\left(\sum_{i=1}^{8} a_n^{i\,\dagger}a_n^i + \sum_{a=1}^{8} b_n^{a\,\dagger}b_n^a\right), \tag{4}$$

where $\omega_n = \sqrt{n^2 + (\alpha'\mu p^+)^2}$. BMN propose that the light-cone vacuum state is dual to a chiral primary operator

$$|\mathrm{vac}\rangle \leftrightarrow \frac{1}{\sqrt{J}\sqrt{N^J}}\mathrm{Tr}[Z^J]. \tag{5}$$

For zero modes or supergravity modes, we insert proper operators with $\Delta - J = 1$ at all possible positions in the vacuum operator. For excitations along the 1,2,3,4 direction we insert $D_i Z$, and for the 5,6,7,8 directions ϕ^i.

B. *pp*-wave light-cone string field theory

The authors of [16] constructed the cubic interaction Hamiltonian H_3 following the light-cone string field theory formalism of [19]. It can be expressed as

$$|H_3\rangle = \hat{h}_3|V\rangle. \tag{6}$$

$|V\rangle$ is just a kinematical three-string delta functional which preserves kinematical symmetries and is common to the other dynamical generators. $\hat{h}_3$ is a prefactor inserted at the interaction point to respect the whole supersymmetry algebra including dynamical symmetries. In [16], the prefactor is claimed to be of the same form as in flat spacetime since the prefactor arises from a worldsheet UV effect and the additional mass term in the pp wave should not affect it. More explicitly, they are given as

$$|V\rangle = E_a E_b|0\rangle, \quad \hat{h}_3 = \mathsf{P}^i\mathsf{P}^j v_{ij}(\Lambda). \tag{7}$$

Let us define $\alpha_r \equiv \alpha' p_r^+$. With $\alpha = \alpha_1\alpha_2\alpha_3$ and $\beta = \alpha_1/\alpha_3$, we have

$$E_a = \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{i=1}^{8} a_r^{\dagger i}M^{rs}a_s^{\dagger i}\right],$$

$$M = \begin{pmatrix} \beta+1 & -\sqrt{-\beta(1+\beta)} & -\sqrt{-\beta} \\ -\sqrt{-\beta(1+\beta)} & -\beta & -\sqrt{1+\beta} \\ -\sqrt{-\beta} & -\sqrt{1+\beta} & 0 \end{pmatrix}, \tag{8}$$

$$E_b = \lambda^1, \ldots, \lambda^8, \quad \lambda = \lambda_1 + \lambda_2 + \lambda_3,$$

$$\mathsf{P}^i = \alpha_1 p_2^i - \alpha_2 p_1^i, \quad \Lambda = \alpha_1\lambda_2 - \alpha_2\lambda_1,$$

$$v_{ij}(\Lambda) = \delta^{ij} + \frac{1}{6\alpha^2}\gamma_{ab}^{ik}\gamma_{cd}^{jk}\Lambda^a\Lambda^b\Lambda^c\Lambda^d + \frac{16}{8!\,\alpha^4}\delta^{ij}\epsilon_{abcdefgh}\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h. \tag{9}$$

Here, i,j and a,b are SO(8) vector and spinor indices, respectively.[3] We take $\alpha_1,\alpha_2>0$ and $\alpha_3<0$ such that $\alpha_1+\alpha_2+\alpha_3=0$. In addition, γ_{ab}^{ik} are the usual antisymmetrizations of gamma matrices and for concreteness we use a basis such that

$$\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}. \tag{10}$$

In this basis, λ takes the following form in terms of harmonic oscillator operators:

$$\lambda_r = \sqrt{\frac{\alpha_r}{2}}(b_r^{\dagger 1}b_r^{\dagger 2}b_r^{\dagger 3}b_r^{\dagger 4}b_r^5 b_r^6 b_r^7 b_r^8)^T, \quad r=1,2, \tag{11}$$

$$\lambda_3 = \sqrt{\frac{-\alpha_3}{2}}(b_3^1 b_3^2 b_3^3 b_3^4 b_3^{\dagger 5}b_3^{\dagger 6}b_3^{\dagger 7}b_3^{\dagger 8})^T. \tag{12}$$

Lastly, the "ground" state $|0\rangle$ is related to the "vacuum" state as [3]

$$|0\rangle = b_1^{\dagger 5}b_1^{\dagger 6}b_1^{\dagger 7}b_1^{\dagger 8}b_2^{\dagger 5}b_2^{\dagger 6}b_2^{\dagger 7}b_2^{\dagger 8}b_3^{\dagger 1}b_3^{\dagger 2}b_3^{\dagger 3}b_3^{\dagger 4}|\mathrm{vac}\rangle. \tag{13}$$

C. H_3 from perturbative SYM

Let $|1\rangle,|2\rangle,|3\rangle$ be free single string states with unit norm and $\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3$ be the corresponding gauge operators with unit two-point function

$$\langle\bar{\mathcal{O}}_i(0)\mathcal{O}_j(x)\rangle = \frac{\delta_{ij}}{(2\pi x)^{2\Delta_i}}. \tag{14}$$

Define C_{123} as

[3]Here the reader should not be confused with E_a, E_b where indices a,b refer to the bosonic and fermionic parts of the prefactor, respectively.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\overline{\mathcal{O}}_3(x_3)\rangle = \frac{\delta_{J_3,J_1+J_2}C_{123}}{(2\pi x_{13})^{\Delta_3+\Delta_1-\Delta_2}(2\pi x_{23})^{\Delta_3+\Delta_2-\Delta_1}(2\pi x_{12})^{\Delta_1+\Delta_2-\Delta_3}} \tag{15}$$

in the planar limit. In the regime of small $g_{YM}^2 N/J^2$, the authors of [8] propose to leading order in $g_{YM}^2 N/J^2$ that

$$\langle 123|H_3\rangle = \mu(\Delta_3-\Delta_1-\Delta_2)C_{123}. \tag{16}$$

This proposal successfully reproduced the mass renormalization of excited string states via second order quantum mechanical perturbation theory, and thereby passed a unitarity check. It is further conjectured that the prefactor reproduces the dressing factor while $|V\rangle$ corresponds to C_{123}. One should note that the proposal has been applied for matrix elements of bosonic string excitations along directions 5–8 only. The authors of [8] claimed that, for general bosonic excitations, the prefactor gives a factor of the form

$$\begin{aligned}&(\text{net No. of insertions along directions } 1\text{–}4)\\ &-(\text{net No. of insertions along directions } 5\text{–}8). \end{aligned} \tag{17}$$

By "net No.," we mean (No. in operator 1+ No. in operator 2− No. in operator 3). Lastly, they argued that the prefactor should lead to a vanishing result for fermionic excitations only.

In the following sections, we confirm this set of proposals for the matrix elements of the cubic interaction Hamiltonian H_3 restricted to the supergravity sector.

III. PREFACTOR

In this section, we illuminate the role of the prefactor term in the three-string interaction Hamiltonian; namely, we validate the conjecture of [8] by showing that the prefactor factorizes as expected in the literature only when one considers bosonic excitations without fermionic ones. In the Appendix, we show that the expression for the prefactor simplifies due to the following relation:

$$\begin{aligned}\mathsf{P}^i\mathsf{P}^j E_a|\text{vac}\rangle = {}&\mu\alpha_1\alpha_2[\alpha_2 a_1^{\dagger i}a_1^{\dagger j}+\alpha_1 a_2^{\dagger i}a_2^{\dagger j}\\ &-\sqrt{\alpha_1\alpha_2}(a_1^{\dagger i}a_2^{\dagger j}+a_1^{\dagger j}a_2^{\dagger i})]E_a|\text{vac}\rangle.\end{aligned} \tag{18}$$

Let us first consider purely bosonic excitations. In this case, v^{ij} in the prefactor simplifies to [18]

$$v^{ij} = \frac{1}{6\alpha^2}\gamma^{ik}_{ab}\gamma^{jk}_{cd}\Lambda^a\Lambda^b\Lambda^c\Lambda^d \tag{19}$$

and only the $t^{ij}_{5678}\Lambda^5\Lambda^6\Lambda^7\Lambda^8$ term survives in the prefactor, where

$$t^{ij}_{5678} \equiv \gamma^{ik}_{[56}\gamma^{jk}_{78]} = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4\end{pmatrix} \tag{20}$$

in the basis of gamma matrices we are using. Hence, for purely bosonic amplitudes, we only need to focus on the case when $i=j$. By employing a similar strategy as outlined in the Appendix, relation (18) with $i=j$ can be written as[4]

$$\mathsf{P}^i\mathsf{P}^i E_a|\text{vac}\rangle = -\mu\alpha(a_1^{\dagger i}a_1^i + a_2^{\dagger i}a_2^i - a_3^{\dagger i}a_3^i)E_a|\text{vac}\rangle. \tag{21}$$

Therefore, the prefactor takes the following form:

$$\begin{aligned}\frac{\mu}{6\alpha}\Bigg[&\sum_{i=1}^{4}(a_1^{\dagger i}a_1^i + a_2^{\dagger i}a_2^i - a_3^{\dagger i}a_3^i)\\ &-\sum_{i=5}^{8}(a_1^{\dagger i}a_1^i + a_2^{\dagger i}a_2^i - a_3^{\dagger i}a_3^i)\Bigg].\end{aligned} \tag{22}$$

The light-cone Hamiltonian is given as $\mu\Sigma_i a^{\dagger i}a^i$, and we have shown that, for only bosonic zero-mode excitations along directions 5–8, the prefactor becomes

$$\hat{h}_3 \sim (p_3^- - p_1^- - p_2^-) \tag{23}$$

up to an overall constant factor. Recall that bosonic excitations along directions 5–8 correspond to insertion of scalar defects on the SYM side. Furthermore, for generic bosonic overlaps, the prefactor is

$$\hat{h}_3 \sim \mu[(\hat{\Delta}_1+\hat{\Delta}_2-\hat{\Delta}_3)-(\tilde{\Delta}_1+\tilde{\Delta}_2-\tilde{\Delta}_3)], \tag{24}$$

where $\hat{\Delta}$ stands for the number of bosonic excitations along directions 1–4 and $\tilde{\Delta}$ stands for the number along directions 5–8.

Next we consider purely fermionic excitations. One can immediately see that three-string Hamiltonian matrix elements vanish due to the creation operator coming from Eq. (18) which acts to the left on the three-string vacuum. Since $\mathsf{P}^i\mathsf{P}^j$ vanishes for all i,j, the three-string fermionic amplitude vanishes.

One should note that the factorization of the prefactor term occurs here because there is no fermionic excitation. Once we include fermionic modes such that the a,b,c,d indices take value in both SO(4) subgroups of SO(8), the t^{ij}_{abcd}

[4]One can also see this relation for the case $i=j$ by using the fact that $(p_1+p_2+p_3)|V\rangle=0$ and $\alpha_1+\alpha_2+\alpha_3=0$. We thank M. Spradlin and A. Volovich for pointing this out.

matrices become nondiagonal. In such cases, the prefactor in general may not factorize in the simple form given in Eq. (24).

IV. SUPERGRAVITY VERTEX AND SYM THREE-POINT FUNCTION

Having clarified the effect of the prefactor in the previous section, let us proceed and compute the on-shell cubic interaction Hamiltonian matrix elements for bosonic supergravity modes. We restrict our attention to the excitations along directions 5–8 in order to match the matrix elements with the gauge theory three-point functions. We have explicitly shown that the prefactor factorizes as in Eq. (24). Dropping this overall factor, an on-shell process involving only the bosons can be evaluated to give

$$\langle\text{vac}|\prod_{i=5}^{8}\frac{(a_1^i)^{l_i}}{\sqrt{l_i!}}\frac{(a_2^i)^{m_i}}{\sqrt{m_i!}}\frac{(a_3^i)^{n_i}}{\sqrt{n_i!}}E_a|\text{vac}\rangle \Big/ \langle\text{vac}|E_a|\text{vac}\rangle = \prod_{i=5}^{8}\delta_{l_i+m_i,n_i}\frac{n_i!}{\sqrt{l_i!m_i!n_i!}}(M^{13})^{l_i}(M^{23})^{m_i} \tag{25}$$

with the on-shell condition $\Sigma_{i=5}^{8}(l_i+m_i-n_i)=0$. The proof is as follows. First of all, let us note that all the directions decompose and we only have to calculate one of eight directions and take their products. Since $M^{33}=0$, the terms expanded from the exponent of E_a that contract with a_3^i involve only a_1^i or a_2^i. Therefore, in order to obtain a nonvanishing value, $l_i+m_i-n_i\leqslant 0$ should be satisfied for all the directions i. Since the sum over i is zero, $l_i+m_i-n_i=0$ should hold for each direction i. In this way, the left-hand side is decomposed with respect to each direction i and the contribution of each can be treated separately. The combinatoric factor $n_i!$ comes from determining which a_3^i is to be chosen as a partner of a_1^i and a_2^i.

Now that we have general three-point overlaps of the bosonic supergravity excitations in light-cone superstring field theory, let us move to the gauge theory side to see whether these three-point functions can be reproduced.

It was proposed [1] that the one and two supergravity mode excitations in the string theory side correspond on the field theory side to

$$a_i^\dagger|\text{vac}\rangle\leftrightarrow\mathcal{O}_0^J\equiv\frac{1}{\sqrt{N^{J+1}}}\text{Tr}\phi_i Z^J, \tag{26}$$

$$a_i^\dagger a_j^\dagger|\text{vac}\rangle\leftrightarrow\mathcal{O}_{00}^J\equiv\frac{1}{\sqrt{N^{J+2}(J+1)}}\times\sum_{l=0}^{J}\text{Tr}\phi_i Z^l\phi_j Z^{J-l}. \tag{27}$$

Here the normalization factor is determined by normalizing the two-point function to 1. In general,

$$\frac{a^{\dagger n}}{\sqrt{n!}}|\text{vac}\rangle\leftrightarrow\mathcal{O}_{0**n}^J\equiv\frac{1}{\mathcal{N}_{J,n}}\sum\text{Tr}\phi^n Z^J. \tag{28}$$

Here the summation runs over all inequivalent operators with n ϕ's and J Z's and the number of them is the combinatoric factor of choosing n out of $J+n$ sites on a circle,

$$\mathcal{N}_{J,n}=\sqrt{N^{J+n}(J+n-1)!/(J!n!)}. \tag{29}$$

To see the correspondence with the string side, let us compute the planar diagram of $\langle\mathcal{O}_{0**l}^{J_1}\mathcal{O}_{0**m}^{J_2}\bar{\mathcal{O}}_{0**n}^{J}\rangle$. $U(1)_J$ charge conservation implies that $J_1+J_2=J$ for the correlation function to not vanish. Since the proposal of [8] holds only for energy-preserving processes, we restrict ourselves to the case $l+m=n$. In any case, the non-energy-preserving three-point functions scale as $1/J$ compared to the on-shell ones and vanish in the pp-wave limit. The only thing to do is to count the number of ways of contracting. First of all, we do not use the cyclic symmetry of the trace for $\bar{\mathcal{O}}_{0**n}^{J}$ and write down all the terms with the factor $1/(J+n)$. Then, we have $(J+n)$ ways to divide the string of the operators $\bar{\mathcal{O}}_{0**n}^{J}$ into the parts to be contracted with $\mathcal{O}_{0**l}^{J_1}$ and $\mathcal{O}_{0**m}^{J_2}$. Since we have included all the operators in $\bar{\mathcal{O}}_{0**n}^{J}$, we have (J_1+l) inequivalent ways to contract $\mathcal{O}_{0**l}^{J_1}$ and the contraction gives us the factor $(J_1+l-1)!/(J_1!l!)$. We also have similar factors for $\mathcal{O}_{0**m}^{J_2}$. Collecting all the factors, we find

$$\frac{1}{\mathcal{N}_{J,n}}\frac{1}{\mathcal{N}_{J_1,l}}\frac{1}{\mathcal{N}_{J_2,m}}\frac{1}{J+n}(J+n)(J_1+l)\times\frac{(J_1+l-1)!}{J_1!l!}(J_2+m)\frac{(J_2+m-1)!}{J_2!m!}N^{J+n-1} \tag{30}$$

in all. To compare with the result of field theory, let us take the limit $J,J_1,J_2\to\infty$ with J_1/J and J_2/J fixed. Since

$$\frac{(J+n-1)!}{J!}\sim J^{n-1}, \tag{31}$$

in the limit $J\to\infty$, the three-point function on the field theory side is

$$\langle\mathcal{O}_{0**l}^{J_1}\mathcal{O}_{0**m}^{J_2}\bar{\mathcal{O}}_{0**n}^{J}\rangle=\sqrt{\frac{n!}{l!m!}}\left(\frac{J_1}{J}\right)^{(l-1)/2}\times\left(\frac{J_2}{J}\right)^{(m-1)/2}\frac{J_1J_2}{N\sqrt{J}}. \tag{32}$$

Divided by the ground state amplitude

$$\langle\mathcal{O}^{J_1}\mathcal{O}^{J_2}\bar{\mathcal{O}}^{J}\rangle=\frac{\sqrt{JJ_1J_2}}{N}, \tag{33}$$

one has

$$\frac{\langle \mathcal{O}^{J_1}_{0**l}\mathcal{O}^{J_2}_{0**m}\overline{\mathcal{O}}^{J}_{0**n}\rangle}{\langle \mathcal{O}^{J_1}\mathcal{O}^{J_2}\overline{\mathcal{O}}^{J}\rangle}=\sqrt{\frac{n!}{l!m!}}\left(\frac{J_1}{J}\right)^{l/2}\left(\frac{J_2}{J}\right)^{m/2}. \quad (34)$$

This is exactly what is expected from the string theory calculation given in Eq. (25).

V. ON-SHELL CUBIC INTERACTION

In this section, we discuss general on-shell cubic interaction Hamiltonian matrix elements. Interestingly, we find that they all vanish. Since we are considering on-shell interactions, v^{ij} in the prefactor simplifies to a single term given in Eq. (19). First of all, consider the bosonic part of the matrix element, which is given as a sum over terms of the form

$$\langle \mathrm{vac}|(a_1)^p(a_2)^q(a_3)^r \mathsf{P}^i\mathsf{P}^j E_a|\mathrm{vac}\rangle, \quad (35)$$

where the excitations can be along any of the eight directions. When $r\geqslant p+q$, the contractions of the annihilation operators with the creation operators in Eq. (18) will bring down at least one factor of $M^{33}=0$. Since every term of the form (18) vanishes, we conclude that the full string amplitudes, even the off-shell ones, vanish when $r\geqslant p+q$ independent of the fermionic part of the interaction Hamiltonian.

Next, let us consider the fermionic part of the three-string Hamiltonian matrix element. It is given as a sum over terms of the form

$$\langle \mathrm{vac}|(b_1)^l(b_2)^m(b_3)^n\Lambda^a\Lambda^b\Lambda^c\Lambda^d\lambda^1\cdots\lambda^8|0\rangle. \quad (36)$$

Let us focus on the string 3 part. First of all, denote

$$(b_3)^n\sim(\hat{b}_3)^{n_1}(\tilde{b}_3)^{n_2}, \quad (37)$$

where $\hat{b}_3$ represents a fermionic zero mode excitation along one of the directions in $\{1,2,3,4\}$ and $\tilde{b}_3$ represents one in $\{5,6,7,8\}$ and $0\leqslant n_1,n_2\leqslant 4$ such that $n_1+n_2=n$. Since Λ involves contributions from strings 1 and 2 only, we can write the string 3 contribution to the fermionic amplitude as

$$(\hat{b}_3)^{n_1}(\tilde{b}_3)^{n_2}(\hat{\lambda})^4(\tilde{\lambda})^4(\hat{b}_3^\dagger)^4|\mathrm{vac}\rangle \quad (38)$$

with $\hat{\lambda}$ and $\tilde{\lambda}$ defined as for the b's. From the expression for λ, it is clear that $\tilde{b}_3$ operators must contract with n_2 of four $\tilde{\lambda}$'s and likewise $\hat{b}_3$'s must contract with n_1 of four $\hat{b}^\dagger$'s to have nonvanishing overlap. Hence, we have

$$(\hat{b}_3)^{n_1}(\tilde{b}_3)^{n_2}(\hat{\lambda})^4(\tilde{\lambda})^4(\hat{b}_3^\dagger)^4|\mathrm{vac}\rangle \sim(\hat{\lambda})^4(\tilde{\lambda})^{4-n_2}(\hat{b}_3^\dagger)^{4-n_1}|\mathrm{vac}\rangle. \quad (39)$$

Notice that this implies that $4-n_1$ of four $\hat{\lambda}$'s must take the form $\hat{b}_3$ in order to have a nonzero contribution. This further implies that

$$(\hat{b}_3)^{n_1}(\tilde{b}_3)^{n_2}(\hat{\lambda})^4(\tilde{\lambda})^4(\hat{b}_3^\dagger)^4|\mathrm{vac}\rangle\sim(\hat{\lambda})^{n_1}(\tilde{\lambda})^{4-n_2}|\mathrm{vac}\rangle, \quad (40)$$

where the $b_{1,2}^\dagger$ parts of $\hat{\lambda}$ and $b_{1,2}$ parts of $\tilde{\lambda}$ can contribute. Therefore, we have

$$(\hat{b}_3)^{n_1}(\tilde{b}_3)^{n_2}(\hat{\lambda})^4(\tilde{\lambda})^4(\hat{b}_3^\dagger)^4|\mathrm{vac}\rangle \sim(b_{1,2}^\dagger)^{n_1}(b_{1,2})^{4-n_2}|\mathrm{vac}\rangle. \quad (41)$$

Denoting $\Lambda^a\Lambda^b\Lambda^c\Lambda^d\sim(b_{1,2})^j(b_{1,2}^\dagger)^{4-j}$ with $0\leqslant j\leqslant 4$, the full fermionic contribution can be written as a sum over j of terms of the form

$$\langle \mathrm{vac}|(b_{1,2})^j(b_{1,2}^\dagger)^{4-j}(b_{1,2})^{l+m}(b_{1,2})^{4-n_2} \times(b_{1,2}^\dagger)^{n_1}(b_{1,2}^\dagger)^8|\mathrm{vac}\rangle. \quad (42)$$

In order to have nonvanishing overlap, the number of $b_{1,2}$'s must balance the number of $b_{1,2}^\dagger$'s. This imposes the following condition:

$$j+l+m+4-n_2=4-j+n_1+8. \quad (43)$$

Since j ranges from 0 to 4, we have

$$n\leqslant l+m\leqslant 8+n. \quad (44)$$

Consider a general on-shell overlap of the form

$$\langle \mathrm{vac}|(a_1)^p(b_1)^l(a_2)^q(b_2)^m(a_3)^r(b_3)^n|H_3\rangle, \quad (45)$$

with $l+m+p+q=n+r$. If $l+m<n$ one immediately sees that the amplitude vanishes trivially. When $l+m\geqslant n$, then $p+q\leqslant r$ and the amplitude again vanishes since the bosonic part vanishes. Therefore, we conclude that, interestingly, all energy-preserving three-string interaction Hamiltonian elements vanish. It would be worthwhile to explore how to generalize the proposal of [8] to the general case we considered in this section.

VI. DISCUSSION

In this article, we have clarified the role of the prefactor and the three-string delta functional in the cubic interaction Hamiltonian H_3 in the zero mode, supergravity, sector. For purely bosonic on-shell excitations of the light-cone vacuum, the prefactor gives the expected dressing factor where 1,2,3,4 and 5,6,7,8 directions contribute with opposite signs. The important point of the calculation is that the prefactor contribution can be factored out and we need only consider the three-string delta functional contribution $\langle 123|V\rangle$. Furthermore, it is shown to correspond to the three-point function C_{ijk} in $\mathcal{N}=4$ SYM gauge theory. Here, it is crucial that two different combinatoric considerations on the string side and on the gauge theory side agree in the large J limit.

We have also shown that for all on-shell supergravity modes including fermionic excitations the cubic interaction Hamiltonian matrix element vanishes. However, for generic supergravity excitations, the role of the prefactor is not as clear as for the purely bosonic ones since it does not factorize for general excitations. It would be interesting to explore this case further.

Note added

In this paper we used the Neumann coefficients of the supergravity vertex $M^{(rs)}$. However, it was discussed in [21] that the supergravity vertex does not match with the zero modes of the string vertex $\overline{N}_{00}^{(rs)}$ in the large $\mu\alpha' p^+$ limit where the perturbative gauge theory computation is valid. In general, one has

$$\frac{\overline{N}_{00}^{(rs)}}{M^{rs}} = 1 + \mu\alpha B^T \frac{1}{\Gamma_+} B \equiv \mathcal{R} \tag{46}$$

for $r,s=1,2$, while $\overline{N}_{00}^{(rs)}/M^{rs}=1$ holds with $r=3$ or $s=3$.[5] One can evaluate the behavior of $\mathcal{R}$ using the methods discussed in [26] and find that $\mathcal{R}\to 0$ as $\mu\to\infty$ and $\mathcal{R}\to 1$ as $\mu\to 0$. Hence, $\overline{N}_{00}^{(rs)}/M^{rs}=1$ $(r,s=1,2)$ holds only when $\mu\to 0$, while for $r=3$ or $s=3$, $\overline{N}_{00}^{(rs)}$ equals M^{rs} for all values of μ. The physical interpretation of this result is clear. The supergravity vertex is constructed by assuming that the zero modes decouple from the higher ones. Generally, this is not true because $X^{(r)}$ $(r=1,2)$ has nonvanishing overlap between the zero modes and the higher ones. Hence, the matching between $\overline{N}_{00}^{(rs)}$ and M^{rs} occurs only in the flat space limit $\mu\alpha' p^+\to 0$.

In the correspondence between the string theory side and the field theory side of Sec. IV, only the nonrenormalized Neumann coefficients $\overline{N}_{00}^{(rs)}$ for $r=3$ or $s=3$ matter. This corresponds to the fact that the three-point functions of the chiral primary operators are not renormalized [23,24]. When we identify the prefactor in the string field theory as Eq. (24) in Sec. II and the Appendix, the interpolating Neumann coefficients $\overline{N}_{00}^{(rs)}$ for $r,s=1,2$ also appear. However, our claim in the present paper does not change if one repeats the analysis using the full string field theory three-string Hamiltonian. Details of these results will appear in our forthcoming paper [27].

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APPENDIX: PREFACTOR

In this appendix, we evaluate $\mathsf{P}^i\mathsf{P}^j E_a|\mathrm{vac}\rangle$. Let us first consider the case when $i=j$ and define[6]

$$\mathsf{P}^i \equiv \pi^i + \pi^{\dagger i} \tag{A1}$$

such that

$$(\mathsf{P}^i)^2 E_a|\mathrm{vac}\rangle = \left(\pi^i\pi^i + 2\pi^{\dagger i}\pi^i + \pi^{\dagger i}\pi^{\dagger i} + \frac{\alpha_1\alpha_2\mu}{4}\alpha_3 \right) E_a|\mathrm{vac}\rangle, \tag{A2}$$

where

$$\pi^i = \frac{1}{2}(\alpha_1\sqrt{\alpha_2\mu}\,a_2^i - \alpha_2\sqrt{\alpha_1\mu}\,a_1^i). \tag{A3}$$

Explicitly, one has

$$(\pi^i)^2 = \frac{\alpha_1\alpha_2\mu}{4}(\alpha_1 a_2^i a_2^i - 2\sqrt{\alpha_1\alpha_2}\,a_1^i a_2^i + \alpha_2 a_1^i a_1^i). \tag{A4}$$

By using the fact that $[a, f(a^\dagger)] = (\partial/\partial a^\dagger) f(a^\dagger)$, one can straightforwardly evaluate Eq. (A2) term by term. For example,

$$a_2^i a_2^i E_a|\mathrm{vac}\rangle = \left(M^{22} + \sum_{rs} M^{2r}M^{2s} a_r^{\dagger i} a_s^{\dagger i} \right) E_a|\mathrm{vac}\rangle, \tag{A5}$$

and one gets similar expression for other terms. Putting all the pieces together, we have

$$\begin{aligned}(\pi^i)^2 E_a|\mathrm{vac}\rangle = \frac{\alpha_1\alpha_2\mu}{4}\{&(\alpha_1 M^{22} - 2\sqrt{\alpha_1\alpha_2}M^{12} + \alpha_2 M^{11}) + (\alpha_1 M^{21}M^{21} - 2\sqrt{\alpha_1\alpha_2}M^{21}M^{11} + \alpha_2 M^{11}M^{11}) a_1^{\dagger i} a_1^{\dagger i} \\ &+ 2(\alpha_1 M^{21}M^{22} - 2\sqrt{\alpha_1\alpha_2}M^{21}M^{21} + \alpha_2 M^{11}M^{12}) a_1^{\dagger i} a_2^{\dagger i} + 2(\alpha_1 M^{21}M^{23} - \sqrt{\alpha_1\alpha_2}(M^{21}M^{13} + M^{23}M^{11}) \\ &+ \alpha_2 M^{11}M^{13}) a_1^{\dagger i} a_3^{\dagger i} + (\alpha_1 M^{22}M^{22} - 2\sqrt{\alpha_1\alpha_2}M^{22}M^{12} + \alpha_2 M^{12}M^{12}) a_2^{\dagger i} a_2^{\dagger i} + 2(\alpha_1 M^{22}M^{23} \\ &- \sqrt{\alpha_1\alpha_2}(M^{22}M^{13} + M^{23}M^{12}) + \alpha_2 M^{12}M^{13}) a_2^{\dagger i} a_3^{\dagger i} + (\alpha_1 M^{23}M^{23} - 2\sqrt{\alpha_1\alpha_2}M^{23}M^{13} \\ &+ \alpha_2 M^{13}M^{13}) a_3^{\dagger i} a_3^{\dagger i}\} E_a|\mathrm{vac}\rangle.\end{aligned} \tag{A6}$$

[5]The definition of B and Γ_+ can be found in [25].

[6]In this appendix, repeated indices are not summed over.

The above expression can be simplified to

$$(\pi^i)^2 E_a|\text{vac}\rangle=\frac{\alpha_1\alpha_2\mu}{4}(-\alpha_3+\alpha_2 a_1^{\dagger i}a_1^{\dagger i}+\alpha_1 a_2^{\dagger i}a_2^{\dagger i}-2\sqrt{\alpha_1\alpha_2}a_1^{\dagger i}a_2^{\dagger i})E_a|\text{vac}\rangle. \quad \text{(A7)}$$

Also,

$$(\pi^{\dagger i})^2|\text{vac}\rangle=\frac{\alpha_1\alpha_2\mu}{4}(\alpha_2 a_1^{\dagger i}a_1^{\dagger i}+\alpha_1 a_2^{\dagger i}a_2^{\dagger i}-2\sqrt{\alpha_1\alpha_2}a_1^{\dagger i}a_2^{\dagger i})E_a|\text{vac}\rangle. \quad \text{(A8)}$$

Note that this has the same expression as above up to a constant. Lastly, we have

$$2\pi^{\dagger i}\pi^i|\text{vac}\rangle=\frac{\alpha_1\alpha_2\mu}{2}(\alpha_2 a_1^{\dagger i}a_1^{\dagger i}+\alpha_1 a_2^{\dagger i}a_2^{\dagger i}-2\sqrt{\alpha_1\alpha_2}a_1^{\dagger i}a_2^{\dagger i})E_a|\text{vac}\rangle. \quad \text{(A9)}$$

Summing over all the contributions, we conclude

$$(\mathsf{P}^i)^2 E_a|\text{vac}\rangle=\mu\alpha_1\alpha_2(\alpha_2 a_1^{\dagger i}a_1^{\dagger i}+\alpha_1 a_2^{\dagger i}a_2^{\dagger i}-2\sqrt{\alpha_1\alpha_2}a_1^{\dagger i}a_2^{\dagger i})E_a|\text{vac}\rangle. \quad \text{(A10)}$$

The proof for the case when $i\neq j$ is analogous to this one and the expression given in Eq. (21) holds for it as well.

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